Math 227 Winter 2009

Solutions to HW #6

Exercise 1. This is a consequence of $\nabla u = \nabla(x^t\, a) = a$.

Exercise 2.

i. Observe that for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$ we have

$$\begin{aligned} L(\lambda\, x + (1-\lambda)\, y) &= \sup_p \left(p^t\,(\lambda\, x + (1-\lambda)\, y) - H(p)\right) \\ &= \sup_p \left(\lambda\,[p^t x - H(p)] + (1-\lambda)[p^t\, y - H(p)]\right) \\ &\leq \lambda L(x) + (1-\lambda)\, L(y) \end{aligned}$$

which implies convexity of L.

ii. For this particular H we have

$$\nabla(p^t\, y - |p|^r/r) = y - |p|^{r-2}\, p$$

which equals zero only at $p = |y|^{\frac{1}{r-1}-1} y$. Since the function decreases as $|p| \to \infty$, this is a maximum. Hence

$$L(y) = |y|^{\frac{1}{r-1}-1}\, |y|^2 - |y|^{\frac{r}{r-1}}/r = |y|^{(1-\frac{1}{r})^{-1}}\, (1 - \frac{1}{r})$$

as required.

iii. In vector notation we have $L(y) = \sup_p \left(p^t (y-b) - \frac{1}{2}\, p^t\, A\, p\right)$ for the symmetric positive-definite and hence invertible matrix $[A]_{ij} = a_{ij}$. Differentiating with respect to p gives

$$\nabla(p^t (y-b) - \frac{1}{2}\, p^t\, A\, p) = y - b - A\, p$$

which equals zero only at the unique $p = A^{-1}(y-b)$. Similarly the Hessian of the argument of the sup is $-A$, negative-definite, so this is a maximum. Consequently

$$L(y) = \frac{1}{2}\, (y-b)^t A^{-1} (y-b).$$

Exercise 3. With u defined as in the question, we certainly have for $h > 0$ that

$$u(x,t) = \min\left\{ \int_{t-h}^{t} L(\dot{w}(\theta))\, d\theta + u(w(0), t-h) : w(t) = x \right\}. \tag{1}$$

From here, for any $z \in \mathbb{R}^d$ taking the particular path $w(s) = x + (\theta - t) z$ for $t - h \leq \theta \leq t$ yields the inequality

$$u(y,t) - u(x - h\, z, t - h) \leq h\, L(z).$$

Dividing by $h > 0$ and taking $h \to 0$ yields

$$u_t(x,t) + z \cdot \nabla u(x,t) - L(z) \leq 0,$$

and since this is true for any z, it follows from $H(x) = \sup_q \{q \cdot x - L(x)\}$ that

$$u_t(x,t) + H(\nabla u(x,t)) = u_t(x,t) + \sup_z \{ z \cdot \nabla u(x,t) - L(z) \} \leq 0. \tag{2}$$

To show the reverse inequality, fix (x, t) and notice that by the result of **Exercise 4** we can find a $y_* \in \mathbb{R}^n$ for which

$$u(x,t) = t\, L(\frac{x - y_*}{t}) + g(y_*).$$

Letting $z = \frac{x - y_*}{t}$, we then still have the path $w(\theta) = x + (t - \theta)\, y_*$ pass through $(x - h\, z, t - h)$, which implies

$$u(x - h\, z, t - h) \leq (t - h)\, L(z) + g(y_*)$$

(actually, equality holds by relationship (1)). Hence

$$u(x,t) - u(x - h\, z, t - h) \leq h\, L(z),$$

so again dividing by h and taking $h \to 0$ will give

$$u_t(x,t) + H(\nabla u(x,t)) \geq u_t(x,t) + z \cdot \nabla u(x,t) - L(z) \geq 0. \tag{3}$$

Combining (2) and (3) yields the result.

Exercise 4. Notice that taking $w(s) = y(1 - s)/t + x\, s/t$ with $0 \leq s \leq t$ we in particular get

$$u(x,t) \leq \int_0^t L(\dot{w}(s))\, ds + g(y) = t\, L(\frac{x - y}{t}) + g(y),$$

so that

$$u(x,t) \leq \inf_y \left[t\, L(\frac{x - y}{t}) + g(y) \right].$$

In other words, we have simply restricted the set of paths over which we minimize, so the new minimum cannot be smaller. On the other hand, it is interesting that it is not strictly greater, either. From the convexity of L, we have Jensen's inequality giving us for any w with $w(0) = y$

$$L\left(t^{-1} \int_0^t \dot{w}(s)\, ds \right) \leq t^{-1} \int_0^t L(\dot{w}(s))\, ds,$$

and consequently

$$t\, L(\frac{x - y}{t}) + g(y) \leq \int_0^t L(\dot{w}(s))\, ds + g(y)$$

implying

$$\inf_y \left[t\, L(\frac{x - y}{t}) + g(y) \right] \leq u(x,t).$$

Exercise 5. First, we will establish that $\nabla H(\nabla L(x)) = x$. From the definition $L(x) = \sup_p \{p \cdot x - H(p)\}$. Differentiating the expression in the supremum with respect to p gives that the maximum is achieved at $p = (\nabla H)^{-1}(x)$. With that p we have

$$L(x) = (\nabla H)^{-1}(x) \cdot x - H((\nabla H)^{-1}(x)),$$

and differentiating with respect to x gives

$$\nabla L(x) = (\nabla H)^{-1}(x) + [x - \nabla H((\nabla H)^{-1}(x))]\, \nabla[(\nabla H)^{-1}(x)] = (\nabla H)^{-1}(x)$$

and the claim follows.

Now consider

$$u(x,t) = \min_{y \in \mathbb{R}^d} \{t\, L(\frac{x - y}{t}) + g(y)\},$$

where the minimum is achieved finitely. Differentiate the argument of the minimum to obtain that it is achieved at some y satisfying

$$\nabla L(\frac{x - y}{t}) = \nabla g(y).$$

Applying $\nabla H(\,\cdot\,)$ to both sides gives

$$\frac{x-y}{t} = \nabla H(\nabla g(y))$$

and consequently $|x-y| \leq t\sup_z |\nabla H(\nabla g(z))| = R$.

Finally, in view of what we have just showed and the previous exercises, formula (1) simplifies to

$$u(x,t) = \min_{y\in B(x,R(t-s))} \{(t-s)\,L(\frac{x-y}{t-s}) + u(y,s)\}.$$

Hence the solution u at the point (x,t) is not affected by the values of the solution u outside of the cone

$$C \equiv \{(z,s): \quad 0 \leq s \leq t, \quad z \in B(x, R\,s)\}.$$

In other words, the information propagates at a finite speed.