Math 227 Winter 2009

Solutions to HW #4

Exercise 1. Applying Itô to B_t^k gives $dB_t^k = k\, B_t^{k-1}\, dB_t + \frac{k\,(k-1)}{2}\, B_t^{k-2}\, dt$. Taking expectations to get rid of the Brownian integral results in

$$\beta_k(t) = \mathbb{E} B_t^k = \mathbb{E} \int_0^t \frac{k\,(k-1)}{2}\, B_s^{k-2}\, ds = \frac{k\,(k-1)}{2} \int_0^t \mathbb{E} B_s^{k-2}\, ds = \frac{k\,(k-1)}{2} \int_0^t \beta_{k-2}(s)\, ds,$$

for $k \geqslant 2$. Since $\beta_0(t) = 1$ and $\beta_1(t) = 0$, it is straightforward to verify by induction that for $k \geqslant 0$

$$\beta_{2k}(t) = \frac{(2\,k)!}{2^k\, k!}\, t^k$$

and obviously $\beta_{2k+1}(t) \equiv 0$.

Exercise 2. Applying Itô to X_t^2 gives

$$X_t^2 = \int_0^t 2\, X_s\, v(s,\omega) dB_s + \int_0^t |v(s,\omega)|^2\, ds, \tag{1}$$

where the first integral (equal to M_t) is a martingale by the properties of the Itô integral from the notes, since for all $t \geqslant 0$

$$\mathbb{E}\left\{ \int_0^t X_s^2\, |v(s,\omega)|^2 ds \right\} \leqslant M^2 \int_0^t \mathbb{E} X_s^2\, ds = M^2 \int_0^t \int_0^s |v(s',\omega)|^2\, ds'\, ds < \infty.$$

Now, if X_t^2 also was a martingale, then $X_t^2 - M_t$ would have to be a martingale, too. But taking, e.g. $v(s,\omega) \equiv 1$, we get that $X_t^2 - M_t = t$, which definitely is not a martingale. Hence in this case X_t^2 cannot be a martingale, either.

Exercise 3.

i. For any $t \in [0,1)$ we have$M_t \equiv \int_0^t (1-s)^{-1} dB_s$ distributed like $N(0, t/(1-t))$. [Write out the integral as a limit of sums like in Exercise 1 of Homework #2]. Hence

$$X_t \equiv (1-t) \int_0^t (1-s)^{-1} dB_s \quad \sim \quad N(0, t\,(1-t))$$

and so $\mathbb{E} X_t^2 \rightarrow 0$, i.e. X_t tends to zero in L^2 and a fortiori in probability, hence $Y_t \rightarrow 1$ in L^2 and in probability, too. To show that $Y_t \rightarrow 1$ a.s. we need to work a bit harder. Again, it is enough to show that $X_t \rightarrow 0$ a.s. To this end we will leverage the fact that M_t is a martingale, so that by Doob's inequality (Theorem 3.1.2 from the notes with $p = 2$) for any $0 \leqslant t_n < t_{n+1} \leqslant 1$ and $\lambda > 0$

$$\mathbb{P}\left\{ \sup_{t_n \leqslant t \leqslant t_{n+1}} |M_t| \geqslant \lambda \right\} \leqslant \frac{\mathbb{E}|M_{t_{n+1}}|^2}{\lambda^2} = \frac{t_{n+1}/(1-t_{n+1})}{\lambda^2},$$

where we used the variance of M_t calculated above. Choosing $t_n = 1 - 2^{-n}$ and $\lambda = \varepsilon\, 2^n$ a little algebra yields

$$\mathbb{P}\left\{ \sup_{t_n \leqslant t \leqslant t_{n+1}} 2^{-n} |M_t| \geqslant \varepsilon \right\} \leqslant 2\, \varepsilon^{-2}\, 2^{-n}.$$

Finally, noticing that $\sup_{t_n \leqslant t \leqslant t_{n+1}} (1-t) = 2^{-n}$, we identify the above as simply saying that for all $\varepsilon > 0$

$$\mathbb{P}\left\{ \sup_{t_n \leqslant t \leqslant t_{n+1}} |X_t| \geqslant \varepsilon \right\} \leqslant 2\, \varepsilon^{-2}\, 2^{-n}. \tag{2}$$

Almost sure converence to zero will follow once we show that there is an event A with $\mathbb{P}(A) = 1$ and such that each $\omega \in A$ is only in finitely many sets $A_n \equiv \{\sup_{t_n \leqslant t \leqslant t_{n+1}} |X_t| \geqslant \varepsilon_n\}$ for some $\varepsilon_n \downarrow 0$. This can be concluded from the first Borel-Cantelli lemma, if we only manage to exhibit a sequence of $\varepsilon_n \downarrow 0$ such that $\sum_n \mathbb{P}(A_n) < \infty$. In view of (2), taking e.g. $\varepsilon_n = 2^{-n/4}$ will give $\mathbb{P}(A_n) = 2^{-n/2+1}$, which certainly makes the sum finite.

ii. E.g. take

$$Y_t = a\,(1-t) + b\,t + (1-t)\int_0^t (1-s)^{-1} dB_s.$$

Exercise 4. Consider $X_t = x\,e^{\beta B_t + (\alpha - \beta^2/2)t}$ and apply Itô to $v(\tau, X_\tau) \equiv u(t-\tau, X_\tau)$ to get

$$u(t-\tau, X_\tau) - u(t,x) = \int_0^\tau \left[\frac{\beta^2 X_s^2}{2}\,\partial_{xx} u + \alpha\,X_s\,\partial_x u - \partial_s u\right] ds + \int_0^\tau \beta\,X_s\,\partial_x u\,dB_s = \int_0^\tau \beta\,X_s\,\partial_x u\,dB_s.$$

Taking expectation, noticing that the integral disappears and taking $\tau = t$ gives

$$u(t,x) = \mathbb{E}u(0, X_t) = \mathbb{E}f(X_t)$$

as required.

Exercise 5.

i. By Girsanov, under the new measure Q given by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left\{-\int_0^t b(X_s)\,dB_s - \frac{1}{2}\int_0^t b^2(X_s)\,ds\right\}$$

X_t^x is a Brownian motion [Check that the Novikov's condition holds, because $b(\,\cdot\,)$ is bounded!] and so $Q(X_t^x \geqslant M) = 1 - \Phi(\frac{M-x}{\sqrt{t}}) > 0$, where Φ is the standard normal distribution function. Since Q is absolutely continuous with respect to P, it follows that this event also has positive probability under P.

ii. Solving the SDE we get $X_t = -t + B_t = -t(1 + B_t/t)$ and since $\operatorname{limsup}_{t \to +\infty} |B_t|/t = 0$ a.s., it follows that $X_t \to -\infty$ a.s., too.

This does not contradict part (i), because we do not require the paths of X_t to tend to $-\infty$ at a uniform rate. At each time there is still an event (with positive measure) of paths that have not fallen below level M.

Exercise 6.

i. Similarly to Exercise 4 on Homework #3, consider

$$r\,x\,u'(x) + \frac{\alpha^2}{2}\,x^2\,u''(x) = -1 \quad \text{for} \quad x \in (a,b) \tag{3}$$

with $u(a) = u(b) = 0$. Supposing this has a solution, applying Itô to $u(X_{t\wedge\tau})$ yields

$$u(X_{t\wedge\tau}) - u(x) = \int_0^{t\wedge\tau} r\,X_s\,u'(X_s) + \frac{\alpha^2 X_s^2}{2}\,u''(X_s)\,ds + \int_0^{t\wedge\tau} \alpha\,X_s u'(X_s)\,dB_s = -t\wedge\tau + M.G.,$$

and taking expectations yields

$$u(x) = \mathbb{E}[t\wedge\tau] + \mathbb{E}u(X_{t\wedge\tau}). \tag{4}$$

Finally, since we can assume $\mathbb{E}\tau(x) < \infty$, it follows that τ is a.s. finite and so $u(X_{t\wedge\tau}) \to u(X_\tau) = 0$ a.s. as $t \to \infty$ and since u is bounded, Dominated Convergence allows us to conclude that the RHS of (4) tends to $\mathbb{E}\tau$ as $t \to \infty$.

For completeness, let us solve the ODE (3) in the interesting case $r, \alpha \neq 0$ and $0 < a < b$. Substitution $x = e^\theta$ allows us to reduce it to a constant-coefficient ODE from which it is easy to see that

$$u(x) = c_1 + c_2\,x^{1-\frac{2r}{\alpha^2}} + \frac{\log x}{\alpha^2/2 - r}$$

unless $\alpha^2/2 - r = 0$, in which case the solution is $u(x) = c_1 + c_2 \log x + \frac{1}{r}(\log x)^2$. The constants can be determined from the boundary conditions $u(a) = u(b) = 0$.

ii. Since $\mathbb{P}(X_{\tau(x)} = b) = \mathbb{E}\mathbf{1}_b(X_{\tau(x)})$ we are drawn to consider the ODE

$$r\,x\,v'(x) + \frac{\alpha^2}{2}x^2\,v''(x) = 0 \quad \text{for} \quad x \in (a, b) \tag{5}$$

with $v(a) = 0$ and $v(b) = 1$. An application of Itô will then give

$$v(x) = \mathbb{E}v(X_\tau) = \mathbb{E}\mathbf{1}_b(X_{\tau(x)}) = \mathbb{P}(X_{\tau(x)} = b),$$

exactly as desired. Finally, this ODE is straightforward to solve: supposing $0 \leqslant a < b$ and $\alpha^2 \neq 2\,r$ it is the same as

$$\frac{v''(x)}{v'(x)} = -\frac{2r}{\alpha^2}x^{-1},$$

from which $\log v'(x) = -(2r/\alpha^2)\log x$ and $v'(x) = x^{-2r/\alpha^2}$. Hence the general solution is $v(x) = c_1 + c_2\,x^{1-2r/\alpha^2}$ where the constants have to solve

$$\begin{cases} 0 = c_1 + c_2\,a^{1-2r/\alpha^2} \\ 1 = c_1 + c_2\,b^{1-2r/\alpha^2} \end{cases}$$

so that

$$c_1 = \frac{-a^{1-2r/\alpha^2}}{b^{1-2r/\alpha^2} - a^{1-2r/\alpha^2}} \qquad\qquad c_2 = \frac{1}{b^{1-2r/\alpha^2} - a^{1-2r/\alpha^2}}$$

and finally

$$v(x) = \frac{x^{1-2r/\alpha^2} - a^{1-2r/\alpha^2}}{b^{1-2r/\alpha^2} - a^{1-2r/\alpha^2}}.$$

iii. The function w is the solution of

$$r\,x\,w'(x) + \frac{\alpha^2}{2}x^2\,w''(x) = -g(x) \quad \text{for} \quad x \in (a, b)$$

with $w(a) = w(b) = 0$. Through substitution $x = e^t$ this can be again reduced to a constant-coefficient inhomogeneous ODE

$$\frac{\alpha^2}{2}\frac{d^2w}{dt^2} + (r - \alpha^2/2)\frac{dw}{dt} = -g(e^t) \quad \text{for} \quad t \in (\log a, \log b),$$

solution to which can be readily expressed as an integral involving g.