Homework 2

1. For $F(s,\omega) \in \mathcal{L}^2$, let
$$Z_t = e^{2\int_0^t F(s,\omega)\,dB_s - \int_0^t |F|^2\,ds} \tag{0.1}$$
Use Itô's formula to derive a stochastic differential equation (SDE) for Z_t. Write your answer in integral form. Is Z_t a martingale or a sub-martingale, or a super-martingale?

2. Suppose that L denotes the differential operator
$$Lu := \frac{\sigma^2(x)}{2} u_{xx} + v(x) u_x, \tag{0.2}$$
and that $X_t(\omega)$ is a one-dimensional stochastic process satisfying
$$dX_t = V(X_t)\,dt + \sigma(X_t) dB_t, \qquad X_0(\omega) = x_0. \tag{0.3}$$
Using Itô's formula, show that the process
$$M(t,\omega) := f(X_t(\omega)) - f(x_0) - \int_0^t Lf(X_s(\omega))\,ds \tag{0.4}$$
is a martingale, if $f \in C^2(\mathbb{R}^d)$ and f' is uniformly bounded. How does that relate to the sitaution in Problem 1?

3. What stochastic differential equation is satisfied by $Y_t = e^{\lambda t}\cos(\alpha B_t)$? Under what conditions on λ and α is Y_t a martingale (with respect to the Brownian filtration)? Now, suppose that X_t solves an SDE
$$dX_t = u(X_t) + \sigma(X_t) dB_t,$$
under what conditions on u and v can $S_t = e^{\lambda t}\cos(\alpha X_t)$ be a martingale for some α and λ?

4. Suppose that $D \subset \mathbb{R}^d$ is a smooth bounded domain. For $x \in D$, let
$$X_t^x(\omega) = x + B_t(\omega), \quad t \geq 0 \tag{0.5}$$
where $B_t(\omega)$ is a d-dimensional Brownian motion. Define the stopping time $\gamma_D^x = \inf\{t > 0 \,|\, X_t^x \in \mathbb{R}^d \setminus D\}$, which is the first hitting time to the boundary ∂D. Define the function
$$w(x) = E\left[\gamma_D^x\right]. \tag{0.6}$$
Supposing that $w(x)$ is finite, what boundary value problem does w satsify? Justify your answer. You may assume existence of a unique classical solution to the appropriate BVP.

5. Suppose that $u(x,t)$ is smooth and satisfies the terminal value problem
$$u_t + u_{xx} - \mu u_x + \lambda u = 0, \quad t < T, \ \ x \in \mathbb{R} \tag{0.7}$$
with terminal data $u(x,T) = \phi(x) \in C_0^\infty(\mathbb{R})$. Use Itô's fromula and the product rule to derive a stochastic representation for $u(x,t)$.

6. Suppose that $D \subset \mathbb{R}^d$ is a smooth bounded domain and that $u(x)$ and $v(x)$ both satisfy the equation
$$\frac{1}{2}\Delta u + b(x)\cdot\nabla u = 0, \quad x \in D \tag{0.8}$$
where $b(x)$ is a smooth, bounded vector field. Suppose that $u(x) = g(x)$ for $x \in \partial D$ and that $v(x) = f(x)$ for $x \in \partial D$, where f and g are smooth functions.

(i) Derive a stochastic representation for $u(x)$ and $v(x)$ in terms of a stochastic process X_t^x.

(ii) Suppose that for any set $\Gamma \subset \partial D$ having positive Lebesgue measure, the process X_t^x has a nonzero probability of hitting the boundary first at Γ. That is, if γ_D^x is the first hitting time of X_t^x to the boundary ∂D, then we suppose that $P(X_{\gamma_D}^x \in \Gamma) > 0$ for all $x \in D$. Use this assumption and your answer in part (i) to prove a strong comparison principle: if $f \geq g$ for all $x \in \partial D$ with $f > g$ somewhere on the boundary, then $v > u$ for all $x \in D$.

7. Suppose that $D \subset \mathbb{R}^d$ is a smooth bounded domain. Suppose that u satisfies

$$\Delta u = 0, \quad x \in D \tag{0.9}$$

Show that if $x \in D$ and $r > 0$ are such that the ball of radius r centered at x (this is $B_r(x) := \{y \in \mathbb{R}^d | \, |x - y| < r\}$) is contained in D, then

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) \, dS(y) \tag{0.10}$$

This is called the mean-value formula for harmonic functions. The expression $|\partial B_r(x)|$ means the surface area of the ball $B_r(x)$. There are various ways to do it, use Brownian motion to make it rather trivial.