Homework 2

1. Use the definition of the Ito integral to compute directly

$$\int_0^t s^2 \, dB_s \tag{0.1}$$

2. Show directly (using the definition of the integral rather than Itô's formula) that

$$(B_t)^2 = \int_0^t 2B_s \, dB_s + t \tag{0.2}$$

3. We have shown in class that Brownian motion paths are nowhere differentiable with probability one. Use the same method to show that there is some $\alpha \in (0,1)$ so that Brownian motion is nowhere locally Hölder with exponent α, that is, there is no s so that

$$|B(t) - B(s)| \leq C|t-s|^\alpha,$$

for all $t \in [0,1]$. Try to make α as small as possible.

4. (i) Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and let $\tau(x)$ be the solution of the boundary value problem

$$-\Delta\tau = 1 \text{ in } \Omega,$$

with the boundary condition $\tau(x) = 0$ for all $x \in \partial\Omega$. Use the maximum principle to show that $\tau(x) > 0$ for all $x \in \Omega$.

(ii) Show that there exists $\lambda_0 > 0$ so that if $0 < \lambda < \lambda_0$ then $\tau(x)$ satisfies

$$-\Delta\tau \geq \lambda e^{\tau(x)} \text{ in } \Omega.$$

(iii) Let $0 < \lambda < \lambda_0$ and consider a sequence of functions $\phi_n(x)$ defined iteratively as follows: $\phi_0(x) \equiv 0$, and $\phi_{n+1}(x)$ is the solution of the boundary value problem

$$-\Delta\phi_{n+1} = \lambda e^{\phi_n(x)} \text{ in } \Omega,$$

with the boundary condition $\phi_{n+1}(x) = 0$ for $x \in \partial\Omega$. Use the maximum principle to show that the sequence $\phi_n(x)$ is increasing in n for all $x \in \Omega$: $\phi_{n+1}(x) \geq \phi_n(x)$. Show also that $\phi_n(x) \leq \tau(x)$ for all $x \in \Omega$ and all $n \geq 1$, also using the maximum principle.

(iv) Show that the nonlinear boundary value problem

$$-\Delta\phi = \lambda e^\phi \text{ in } \Omega, \ \phi(x) = 0 \text{ for } x \in \partial\Omega,$$

has a positive solution for all $\lambda \in (0,]\lambda_0)$.

5. The notation

$$\int_0^t f(s,\omega) \circ dB_s$$

denotes the Stratonovich integral. Compute

$$\int_0^t B_s \circ dB_s.$$

6. Given $t > 0$ set $\Delta t_k = t/2^k$ and $t_j = j\Delta t_k$, $0 \le j \le 2^k$. Define also

$$Y(t,\omega) = \sum_{j=0}^{2^k-1} (B_{t_{k+1}} - B_{t_k})^2.$$

Show that

$$E\left(Y(t,\omega) - t\right)^2 = 2\sum_{j=0}^{2^k-1} (\Delta t_k)^2,$$

and deduce that the random variable $Y(t,\omega)$ converges to a deterministic function t in $L^2(\Omega)$ as $\Delta t_k \to 0$.

7. Show that the law of a d-dimensional Brownian motion is invariant with respect to rotation. That is, if B_t is a d-dimensional Brownian motion, and Θ is an orthogonal matrix ($\Theta^{-1} = \Theta^T$), then $P(B_t \in A) = P(\tilde{B}_t \in A)$ where $\tilde{B}_t = \Theta B_t$.