Homework 5

1. A digital option is one with payoff

$$g(x) = \begin{cases} 0, & x < K, \\ b & x \geq K \end{cases} \tag{0.1}$$

at the expiration time T. By transforming the Black-Scholes equation to the heat equation and solving the resulting initial value problem explicitly, find the price of such option.

2. Using the result from the preceeding question and the linearity of the Black-Scholes equation, compute the price of the option with payoff

$$g(x) = \begin{cases} b, & x \in [K_1, K_2], \\ 0 & x \in (0, K_1) \cup (K_2, \infty) \end{cases} \tag{0.2}$$

where $0 < K_1 < K_2$.

3. In the previous homeworks you showed that a harmonic function satisfies the mean value property. This means that if $\Delta u = 0$ for all $x \in D$ and if $B_r(x_0) \subset D$,

$$u(x_0) = \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u(x)\, dS(x) \tag{0.3}$$

where $B_r(x_0) = \{x \in \mathbb{R}^d \| x - x_0| < r\}$ is the ball or radius r centered at x_0, $\partial B_r(x_0)$ is the surface of the ball. In this exercise you will explore this idea a bit more.

   (i) Use (0.3) to show that

$$u(x_0) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(x)\, dx \tag{0.4}$$

   Here $|B_r(x_0)|$ denotes the volume of the ball $B_r(x_0)$. This says $u(x_0)$ is also equal to the *volume average* of u over the ball centered at x_0, assuming $B_r(x_0) \subset D$. It may help to recall that there is a constant $c(d)$, depending only on the dimension $d \geq 1$, such that $|B_r(x)| = c(d)r^d$ and $|\partial B_r(x)| = \frac{d}{r}|B_r(x)| = dc(d)r^{d-1}$.

   (ii) Use (0.4) to prove the following: Fix $R > 0$ and $r \in (0, R)$. There exists a constant $K > 0$ depending only on R, r, and d, such that

$$\inf_{x \in B_r(0)} u(x) \geq K \sup_{x \in B_r(0)} u(x) \tag{0.5}$$

   for any function u that is non-negative ($u \geq 0$) and harmonic in $B_R(0)$ ($\Delta u = 0$ for all $x \in B_R(0)$). Obviously $K \leq 1$. The point is that the constant is independent of u! In one-dimension, the result is trivial, since $\Delta u = 0$ implies that u is a line, non-negative in $[-R, R]$.

4. Let $\sigma > 0$. Use separation of variables to solve

$$\begin{aligned} u_t &= \frac{\sigma^2}{2} u_{xx} \quad x \in [-L, L],\ t > 0 \\ u(-L, t) &= 0 = u(L, t), \quad t > 0 \\ u(x, 0) &= 1. \end{aligned} \tag{0.6}$$

How fast does the solution converge to zero? You may note that the initial data does not satisfy the boundary conditions. Approximate $u(x,0)$ by functions $g_n(x)$ of the form

$$g_n(x) = \begin{cases} 1, & \text{for } x \in (-L+1/n, L-1/n) \\ & 0, \text{otherwise}, \end{cases}$$

that do satisfy the boundary conditions and pass to the limit $n \to +\infty$. Show that the limit exists and satisfies (0.6).

5. Suppose $x \in (0,L)$. Let $\{B_t(\omega)\}_{t\geq 0}$ be a Brownian motion and let $\gamma_D^x(\omega) \geq 0$ be the first time that $x + B_t$ exits the domain $D = [0,L]$. Show that for $t > 1$,

$$P(\gamma_D^x \geq t) \leq Ce^{-\lambda_1 t} \tag{0.7}$$

where $\lambda_1 > 0$ is the smallest eigenvalue (i.e. the principal eigenvalue) associated with the operator $\frac{1}{2}\Delta$ on the domain $[0,L]$. That is, λ_1 is the smallest positive number such that

$$\begin{aligned} -\frac{1}{2}\Delta u &= \lambda_1 u \\ u(0) = u(L) &= 0 \end{aligned} \tag{0.8}$$

has a nontrivial solution. $C > 0$ is some constant. Hint: what PDE does $P(\gamma_D^x \geq t)$ satisfy, as function of x and t? You should be able to represent this quanity explicitly as a Fourier series.