Math 227 Winter 2009

Solutions to HW #3

Exercise 1. Define $X_t(\omega) := 2\int_0^t F_s\,dB_s - \int_0^t |F_s|^2\,ds$. Then $Z_t = f(X_t) = e^{X_t}$ and applying Itô gives

$$\begin{aligned} dZ_t &= \partial_x f(X_t)\,dX_t + \frac{1}{2}\partial_{xx} f(X_t)\;d[X]_t \\ &= Z_t\,\{2\,F_t\,dB_t - |F_t|^2\,dt\} + 2\,Z_t\,|F|^2\;dt \\ &= 2\,Z_t\,F\,dB_t + Z_t\,|F|^2\;dt \end{aligned}$$

since the quadratic variation of X_t is given by $d[X]_t = 4\,|F_t|^2\,dt$. In the integral form this states

$$Z_t - Z_0 = \int_0^t 2\,Z_s\,F_s\,dB_s + \int_0^t Z_s\,|F_s|^2\;ds,$$

where the drift term is non-negative, so this is a submartingale (in particular a martingale if $F \equiv 0$).

Exercise 2. We apply Itô's formula (in the integral form) and get

$$f(X_t) - f(x_0) = \int_0^t \left\{ v(X_t)\,f'(X_t) + \frac{\sigma^2(X_t)}{2} f''(X_t) \right\} dt + \int_0^t \sigma\,(X_t)\;f'(X_t)\,dB_t$$

where the curly bracket is exactly $L\,f(X_t)$, so now it is enough to show that the last integral on the right-hand side is a martingale. By the Properties of Itô integral, as listed in the notes, this will follow once we show that $f'(X_t) \in \mathcal{L}^2([0,T])$.

If we only know that the second derivatives of f are all unformly bounded, we have a mean-value inequality $\|f'(x) - f'(0)\| \leqslant K|x|$ for all $x \in \mathbb{R}$ and some constant K. Thus for some further constants A, B, C we have

$$\mathbb{E}\left(\int_0^T \|f'(X_t)\|^2\,dt \right) \leqslant \mathbb{E}\left(\int_0^T A + B\,\|X_t\| + C\,\|X_t\|^2\,dt \right)$$

which is finite, because all the moments of B.M. are bounded on $[0,T]$ (and the velocity V must be Lipschitz for the definition (0.5) from the exercise statement to make sense).

[If, on the other hand, we know that the first derivatives of f are all uniformly bounded, we automatically have $f'(X_t) \in \mathcal{L}^2([0,T])$ by monotonicty of the expectation.]

Exercise 3. Applying Itô's formula we get

$$dY_t = \lambda\,Y_t\,dt - \alpha\,e^{\lambda t}\sin(\alpha\,B_t)\,dB_t - \frac{1}{2}\alpha^2\,Y_t\,dt = \left(\lambda - \frac{\alpha^2}{2}\right)Y_t\,dt - \alpha\,e^{\lambda t}\sin(\alpha\,B_t)\,dB_t$$

Notice that for each T

$$\mathbb{E}\left(\int_0^T e^{2\lambda t}\sin^2(\alpha\,B_t)\,dt \right) \leqslant \int_0^T e^{2\lambda t}\,dt < \infty,$$

so the integral against Bm is a martingale and for $\{Y_t\}$ to be a martingale too suffices for the drift term to disappear, that is $\lambda = \alpha^2/2$.

For the second part, applying Itô's formula we get

$$\begin{aligned} dS_t &= \lambda\,X_t\,dt - \alpha\,e^{\lambda t}\sin(\alpha\,X_t)\,dX_t - \frac{1}{2}\alpha^2\,X_t\;\sigma^2(X_t)\,dt \\ &= e^{\lambda t}\left(\lambda\cos(\alpha\,X_t) - \frac{\alpha^2\,\sigma^2(X_t)}{2}\cos(\alpha\,X_t) - \alpha\sin(\alpha\,X_t)\,u(X_t) \right) dt - \alpha\,e^{\lambda t}\sin(\alpha\,X_t)\,\sigma^2(X_t)\,dB_t. \end{aligned}$$

For $\{S_t\}$ to be a martingale on $[0,T]$ we would thus require u and σ to satisfy

$$\mathbb{E}\left(\int_0^T e^{2\lambda t}\sin^2(\alpha X_t)\ \sigma^2(X_t)\,dt\right) < \infty$$

$$\left(\lambda - \frac{\alpha^2\sigma^2(x)}{2}\right)\cos(\alpha x) - \alpha\sin(\alpha x)\,u(x) = 0 \qquad \text{for all } x\in\mathbb{R}.$$

Exercise 4. To identify the correct BVP we study formula (2.28) from the notes (probabilistic representation of a general BVP). Observe that taking $g(x)=c(x)=0$ and $f(x)=-1$ for all $x\in D$ the formula simplifies to

$$w(x)=\mathbb{E}\left[\int_0^{\gamma_D^x} ds\right]=\mathbb{E}[\gamma_D^x],$$

which is exactly what we are looking for.

Thus we consider the following BVP for $v\in C^2(\overline{D}\,)$

$$\frac{1}{2}\Delta v = -1$$

with $v(x)=0$ for $x\in\partial D$. By our initial assumption it has a unique classical solution, call it v. Then it follows from the notes that indeed $v(x)=\mathbb{E}[\gamma_D^x]=:w(x)$. Hence w must solve the above stated BVP.

We should also mention here that in this problem, with D smooth, it can be proved that our assumption holds.

Exercise 5. Consider the function $f(x,\,t)=e^{\lambda t}\ u(x,\,t)$ and the Itô process defined on $[t,\,T]$ through $dX_s=-\mu\,ds+\sqrt{2}\,dB_s$, $X_t=x$. Applying Itô's formula to $f(X_t,t)$ we get

$$f(X_T,T)-f(X_t,t)=e^{\lambda T}\phi(X_T)-e^{\lambda t}u(x,t)=\int_t^T e^{\lambda s}\{u_s+\lambda u-\mu u_x+u_{xx}\}\,dt+\int_t^T\sqrt{2}\,e^{\lambda s}u_x\,dB_s.$$

where we suppressed the arguments (X_s,s) of u and its derivatives. The first integrand equals zero by our assumption on u, while the second integral has expectation equal to zero and we have

$$u(x,t)=\mathbb{E}\Big[e^{\lambda(T-t)}\,\phi(X_T)\Big].$$

Exercise 6.

i. Let $dX_t=b(x)\,dt+dB_t$, $X_0=0$ and define $\gamma_D=\inf\{t\geqslant 0\colon X_t\in\mathbb{R}^d\backslash D\}$. Then applying the stopped Itô's formula to $v(X_t)$ we get

$$\begin{aligned} v(X_{t\wedge\gamma_D})-v(x) &= \int_0^{t\wedge\gamma_D}\left\{b(X_s)\cdot\nabla v(X_s)+\frac{1}{2}\Delta v(X_s)\right\}ds+\int_0^{t\wedge\gamma_D}[\nabla v(X_s)]\cdot dB_s\\ &= \int_0^{t\wedge\gamma_D}[\nabla v(X_s)]\cdot dB_s. \end{aligned}$$

Taking expectations of both sides and using the martingale stopping theorem on the right-hand side we get

$$v(x)=\mathbb{E}v(X_{t\wedge\gamma_D}).$$

Now since D is smooth and bounded, we have $\lim_{t\to\infty}\gamma_D\wedge t=\gamma_D$ a.s. and v bounded on D. Hence Dominated Convergence Theorem gives us

$$\lim_{t\to\infty}\mathbb{E}v(X_{t\wedge\gamma_D})=\mathbb{E}v(X_{\gamma_D})=\mathbb{E}f(X_{\gamma_D}).$$

Thus we have $v(x)=\mathbb{E}f(X_{\gamma_D})$ and $u(x)=\mathbb{E}g(X_{\gamma_D})$.

ii. Suppose indeed for some $\varepsilon>0$ we have $f(x)>g(x)+2\,\varepsilon$ for some $x\in\partial D$. Then by continuity of f and g there exists a $\delta>0$ such that for all $y\in\Gamma$, $\Gamma:=\{y\in\partial D\colon\|x-y\|<\delta\}$ we have

$$f(y)>g(y)+\varepsilon.$$

Thus by monotonicity of the integral we must have

$$\mathbb{E}[f(X_{\gamma_D}); X_{\gamma_D} \in \Gamma] \geqslant \mathbb{E}[g(X_{\gamma_D}) + \varepsilon; X_{\gamma_D} \in \Gamma] = \mathbb{E}[g(X_{\gamma_D}); X_{\gamma_D} \in \Gamma] + \varepsilon\, \mathbb{P}(X_{\gamma_D} \in \Gamma).$$

Because $\mathrm{Leb}(\Gamma) > 0$, we have $\mathbb{P}(X_{\gamma_D} \in \Gamma) > 0$ and so

$$\mathbb{E}[f(X_{\gamma_D}); X_{\gamma_D} \in \Gamma] > \mathbb{E}[g(X_{\gamma_D}); X_{\gamma_D} \in \Gamma].$$

Since $f \geqslant g$ on Γ implies that $\mathbb{E}[f(X_{\gamma_D}); X_{\gamma_D} \in \partial D \backslash \Gamma] \geqslant \mathbb{E}[g(X_{\gamma_D}); X_{\gamma_D} \in \partial D \backslash \Gamma]$, adding the two inequalities we finally get for all $x \in D$

$$v(x) = \mathbb{E} f(X_{\gamma_D}) > \mathbb{E} g(X_{\gamma_D}) = u(x)$$

as desired.

Exercise 7. By Feynman-Kac we have

$$u(x) = \mathbb{E}[u(B(\gamma) + x)] = \int_{\partial B_r(0)} u(y + x)\, f_{B(\gamma)}(y)\, dS(y)$$

where $\{B(t)\}$ is standard Brownian motion, $\gamma = \inf\{t \geqslant 0: \|B(\gamma)\| \geqslant r\}$ is a stopping time and $f_{B(\gamma)}$ is the density of $B(\gamma)$. Recall that in the last exercise of Homework #2 we showed that the law of the d-dimensional Brownian motion is symmetric with respect to rotations. It follows that $B(\gamma)$ is uniformly distributed on $\partial B_r(0)$ and so $f_{B(\gamma)}(y) = 1/|\partial B_r(0)| = 1/|\partial B_r(x)|$ as required.