Math 227 Winter 2009

Solutions to HW #2

Exercise 1. Let $\{s_j\}_{j=0\ldots N}$ be a partition of the interval $[0, t]$. We will first exhibit a sequence of suitable simple functions to be able to evaluate the integral. Define $\phi_N(t,\omega) := {s_j}^2\, \mathbf{1}_{\{t\in[s_j,s_{j+1})}$. Then we see that

$$\mathbb{E}\int_0^t (s^2-\phi(s,\omega))^2\, ds \leqslant \sum_{j=0}^{N-1} (s_{j+1}^2-s_j^2)^2 \leqslant \left[\max_j |s_{j+1}-s_j|\right] \sum_{j=0}^{N-1} 4\, s_{j+1}^2 (s_{j+1}-s_j),$$

so when we take a sequence of partitions with mesh going to zero as $N\to\infty$, the maximum on the RHS will tend to and the sum will tend to $\int_0^t s^2\, ds = t^3/3 < \infty$. Hence our integral is the L^2 limit as $N\to\infty$ of

$$\sum_{j=0}^{N-1} s_j^2 (B_{s_{j+1}}-B_{s_j}) \sim N\left(0, \sum_{j=0}^{N-1} s_j^4\, (s_{j+1}-s_j)\right),$$

where we recognize the variance to be a Riemann sum approximation to the integral $\int_0^t s^4\, ds = t^5/5$. Thus finally

$$\int_0^t s^2\, dB_s \sim N(0, t^5/5).$$

Exercise 2. With the partition as above, we write $(B_{s_{j+1}})^2 - (B_{s_j})^2 = 2\, B_{s_j} (B_{s_{j+1}} - B_{s_j}) + (B_{s_{j+1}} - B_{s_j})^2$ and summing over j gives

$$(B_{s_N})^2 = \sum_{j=0}^{N-1} \{(B_{s_{j+1}})^2 - (B_{s_j})^2\} = \sum_{j=0}^{N-1} 2\, B_{s_j} (B_{s_{j+1}} - B_{s_j}) + \sum_{j=0}^{N-1} (B_{s_{j+1}} - B_{s_j})^2 \tag{1}$$

Defining $\phi(t,\omega) := B_{s_j}\, \mathbf{1}_{\{t\in[s_j,s_{j+1})\}}$ we have

$$\mathbb{E}\int_0^t (B_s - \phi(s,\omega))^2\, ds = \sum_{j=0}^{N-1} \int_{s_j}^{s_{j+1}} \mathbb{E}(B_s - B_{s_j})^2\, ds = \sum_{j=0}^{N-1} \int_{s_j}^{s_{j+1}} (s - s_j)\, ds = \frac{\sum_{j=0}^{N-1} (s_{j+1}-s_j)^2}{2}$$

which goes to zero as $N\to\infty$ since the mesh of the partitions goes to zero. Hence the first sum on the right-hand side of (1) tends to an Itô integral as before.

For the second sum right-hand side of (1) we need a special argument. Observe that, writing $\Delta B_j = B_{s_{j+1}} - B_{s_j}$ and $\Delta s_j = s_{j+1} - s_j$

$$\begin{aligned}
\mathbb{E}\left\{\sum_{j=0}^{N-1} (\Delta B_j)^2 - t\right\}^2 &= \mathbb{E}\left\{\sum_{j=0}^{N-1} (\Delta B_j)^4 + \sum_{j,k=0}^{N-1} (\Delta B_j)^2 (\Delta B_k)^2 - 2\, t \sum_{j=0}^{N-1} (\Delta B_j)^2 + t^2\right\} \\
&= \sum_{j=0}^{N-1} 3\, (\Delta s_j)^2 + \sum_{j,k=0;\, j\neq k}^{N-1} \Delta s_j \Delta s_k - 2\, t \sum_{j=0}^{N-1} \Delta s_j + t^2 \\
&\leqslant 3\, t \max_j |\Delta s_j| + t^2 - 2\, t^2 + t^2
\end{aligned}$$

which tends to zero as $N\to\infty$ and the mesh of the partitions goes to zero. Hence the second sum in (1) converges in L^2 to t and so the whole expression (1) becomes

$$(B_t)^2 = \int_0^t 2\, B_s\, dB_s + t$$

as required.

Exercise 3. Fix $\beta > 0$, $\alpha \in (1/2, 1)$ and an integer $l = \left\lceil \frac{1}{\alpha - 1/2} \right\rceil > 0$. For all $n > 2\,(l+1)$ let $A_n \subseteq C[0,1]$ be the set of all functions f for which there exists an s such that for all t with $|s-t| \leqslant \frac{l+1}{n}$ we have

$$|B_t - B_s| \leqslant \beta\, |t-s|^{\alpha}. \tag{2}$$

We will show that $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$. For any $f \in A_n$ define $k' = \max\{j : j/n \leqslant s\}$ (where s is the special s for which inequality (2) holds) and $k = \max\{k' - l, 0\}$. We then have $0 \leqslant k \leqslant n - l$ and moreover by triangle inequality we have

$$|f(\frac{k+i}{n}) - f(\frac{k+i-1}{n})| \leqslant |f(\frac{k+i}{n}) - f(s)| + |f(s) - f(\frac{k+i-1}{n})| \leqslant \beta\, |\frac{k+i}{n} - s|^{\alpha} + \beta |\frac{k+i-1}{n} - s|^{\alpha}$$

for $i = 1 \ldots l$ and bounding the RHS we get

$$\max\left\{\left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right|, \left|f\left(\frac{k+2}{n}\right) - f\left(\frac{k+1}{n}\right)\right|, \ldots, \left|f\left(\frac{k+l}{n}\right) - f\left(\frac{k+l-1}{n}\right)\right|\right\} \leqslant 2\,\beta \left(\frac{l+1}{n}\right)^{\alpha}.$$

Denote by B_n the set of all functions f for which this is true for some $k \in \{0, \ldots, n - 2\,l\}$, so that $A_n \subseteq B_n$. Now it is enough to show that $\lim_{n\to\infty} \mathbb{P}(B_n) = 0$. To this end we use the independent increments property of Brownian motion as follows

$$\begin{aligned}
\mathbb{P}(B_n) &\leqslant \sum_{k=0}^{n-2l} \mathbb{P}\left[\max\left\{\left|B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right)\right|, \ldots, \left|B\left(\frac{k+l}{n}\right) - B\left(\frac{k+l-1}{n}\right)\right|\right\} \leqslant 2\,\beta \left(\frac{l+1}{n}\right)^{\alpha}\right] \\
&\leqslant n\, \mathbb{P}\left[\max\left\{\left|B\left(\frac{1}{n}\right) - B(0)\right|, \ldots, \left|B\left(\frac{l}{n}\right) - B\left(\frac{l-1}{n}\right)\right|\right\} \leqslant 2\,\beta \left(\frac{l+1}{n}\right)^{\alpha}\right] \\
&\leqslant n\, \mathbb{P}\left[\left|B\left(\frac{1}{n}\right)\right| \leqslant 2\,\beta \left(\frac{l+1}{n}\right)^{\alpha}\right]^{l} \\
&\leqslant n\left[\sqrt{\frac{n}{2\,\pi}} \int_{-2\beta[(l+1)/n]^{\alpha}}^{2\beta[(l+1)/n]^{\alpha}} e^{-x^2 n/2}\, dx\right]^{l} \\
&\leqslant n\left[\sqrt{\frac{n}{2\,\pi}}\, 4\,\beta \left(\frac{l+1}{n}\right)^{\alpha}\right]^{l} \\
&\leqslant C n^{1 + l(1/2 - \alpha)}.
\end{aligned}$$

Thus since $l = \left\lceil \frac{1}{\alpha - 1/2} \right\rceil$, we have $1 - l\,(1 - 2\alpha) = -\,\delta$ for some $\delta > 0$ and so $\mathbb{P}(B_n) \leqslant C n^{-\delta} \to 0$, as $n \to \infty$.

Since $\beta > 0$ was arbitrary, this shows that almost surely Brownian motion is nowhere locally Hölder continuous with exponent α for any $\alpha \in (1/2, 1)$. A different method can show that this is also true for $\alpha = 1/2$. At the same time, Brownian motion actually is α-Hölder continuous for all $\alpha \in (0, 1/2)$.

Exercise 4. In this question we are dealing with a smooth and bounded domain $\Omega \subset \mathbb{R}^n$.

i. By the maximum principle in the form of, e.g., Theorem 3, Section 6.4.2., L.C. Evans, *Partial Differential Equations* for $L = -\Delta$ (or Theorem 2.7.1 in the notes where we take $u(x,t) = \tau(x)$ for all (x,t), so that $u_t = 0 \geqslant -1 = \Delta\tau = \Delta u$) it follows that either τ is constant or $\tau(x) > 0$ on Ω. But τ constant would mean $\tau \equiv 0$ on $\overline{\Omega}$, implying $\Delta\tau \equiv 0$, which contradicts the definition of τ. Hence we must have $\tau(x) > 0$ on Ω.

ii. Clearly it is enough to show that τ is bounded on Ω. Then taking $0 < \lambda_0 \leqslant \exp[-\max_{x\in\Omega} \tau(x)]$ means that for any $0 < \lambda < \lambda_0$

$$-\Delta\tau = 1 \geqslant e^{-\max_{x\in\Omega}\tau(x)}\, e^{\tau(x)} \geqslant \lambda\, e^{\tau(x)}.$$

That τ is indeed bounded follows from the fact that it is continuous on the bounded domain $\overline{\Omega}$ and $\tau(x)=0$ on $\partial\Omega$.

iii. We will proceed by induction on n. First, for $n=1$ note that $-\Delta\phi_1=\lambda>0$, so just like in part (i), it follows that $\phi_1\geqslant 0\equiv\phi_0$ on Ω. For the inductive step, notice that we always have

$$-\Delta(\phi_{n+1}-\phi_n)=\lambda\,(e^{\phi_n(x)}-e^{\phi_{n-1}(x)})\geqslant 0,$$

where the RHS inequality follows from the inductive assumption of $\phi_n\geqslant\phi_{n-1}$ on Ω. So again by the maximum principle in the form used in part (i) but this time applied to $u(x)=\phi_{n+1}(x)-\phi_n(x)$, we conclude that $\phi_{n+1}\geqslant\phi_n$ on Ω. Hence we have $\phi_{n+1}\geqslant\phi_n$ on Ω for all $n\geqslant 1$.

Finally, considering that $\tau>0\equiv\phi_0$ and

$$-\Delta(\tau-\phi_n)\geqslant\lambda\,(e^{\tau(x)}-e^{\phi_{n-1}(x)})\geqslant 0$$

it is clear that induction also shows that $\tau\geqslant\phi_n$ on Ω for any n.

iv. Define $\phi(x)\equiv\lim_{n\to\infty}\phi_n(x)$ for $x\in\overline{\Omega}$, noting that this limit exists and is finite, since the sequence is increasing and bounded. Clearly $\phi(x)=0$ for $x\in\partial\Omega$ and $\phi(x)\geqslant 0$ for $x\in\Omega$, since this is true for all ϕ_n. We will first show that ϕ is a solution to the nonlinear problem in the weak sense.

Let $\xi\in L^2(\Omega)$ be any smooth function on Ω equal to zero on the boundary. Then for any $n\geqslant 1$ integration by parts gives

$$\int_\Omega -\phi_n\,\Delta\xi\,dx=\int_\Omega -\Delta\phi_n\,\xi\,dx=\int\lambda\,e^{\phi_{n-1}}\,\xi\,dx.$$

Now notice that in view of $\tau\geqslant\phi_n$, both integrands are dominated by integrable functions. Hence taking $n\to\infty$ dominated convergence yields

$$\int_\Omega \phi\,\Delta\xi\,dx=\int\lambda\,e^{\phi}\,\xi\,dx,$$

i.e. ϕ is a weak solution to the nonlinear PDE $-\Delta u=\lambda\,e^u$.

It remains to show that ϕ is an actual solution (up to a set of measure zero), which is somewhat technical and draws on the material in Part II of Evans. We are going to leverage the fact that the PDE is elliptic and clearly $f\equiv\lambda\,e^{\phi}\in L^2(\Omega)$, from which (Theorem 1, Section 6.3.1 in Evans) it will follow that $\phi\in H^2(\Omega)$. But this implies that also $f\in H^2(\Omega)$. Now, by Theorem 2, Section 6.3.1 in Evans, $f\in H^m(\Omega)$ implies $\phi\in H^{m+2}(\Omega)$, and so in turn $f\in H^{m+2}(\Omega)$. Thus actually $\phi\in H^m(\Omega)$ for all $m\geqslant 1$, so that Theorem 6, Section 5.6.3 in Evans gives that $\phi\in C^\infty(\Omega)$, i.e. ϕ is smooth.

Exercise 5. From the definition of the Stratonovich integral, letting $\{s_j\}_{j=0\ldots N}$ be a partition of the interval $[0,t]$, we have

$$\begin{aligned}\int_0^t B_s\circ d\,B_s &= \lim_{\Delta s_j\to 0}\sum_{j=0}^{N-1}\frac{1}{2}\,(B_{s_{j+1}}+B_{s_j})\,(B_{s_{j+1}}-B_{s_j})\\ &= \lim_{\Delta s_j\to 0}\frac{1}{2}\sum_{j=0}^{N-1}\,(B_{s_{j+1}}^2-B_{s_j}^2)\\ &= \lim_{\Delta s_j\to 0}\frac{1}{2}\,(B_t^2-B_0^2)\\ &= B_t^2/2.\end{aligned}$$

Exercise 6. This was already shown as part of **Exercise 2** for a general partition.

Exercise 7. Observe that

$$\begin{aligned}
\mathbb{P}(\tilde{B}_t \in A) &= \mathbb{P}(\Theta\, B_t \in A) \\
&= \mathbb{P}(B_t \in \Theta^T A) \\
&= \int_{\boldsymbol{x} \in \Theta^T A} (2\,\pi t^n)^{-1/2} \exp\big\{-\boldsymbol{x}^T \boldsymbol{x}/(2\,t)\big\}\, d\boldsymbol{x} \\
&= \int_{\boldsymbol{x} \in \Theta^T A} (2\,\pi t^n)^{-1/2} \exp\big\{-\boldsymbol{x}^T \Theta^T\, \Theta\, \boldsymbol{x}/(2\,t)\big\} \det(\Theta^T)\, d\boldsymbol{x} \\
&= \int_{\boldsymbol{y} \in A} (2\,\pi t^n)^{-1/2} \exp\big\{-\boldsymbol{y}^T \boldsymbol{y}/(2\,t)\big\}\, d\boldsymbol{y} \\
&= \mathbb{P}(B_t \in A),
\end{aligned}$$

where we made substitution $\boldsymbol{y} = \Theta\, \boldsymbol{x}$ and used the fact taht orthogonality implies $\det(\Theta) = 1$.

Obviously, this only shows that the laws of $\tilde{B}_t$ and B_t are the same. To show that the law of Brownian motion is invariant with respect to rotation, we would need to show that the laws of $(\tilde{B}_{t_1}, \ldots \tilde{B}_{t_n})$ and $(B_{t_1}, \ldots B_{t_n})$ are the same for any $0 = t_0 \leqslant t_1 < \cdots < t_n$ and $n > 0$. But since the increments of Brownian motion are independent, this will follow once we establish that for all $i = 1 \ldots n$ $B_{t_i} - B_{t_{i-1}}$ has the same law as $\tilde{B}_{t_i} - \tilde{B}_{t_{i-1}}$. Since these have the same laws as, respectively, $B_{t_i - t_{i-1}}$ and $\tilde{B}_{t_i - t_{i-1}}$, this boils down to the initial calculation.