Math 227 Winter 2009

Solutions to HW #5

Exercise 1. We need to solve

$$u_t + \frac{\sigma^2}{2} x^2 u_{x x} + r\, x\, u_x - r\, u = 0, \quad t < T, \quad x > 0 \tag{1}$$

subject to the terminal condition $u(x, T) = b\, \mathbf{1}_{x \geqslant K}$.

Transforming exactly like in Section 6.2 of the lecture notes, we obtain

$$u(x, t) = p^{-1} e^{-\alpha \log(x/\lambda) - \beta \delta^{-1} (T - t)} v(\log(x/\lambda), \delta^{-1} (T - t)),$$

where p, λ are parameters to be chosen soon,

$$\begin{aligned} \alpha &= \frac{1}{2}\left(\frac{2\, r}{\sigma^2} - 1\right) \\ \beta^2 &= \frac{1}{2}\left(\frac{2\, r}{\sigma^2} + 1\right) \\ \delta &= \frac{2}{\sigma^2} \end{aligned}$$

and $v = v(y, \tau)$ solves $v_\tau = \Delta v$ in $\mathbb{R} \times [0, T]$ with initial condition $v(y, 0) = p\, e^{\alpha y} b\, \mathbf{1}_{\lambda e^y \geqslant K}$.

Setting $\lambda = K$ and $p = b^{-1}$ we get $v(x, 0) = e^{\alpha y} \mathbf{1}_{y \geqslant 0}$, from where the convolution formula gives

$$v(y, \tau) = (4\, \pi\, \tau)^{-1/2} \int_0^\infty e^{\alpha z} e^{-\frac{(y - z)^2}{4 \tau}} d z = e^{\alpha^2 \tau + \alpha y}\, \Psi\!\left(\frac{y + 2\, \alpha\, \tau}{\sqrt{2\, \tau}}\right),$$

with Ψ the cumulative distribution function of the standard normal random variable and where we used calculations (2.13) and (2.14) from Section 6.2 in the notes.

Hence, since

$$\begin{aligned} -\alpha \log(x/K) - \frac{\beta}{\delta}(T - t) + \frac{\alpha^2}{\delta}(T - t) + \alpha \log(x/K) &= \frac{T - t}{4\, \delta}\left(\frac{2\, r}{\sigma^2} - 1 - \frac{2\, r}{\sigma^2} - 1\right)\left(\frac{2\, r}{\sigma^2} - 1 + \frac{2\, r}{\sigma^2} + 1\right) \\ &= -\frac{T - t}{\delta} \frac{2\, r}{\sigma^2} \\ &= -(T - t)\, r \end{aligned}$$

we finally obtain

$$u(x, t) = b\, e^{-(T - t) r}\, \Psi\!\left(\frac{\log(x/K) + (r - \sigma^2/2)\,(T - t)}{\sigma \sqrt{T - t}}\right).$$

Exercise 2. Observe that $g(x) = b\,(\mathbf{1}_{x \geqslant K_1} - \mathbf{1}_{x \geqslant K_2})$. [We neglect the term $\mathbf{1}_{x = K_2}$. If you are worried about it, consider the probabilistic interpretation of the problem. What is the probability of the event $\{W_T = a\}$ for a fixed a?]

By linearity, the solution u of eqn. (1) with terminal condition g satisfies $u = u_1 + u_2$, where u_1 is the solution of eqn. (1) with the terminal condition $u_1(x, T) = b\, \mathbf{1}_{x \geqslant K_1}$ and u_2 is the solution of eqn. (1) with terminal condition $u_2(x, T) = -b\, \mathbf{1}_{x \geqslant K_2}$. Thus **Exercise 1** gives

$$u(x, t) = b\, e^{-(T - t) r}\, \{\Psi(d(K_1)) - \Psi(d(K_2))\},$$

where

$$d(K) = \frac{\log(x/K) + (r - \sigma^2/2)\,(T - t)}{\sigma \sqrt{T - t}}.$$

Exercise 3.

i. Here we use the fact that for any continuous function f

$$\int_{B_r(x_0)} f(x)\,dx = \int_0^r \int_{\partial B_\rho(x_0)} f(x)\,dS(x)\,d\rho.$$

It follows that

$$\begin{aligned}\int_{B_r(x_0)} u(x)\,dx &= \int_0^r |\partial B_\rho(x_0)|\,\frac{1}{|\partial B_\rho(x_0)|}\int_{\partial B_\rho(x_0)} u(x)\,dS(x)\,d\rho\\ &= \int_0^r |\partial B_\rho(x_0)|\;u(x_0)\,d\rho\\ &= u(x_0)\int_0^r d\,c(d)\,\rho^{d-1}\,d\rho \;=\; u(x_0)\,c(d)r^d \;=\; u(x_0)|B_r(x_0)|.\end{aligned}$$

ii. First, we will establish that for any two points $x, y \in B_R(0) \subseteq \mathbb{R}^d$ for which

$$B_{2\|x-y\|}(x) \subseteq B_R(0) \tag{2}$$

(recall that all balls are open balls) and for all u such that $\Delta u = 0$ on $B_R(0)$ we have

$$u(x) \geqslant 2^{-d}\,u(y) \tag{3}$$

Setting $\rho := \|x-y\|$, this follows from

$$|B_{2\rho}(x)|\,u(x) = \int_{B_{2\rho}(x)} u(z)\,dz \geqslant \int_{B_\rho(y)} u(z)\,dz = |B_\rho(y)|\,u(y) = |B_{2\rho}(x)|\,2^{-d}\,u(y)$$

where the inequality is a consequence of $B_{2\rho}(x) \supseteq B_\rho(y)$ and where we used $|B_{2\rho}(0)| = 2^d\,|B_\rho(0)|$.

Now, observe that for any $r \in (0, R/3)$ and $x, y \in B_r(0)$ we have (2) and thus also (3). Taking infimum of both sides of (3) over $x \in B_r(0)$ and noticing that the right-hand side does not depend on x we obtain

$$\inf_{x \in B_r(0)} u(x) \geqslant 2^{-d}\,u(y).$$

Similarly, taking supremum over $y \in B_r(0)$ we finally get that for all $r \in (0, R/3)$

$$\inf_{x \in B_r(0)} u(x) \geqslant 2^{-d} \sup_{y \in B_r(0)} u(y).$$

We have achieved a considerable success. However, this is NOT the result the question is asking for, because we only proved this bound for $r \in (0, R/3)$. What can we do for $r \in [R/3, R)$?

Here is an approach that will work for all $r \in (0, R)$. Observe that we can apply the bound (3) on (potentially much smaller) balls of radius $\rho := \frac{R-r}{2}$ and therefore such that $B_{2\rho}(x) \subseteq B_R(0)$ for all $x \in B_r(0)$. Fix $x, y \in B_r(0)$. Then, defining a sequence of points $x_i := \frac{i}{N}(x-y) + x$ for $i = 0, \ldots, N$ with $N = \lceil r/\rho \rceil$ we have $x_0 = x$, $x_N = y$ and for all $i = 0, \ldots, N-1$

$$B_{2\|x_i - x_{i+1}\|}(x_i) \subseteq B_R(0)$$

[Make sure you understand why this is; it follows from convexity of the ball and from the fact that $\|x_i - x_{i+1}\| \leqslant \rho$.] Hence we have $u(x_i) \geqslant 2^{-d}\,u(x_{i+1})$ for all $i = 0, \ldots, N-1$, and combining the N bounds together gives

$$u(x) = u(x_0) \geqslant 2^{-dN}\,u(x_N) = 2^{-dN}\,u(y).$$

Hence, taking inf and sup as before and making the constant explicit results in

$$\inf_{x \in B_r(0)} u(x) \geqslant 2^{-d\left\lceil \frac{2r}{R-r} \right\rceil} \sup_{y \in B_r(0)} u(y).$$

[Notice that in the case $r < R/3$ this reduces to the bound we obtained before.]

Exercise 4. One way of approaching this problem is to transform to $v(x,\, t) = u(\frac{\sigma}{\sqrt{2}}\ (x - L),\, t)$ which solves the heat equation $v_t = v_{xx}$ on $[0, 2\,L]$ putting us in the setup from Section 5.3 of the lecture notes. We will, however, do everything by hand, which could serve as a demonstration of how to approach these questions in an exam.

We are looking at the original problem $u_t = \sigma^2/2\, u_{xx}$ on the interval $[-L, L]$ with the boundary condition $u(-L, t) = 0 = u(L, t)$ and initial condition $u(x, 0) = g_n(x) = \mathbf{1}_{|x| < L - 1/n}$. Call its solution u_n. We expect to have

$$u_n(x,t) = \sum_{l=1}^{\infty} \theta_l(t)\, v_l(x)$$

where each $\theta_l(t)\, v_l(x)$ solves the same PDE problem, except with a different initial conditon. Hence substituting this into the PDE and rearranging to have the t-dependent terms on the R.H.S. and the x-dependent terms on the L.H.S., we get

$$-\frac{\theta_l'(t)}{\theta_l(t)} = -\frac{\sigma^2}{2}\frac{v_l''(x)}{v_l(x)} = \text{const} =: \lambda_l \tag{4}$$

i.e. the two functions must be constant and of the same value. It follows then that $\theta_l(t) = b_l\, e^{-\lambda_l t}$ for some b_l to be determined from the initial condition, while λ_l is determined from the problem

$$v_l''(x) = -\frac{2\,\lambda_l}{\sigma^2}\, v_l(x), \qquad v_l(-L) = 0 = v_l(L).$$

We are only interested in even solutions, since the contribution of the odd ones will necessarily be zero due to the symmetry of the initial condition. It is easy to see that the only even functions solving the PDE and the boundary conditions are $\cos(\frac{\sqrt{2\,\lambda_l}\, x}{\sigma})$, where $\sqrt{2\,\lambda_l}/\sigma = \pi\,(2\,k - 1)/(2\,L)$ for some integer $k \geqslant 1$. Renumbering conveniently to $l = k$ we finally have

$$v_l(x) = \cos\!\left(\frac{\pi\,(2\,l - 1)\,x}{2\,L}\right)$$

and $\lambda_l = \left(\frac{\pi\,(2\,l-1)}{2\,L}\right)^2 \frac{\sigma^2}{2}$. It remains to determine b_l from

$$\mathbf{1}_{|x| < L - 1/n} = \sum_{l=1}^{\infty} b_l\, v_l(x) \tag{5}$$

Observe that we know that we can find b_l's such that this equality will hold (at least almost everywhere) because of the completeness of the set of the eigenfunctions in $L^2([-L, L])$ and the fact that the eigenfunctions not included in the R.H.S of (5) have odd symmetry (so they would come in with a zero coefficient anyway). Since we have

$$\begin{aligned}\int_{-L}^{L} \cos\!\left(\frac{\pi\,(2\,l-1)\,x}{2\,L}\right)\cos\!\left(\frac{\pi\,(2\,l'-1)\,x}{2\,L}\right) d\,x &= \frac{1}{2}\int_{-L}^{L} \cos\!\left(\frac{\pi\,(l+l')x}{L}\right) + \cos\!\left(\frac{\pi\,(l-l')\,x}{L}\right) d\,x \\ &= L\,\mathbf{1}_{l=l'},\end{aligned}$$

i.e. the v_l's are orthogonal, we multiply both sides by v_l and integrate on $[-L, L]$ to reduce the R.H.S to $b_l\, L$, while the L.H.S. becomes

$$\begin{aligned}\int_{-L+1/n}^{L-1/n} \cos\!\left(\frac{\pi\,(2\,l-1)\,x}{2\,L}\right) dx &= \frac{4\,L}{\pi\,(2\,l-1)} \sin\!\left(\frac{\pi\,(2\,l-1)\ (L - 1/n)}{2\,L}\right) \\ &= \frac{4\,L}{\pi\,(2\,l-1)} \sin\!\left(\pi\left(l - \frac{1}{2}\right)\left(1 - \frac{L}{n}\right)\right).\end{aligned}$$

Finally, putting all of the above together we have established that

$$u_n(x,t) = \sum_{l=1}^{\infty} \frac{4\,L}{\pi\,(2\,l-1)} \sin\!\left(\pi\left(l - \frac{1}{2}\right)\left(1 - \frac{L}{n}\right)\right) \exp\!\left(-\left(\frac{\pi\,(2\,l-1)}{2\,L}\right)^2 \frac{\sigma^2}{2}\, t\right) \cos\!\left(\frac{\pi\,(2\,l-1)\,x}{2\,L}\right),$$

and so we have

$$\|u_n(\,\cdot\,,t)\|_2 \leqslant \frac{4\,L}{\pi}\left\{\sum_{l=1}^{\infty}\frac{1}{(2\,l-1)^2}\right\}^{1/2} e^{-\lambda_1 t} < \text{const.} \times e^{-\lambda_1 t}, \tag{6}$$

i.e. the solution converges to zero exponentially fast at the rate $\frac{1}{2}\left(\frac{\pi\,\sigma}{2\,L}\right)^2$. Notice that this rate increases as σ gets bigger, which is in agreement with the probabilistic interpretation of our PDE: taking σ bigger corresponds to the underlying diffusion being more volatile and hence to the heat energy dissipating quicker.

Finally using the bound in (6) we see that taking $n\to\infty$ we have $u_n \to u$ in L^2 where

$$u(x,t) = \sum_{l=1}^{\infty}\frac{4\,L\,(-1)^{l-1}}{\pi\,(2\,l-1)}\exp\left(-\left(\frac{\pi\,(2\,l-1)}{2\,L}\right)^2\frac{\sigma^2}{2}\,t\right)\cos\left(\frac{\pi\,(2\,l-1)\,x}{2\,L}\right),$$

with $u(L,t) = u(-L,t) = 0$ while for $x \in (-L,L)$ we have by the calculation above that $u(x,t)\to 1$ in L^2 as $t\to 0$.

Exercise 5. Observe that $u(x,t) = \mathbb{P}(\gamma_D^x \geqslant t)$ satisfies

$$u_t = \frac{1}{2}\,u_{x\,x}, \qquad (x,t) \in (0,L)\times\mathbb{R}_+$$

with intial condition $u(x,0) = 1$ on $(0,L)$ and boundary conditon $u(0,t) = 0 = u(L,0)$. For this PDE we have the representation

$$u(x,t) = \sum_{n=1}^{\infty}\theta_n(t)\,v_n(x)$$

where the function θ_n & v_n satisfy equation (3) with $\sigma = 1$ on $(0,L)$. Hence familiar calculations give

$$\theta_n(t) = b_n\,e^{-\lambda_n t}, \quad v_n(x) = \sin\left(\frac{\pi\,n\,x}{L}\right), \quad \lambda_n = \frac{1}{2}\left(\frac{\pi\,n}{L}\right)^2,$$

where b_n can be obtained from

$$1 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi\,n\,x}{L}\right).$$

Using orthgonality this implies

$$b_n = \frac{\int_0^L \sin\left(\frac{\pi\,n\,x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{\pi\,n\,x}{L}\right)dx} = \frac{\frac{L}{\pi\,n}\,(1-\cos(\pi\,n))}{\frac{1}{2}\int_0^L\left(1-\cos\left(\frac{2\,\pi\,n\,x}{L}\right)\right)dx} = \frac{2}{\pi\,n}\,(1-(-1)^n)$$

so that $b_n = 0$ for even n. Thus

$$u(x,t) = \sum_{k=1}^{\infty}\frac{4}{\pi\,(2\,k-1)}\,e^{-\lambda_{2k-1}t}\sin\left(\frac{\pi\,(2\,k-1)\,x}{L}\right),$$

where the first term has the slowest decay in time, so that

$$\|u(\,\cdot\,,t)\|_2 \leqslant \left\{\sum_{n=1}^{\infty}\frac{16}{\pi^2\,(2\,n-1)^2}\right\}^{1/2} e^{-\lambda_1 t} = \sqrt{2}\,e^{-\lambda_1 t}$$

with $\lambda_1 = (\pi/L)^2/2$, and the desired conclusion follows.