Homework 4

1. Define $\beta_k(t) = \mathbb{E}(B_t^k))$, where B_t is the standard Brownian motion. Use Ito's formula and induction to find an expression for β_k for all positive integers $k \geq 1$.

2. Let X_t be an Ito integral
$$X_t = \int_0^t v(s,\omega)dB_s,$$
with a bounded function v, that is, $|v(t,\omega)| \leq M$ for all $t \geq 0$ and all $\omega \in \Omega$. Then X_t is a martingale. Give an example of $v(t,\omega)$ such that X_t^2 is not a martingale. Show that if v is bounded then
$$M_t = X_t^2 - \int_0^t |v(s,\omega)|^2 ds$$
is a martingale.

3. (i) Let
$$Y_t = t + (1-t)\int_0^t \frac{dB_s}{1-s}$$
and show that $\lim_{t\to 1} Y_t = 1$ almost surely. Hence, Y_t connects $Y_0 = 0$ and $Y_1 = 1$.
(ii) Generalize (i) to construct a process of the same kind that connects $Y_0 = a$ and $Y_1 = b$ for arbitrary $a, b \in \mathbb{R}$.

4. Show that the solution $u(t,x)$ of the initial value problem
$$\frac{\partial u}{\partial t} = \frac{\beta^2 x^2}{2}\frac{\partial^2 u}{\partial x^2} + \alpha x \frac{\partial u}{\partial x}, \quad x \in \mathbb{R},$$
with the initial data $u(0,x) = f(x)$ may be written as
$$u(t,x) = \mathbb{E}\left\{ f(xe^{\beta B_t + (\alpha - \beta^2/2)t} \right\}.$$

5. Let $b(x)$, $x \in \mathbb{R}$, be a smooth bounded function and define the process X_t by
$$dX_t = b(X_t)dt + dB_t, \quad X_0 = x.$$
(i) Prove that for all $M > 0$, $x \in \mathbb{R}$ and $t > 0$ we have $P(X_t^x \geq M) > 0$. Hint: Girsanov's theorem is helpful here. (ii) Assume that $b(x) \leq -1$ for all x. Show that $X_t^x \to -\infty$ as $t \to +\infty$ almost surely. Why does this not contradict (i)?

6. Let (a,b) be a bounded interval, set
$$dX_t = rX_t + \alpha X_t dB_t, \quad X_0 = x \in (a,b).$$
(i) Let $\tau(x)$ be the exit time for X_t^x from (a,b). Find an equation for $u(x) = \mathbb{E}(\tau(x))$. (ii) Compute $P(X_{\tau(x)} = b)$. (iii) Let g be a bounded continuous function defined on (a,b). Find
$$w(x) = \mathbb{E}\left[\int_0^{\tau(x)} g(X_t)dt\right].$$