Homework 6

1. Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ and for $x \in \mathbb{R}^n$, $t > 0$ set $u(x,t) = a \cdot x - tH(a) + b$. Show that the function $u(x,t)$ satisfies the Hamilton-Jacobi equation
$$u_t + H(\nabla u) = 0.$$

2. Let $H(p)$, $p \in \mathbb{R}^n$, be a convex function and set $L(y) = \sup_{p \in \mathbb{R}^n}(p \cdot y - H(p))$, $y \in \mathbb{R}^n$. (i) Show that $L(y)$ is a convex function. (ii) Let $H(p) = |p|^r/r$, for some $r \in (1,\infty)$. Show that $L(y) = |y|^s/s$ with $s \in (1,\infty)$ satisfying $1/r + 1/s = 1$. (iii) Let a_{ij} be a symmetric positive-definite matrix, and $b \in \mathbb{R}^n$, and set
$$H(p) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}p_i p_j + \sum_{i=1}^{n} b_i p_i.$$
Compute $L(y)$.

3. Let $L(q)$ be a convex function, related to the function $H(p)$ as in Problem 2. Derive the following formula for solution of the Hamilton-Jacobi equation
$$u_t + H(\nabla u) = 0$$
with the initial data $u(x,0) = g(x)$:
$$u(x,t) = \min\left(\int_0^t L(\dot{w}(s))ds + g(y) \middle|\ w(0) = y,\ w(t) = x\right),$$
with the infimum taken over all C^1-functions $w(t)$ with $w(t) = x$.

4. Show that expression for $u(x,t)$ given in Problem 3 can be simplified to
$$u(x,t) = \min_{y \in \mathbb{R}^n}\left(tL\left(\frac{x-y}{t}\right) + g(y)\right).$$

5. Finally, show that expression for $u(x,t)$ given in Problem 4 can be simplified to
$$u(x,t) = \min_{y \in B(x,Rt)}\left(tL\left(\frac{x-y}{t}\right) + g(y)\right),$$
where $R = \sup_{x \in \mathbb{R}^n} |\nabla H(\nabla g(x))$. Use this to show that Hamilton-Jacobi equations have a finite speed of propagation.