

**Class notes for Ma 142**  
Winter/Spring 2003

01/07

Let's first study the free Schrödinger equation:

$$\begin{cases} i\partial_t \Psi + \frac{1}{2}\Delta \Psi = 0 \\ \Psi|_{t=0} = \Psi_0 . \end{cases}$$

Throughout these notes we will use the following convention for the Fourier transform (FT) and its inverse:

$$\begin{aligned} \hat{f}(\xi) &= \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx , \\ f(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \hat{f}(\xi) d\xi . \end{aligned}$$

After applying the FT, the equation becomes

$$\begin{cases} i\partial_t \hat{\Psi}(t, \xi) - \frac{1}{2}|\xi|^2 \hat{\Psi}(t, \xi) = 0 \\ \hat{\Psi}(0, \xi) = \hat{\Psi}_0(\xi) , \end{cases}$$

which can be solved as  $\hat{\Psi}(t, \xi) = e^{-\frac{i}{2}t|\xi|^2} \hat{\Psi}_0(\xi)$ .

Applying the inverse FT,

$$\Psi(t, x) = (2\pi)^{-d} \int e^{i(x \cdot \xi - \frac{1}{2}t|\xi|^2)} \hat{\Psi}_0(\xi) d\xi .$$

Obviously, for  $t > 0$ , we have the property that  $\|\Psi(t, \cdot)\|_2 = \|\Psi_0\|_2$  (use the FT).

Denote  $H^\gamma = W^{\gamma, 2}$  with the norm

$$\|f\|_{H^\gamma} = \left( \int (1 + |\xi|^2)^\gamma |\hat{f}(\xi)|^2 d\xi \right)^{1/2}, \quad \gamma \in \mathbb{R} .$$

The corresponding homogeneous norm is

$$\|f\|_{\dot{H}^\gamma} = \left( \int |\xi|^{2\gamma} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} .$$

Upon scaling,  $f(x) \rightarrow f(\lambda x)$ ,  $\hat{f}(\xi) \rightarrow \lambda^{-d} \hat{f}(\lambda^{-1} \xi)$ ,

$$\begin{aligned} \|f(\lambda \cdot)\|_{\dot{H}^\gamma} &= \left( \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\lambda^{-d} \hat{f}(\xi \lambda^{-1})|^2 d\xi \right)^{1/2} \\ &= (\lambda^{-2d} \lambda^d \lambda^{2\gamma})^{1/2} \|f\|_{\dot{H}^\gamma} \\ &= \lambda^{\gamma - d/2} \|f\|_{\dot{H}^\gamma} . \end{aligned}$$

**Lemma.**

$$\begin{aligned}\|\Psi(t, \cdot)\|_{H^\gamma} &= \|\Psi_0\|_{H^\gamma} \\ \|\Psi(t, \cdot)\|_{\dot{H}^\gamma} &= \|\Psi_0\|_{\dot{H}^\gamma}\end{aligned}$$

*Proof.*

$$\begin{aligned}\|\Psi(t, \cdot)\|_{\dot{H}^\gamma}^2 &= \int_{\mathbb{R}^d} |\xi|^{2\gamma} |e^{-\frac{1}{2}t|\xi|^2} \hat{\Psi}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^d} |\xi|^{2\gamma} |\hat{\Psi}_0(\xi)|^2 d\xi \\ &= \|\Psi_0\|_{\dot{H}^\gamma}^2\end{aligned}$$

and the same is true for the inhomogeneous norm after replacing  $|\xi|^{2\gamma}$  by  $(1 + |\xi|^2)^\gamma$ .  $\square$

Let us study some special solutions (for  $t > 0$ ).

(1) Gaussians

$$\begin{aligned}\Psi_0(x) &= e^{-|x|^2/2} \\ \hat{\Psi}_0(\xi) &= \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-|x|^2/2} dx \\ &= \int_{\mathbb{R}^d} e^{-|x+i\xi|^2/2} dx \cdot e^{-|\xi|^2/2} \\ &= \int_{\mathbb{R}^d} e^{-|x|^2/2} dx \cdot e^{-|\xi|^2/2} \\ &\quad \text{(shift the contour)} \\ &= (2\pi)^{d/2} e^{-|\xi|^2/2}\end{aligned}$$

$$\begin{aligned}\Psi(t, x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - \frac{t}{2}|\xi|^2)} e^{-|\xi|^2/2} d\xi \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-(1+it)|\xi|^2/2} e^{ix \cdot \xi} d\xi \\ &= (1+it)^{-d/2} e^{-|x|^2/2(1+it)} \\ |\Psi(t, x)|^2 &= (1+t^2)^{-d/2} e^{-|x|^2/(1+t^2)}\end{aligned}$$

(2) Modulated gaussians

$$\begin{aligned}\Psi_0(x) &= e^{-|x|^2/2} e^{ix \cdot v} \\ \hat{\Psi}(\xi) &= (2\pi)^{d/2} e^{-|\xi-v|^2/2}\end{aligned}$$

$$\begin{aligned}
\Psi(t, x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - \frac{1}{2}t|\xi|^2)} e^{-|\xi - v|^2/2} d\xi \\
&= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x(\xi+v) - \frac{1}{2}t|\xi+v|^2)} e^{-|\xi|^2/2} d\xi \\
&= e^{ix \cdot v} e^{-\frac{1}{2}t|v|^2} (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i(x-tv) \cdot \xi} e^{-(1+it)|\xi|^2/2} d\xi \\
&= e^{ix \cdot v} e^{-\frac{1}{2}t|v|^2} (1+it)^{-d/2} e^{-|x-tv|^2/2(1+it)}
\end{aligned}$$

(3) Fundamental solution. Let us derive it from scaling considerations. By translation invariance, we must have

$$\Psi(t, x) = \int_{\mathbb{R}^d} K(t, x-y) \Psi_0(y) dy$$

In addition,  $K(0, x) = \delta_0(x)$ . Consider the scaling

$$\Psi_0(x) \rightarrow \Psi_0(\lambda x) \quad , \quad \Psi(t, x) \rightarrow \Psi(\lambda^2 t, \lambda x) .$$

Then

$$\begin{aligned}
\Psi(\lambda^2 t, \lambda x) &= \int_{\mathbb{R}^d} K(t, x-y) \Psi_0(\lambda y) dy \\
\Psi(t, x) &= \int_{\mathbb{R}^d} \lambda^{-d} K(\lambda^{-2} t, \lambda^{-1}(x-y)) \Psi_0(y) dy . \\
&\Rightarrow K(t, x-y) = \lambda^{-d} K(\lambda^{-2} t, \lambda^{-1}(x-y)) \\
\text{or } K(t, x-y) &= \lambda^d K(\lambda^2 t, \lambda x) .
\end{aligned}$$

It is reasonable to look for  $\lambda^d K(\lambda^2 t, \lambda x) = t^{-d/2} \Phi(\frac{x^2}{t})$  and let  $u = x^2/t$ . Since  $i\partial_t K + \frac{1}{2}\Delta K = 0$ , the equation for  $\Phi$  is

$$\begin{aligned}
-i \frac{d}{2} t^{-\frac{d}{2}-1} \Phi(u) - it^{-\frac{d}{2}} \Phi'(u) \cdot \frac{u}{t} + \frac{1}{2} t^{-\frac{d}{2}} \operatorname{div}_x \left( 2 \frac{x}{t} \Phi'(u) \right) &= 0 \\
\Rightarrow -i \frac{d}{2} \Phi(u) - iu \Phi'(u) + d\Phi'(u) + 2u\Phi''(u) &= 0 \\
d(\Phi'(u) - \frac{i}{2} \Phi(u)) + 2u \frac{d}{du} (\Phi'(u) - \frac{i}{2} \Phi(u)) &= 0 .
\end{aligned}$$

This at least admits the solution  $\Phi(u) = e^{\frac{i}{2}u}$ , i.e.,  $K(t, x) = t^{-d/2} e^{\frac{i}{2} \frac{|x|^2}{t}}$ .

We could have guessed this result from

$$\begin{aligned}
K(t, x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi - \frac{1}{2}t|\xi|^2)} d\xi \\
&= (2\pi)^{-d} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{-(\varepsilon+it)|\xi|^2/2} |\xi|^2 d\xi
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0^+} (\varepsilon + it)^{-d/2} (2\pi)^{-d/2} e^{-|x|^2/2(\varepsilon+it)} \\
&= (2\pi it)^{-d/2} e^{-|x|^2/2it} .
\end{aligned}$$

No problem with the definition of this quantity, take the branch cut on the negative real  $t$ -axis. Therefore

$$(1) \quad \Psi(t, x) = (2\pi it)^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2it}} \Psi_0(y) dy .$$

**Theorem 1.** *Let  $\Psi_0 \in \mathcal{S}(\mathbb{R}^2)$ . Then there exists a solution to*

$$\begin{cases} i\Psi_t + \frac{1}{2}\Delta\Psi = 0 , \\ \Psi|_{t=0} = \Psi_0 , \end{cases}$$

*unique on  $\mathcal{S}(\mathbb{R}^d)$ , and given by (1).*

We have obtained the *a priori* estimate  $\|\Psi(t, \cdot)\|_2 = \|\Psi_0\|_2$ , for  $\Psi \in \mathcal{S}$  and  $t > 0$ . This indicates that a natural setting is the space  $L^\infty((0, \infty), L_x^2)$ .

We can combine it with the dispersive estimate  $\|\Psi(t, \cdot)\|_\infty \leq |2\pi t|^{-d/2} \|\Psi_0\|_1$ , to obtain by interpolation

$$\|\Psi(t, \cdot)\|_{L^{p'}(\mathbb{R}^d)} \leq C(d, p) t^{-d(\frac{1}{p} - \frac{1}{2})} \|\Psi_0\|_{L^p(\mathbb{R}^d)} ,$$

valid for  $1 \leq p \leq 2$  and  $t \geq 0$ .

In the sequel, we will consider the inhomogeneous Schrödinger equation

$$\begin{cases} i\Psi_t + \frac{1}{2}\Delta\Psi = F & , \quad F \in \mathcal{S}_{t,x} \\ \Psi|_{t=0} = \Psi_o & , \quad \Psi_o \in \mathcal{S} . \end{cases}$$

Its solution is explicitly given by the Duhamel formula

$$\Psi(t) = e^{\frac{i}{2}t\Delta}\Psi_0 - i \int_0^t e^{i(t-s)\Delta/2} F(s) ds .$$

The notation  $e^{\frac{i}{2}t\Delta}\Psi_0$  should be understood as  $(e^{-\frac{i}{2}t|\xi|^2}\hat{\Psi}_o)^\vee$ .

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**Theorem 2.** *(Strichartz estimates for the inhomogeneous Schrödinger equation)*

$$\|\Psi\|_{L_{t,x}^{p'}(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\Psi_0\|_{L_x^2(\mathbb{R}^d)} + \|F\|_{L_{t,x}^p(\mathbb{R} \times \mathbb{R}^d)}$$

for  $p' = 2 + 4/d$ .

*Remark 3.* This value of  $p$  is the only possible one, and it is necessary to have conjugate exponents  $p', p$  on the lhs and rhs respectively. This can be seen from the following scaling argument:

$$\begin{aligned}
\Psi_0(x) &\rightarrow \Psi_0(\lambda x) \\
\Psi(t, x) &\rightarrow \Psi(\lambda^2 t, \lambda x) \\
F(t, x) &\rightarrow \lambda^2 F(\lambda^2 t, \lambda x) \\
\| \Psi \|_{L_{t,x}^{p'}} &\rightarrow \lambda^{-d/p'} \lambda^{-2/p'} \| \Psi \|_{L^{p'}} \\
\| \Psi_0 \|_{L_x^2} &\rightarrow \lambda^{-d/2} \| \Psi_0 \|_{L^2} \\
&\Rightarrow \frac{d+2}{p'} = \frac{d}{2} \Rightarrow p' = 2 + \frac{4}{d} \\
\text{and } p &= \left(1 - \frac{d}{2d+4}\right)^{-1} = \frac{2d+4}{d+4}.
\end{aligned}$$

Put  $q$  the exponent to consider in  $\| F \|_{L_{t,x}^q(\mathbb{R} \times \mathbb{R}^d)}$ . Then

$$\begin{aligned}
&\| F \|_{L_{t,x}^q} \rightarrow \lambda^2 \lambda^{-\frac{d+2}{q}} \| F \|_{L_{t,x}^q} \\
&\Rightarrow 2 - \frac{d+2}{q} = -\frac{d}{2} \Rightarrow q = \frac{2d+4}{d+4} = p.
\end{aligned}$$

**Lemma 1.** Let  $\Psi(t) = e^{\frac{i}{2}t\Delta}\Psi_0$ . Then

$$\| \Psi(t) \|_{L_x^{q'}(\mathbb{R}^d)} \lesssim |t|^{-d\left(\frac{1}{q}-\frac{1}{2}\right)} \| \Psi_0 \|_{L_x^q(\mathbb{R}^d)},$$

for all  $1 \leq q \leq 2$ .

**Lemma 2.** (Fractional integration, Hardy-Littlewood-Sobolev).

Let  $0 < \alpha < 1$ ,  $g \in \mathcal{S}(\mathbb{R}^d)$ , and

$$(T_\alpha g)(t) = \int_{-\infty}^{\infty} |t-s|^{-\alpha} g(s) ds$$

$$\text{Then } \| T_\alpha g \|_{L^q(\mathbb{R})} \leq C \| g \|_{L^p(\mathbb{R})}$$

where  $1 + \frac{1}{q} = \alpha + \frac{1}{p}$ ,  $1 < p < q < \infty$ .

[Note: one gains regularity by fractional integration. We could not expect  $p = 1$  above because the kernel would not be in  $L^q(\mathbb{R})$ , only in weak  $-L^q(\mathbb{R})$ ].

*Proof.* Strichartz ( $T - T^*$  argument)

Put  $Uf = e^{\frac{i}{2}t\Delta}f$ , then  $U : L_x^2(\mathbb{R}^d) \rightarrow L_t^\infty(L_x^2)$ . Then the adjoint is given by

$$U^* : L_t^1(L_x^2) \rightarrow L_x^2,$$

$$\begin{aligned}
\langle f, U^*G \rangle_{L_x^2} &= \langle Uf, G \rangle_{L_{t,x}^2} \\
&= \iint (e^{\frac{i}{2}t\Delta}f)(x) \bar{G}(t, x) dt dx \\
&= \int f(x) \int \overline{(e^{-\frac{i}{2}t\Delta}G(t, \cdot))}(x) dt dx
\end{aligned}$$

$$\Rightarrow U^*G(x) = \int (e^{-\frac{i}{2}s\Delta}G(s))(x) ds$$

(this indeed belongs to  $L_x^2$  by the assumptions on  $G$ ).

$$(UU^*G)(t, x) = \int_{-\infty}^{\infty} (e^{\frac{i}{2}(t-s)\Delta}G(s))(x) ds$$

such that  $UU^* : L_t^1(L_x^2) \rightarrow L_t^\infty(L_x^2)$ .

Let us show that  $\| UU^*G \|_{L_{t,x}^{p'}} \leq C \cdot \| G \|_{L_{t,x}^p}$

$$\begin{aligned} \| UU^*G \|_{L_x^{p'}} &\leq \int_0^\infty \| e^{\frac{i}{2}(t-s)\Delta}G(s) \|_{L_x^{p'}} ds \\ &\leq C \cdot \int_{-\infty}^{\infty} |t-s|^{-d(\frac{1}{p}-\frac{1}{2})} \| G(s) \|_{L_x^p} ds \end{aligned}$$

by lemma 1.

By lemma 2, we get precisely  $\| UU^*G \|_{L_t^q(L_x^p)} \leq C \cdot \| G \|_{L_{t,x}^p}$  for  $q$  that satisfies

$$1 + \frac{1}{q} = d\left(\frac{1}{p} - \frac{1}{2}\right) + \frac{1}{p} \Rightarrow q = 2 + \frac{4}{d} = p'$$

(and we check that  $0 < \alpha = d(\frac{1}{p} - \frac{1}{2}) < 1$ ).

Let us show Strichartz for  $F = 0$ :

$$\begin{aligned} \| U^*G \|_{L_x^2}^2 &= \langle UU^*G, G \rangle \\ &\leq \| UU^*G \|_{L_{t,x}^{p'}} \| G \|_{L_{t,x}^p} \\ &\leq C \cdot \| G \|_{L_{t,x}^p}^2 \end{aligned}$$

so that  $U^* : L_{t,x}^p \rightarrow L_x^2$  boundedly,

therefore  $U : L_x^2 \rightarrow L_{t,x}^{p'}$  boundedly,

$$\| U\Psi_0 \|_{L_{t,x}^{p'}} \leq C \cdot \| \Psi_0 \|_{L_x^2} .$$

When  $F \neq 0$ ,

$$\Psi(t) = e^{\frac{i}{2}t\Delta}\Psi_0 - i \int_0^t e^{\frac{i}{2}(t-s)\Delta}F(s) ds$$

$$\begin{aligned} \| \Psi \|_{L_{t,x}^{p'}} &\leq C \cdot \| \Psi_0 \|_2 + \left\| \int_0^t e^{\frac{i}{2}(t-s)\Delta}F(s) \|_{L_x^{p'}} ds \right\|_{L_t^{p'}} \\ &\leq C \cdot \| \Psi_0 \|_2 + \left\| \int_{-\infty}^{\infty} |t-s|^{-d(\frac{1}{p}-\frac{1}{2})} \| F(s) \|_{L_x^p} ds \right\|_{L_t^{p'}} \end{aligned}$$

by lemma 1

$$\leq C \cdot (\| \Psi_0 \|_2 + \| F \|_{L^p_{s,x}}) \quad \text{by lemma 2}$$

□

*Remark 3.* The (free) Strichartz estimate can also be derived from the Stein-Tomas restriction theorem. Let  $S \subset \mathbb{R}^n$ ,  $n = d + 1$ , by a hypersurface with nonvanishing gaussian curvature, and  $\sigma_s$  be the corresponding surface measure. Let  $\phi$  be compactly supported on  $S$ . Then the Stein-Tomas theorem says that

$$\| \widehat{\phi \sigma_s} \|_{L^r(\mathbb{R}^n)} \leq C \cdot \| \phi \|_{L^2(\sigma_s)}$$

when  $r = \frac{2n+2}{n-1}$ .

In our case,  $\Psi(t, x) = \int e^{i[x \cdot \xi - \frac{1}{2}t|\xi|^2]} \hat{\Psi}_0(\xi) d\xi$  and  $S$  is the paraboloid

$$S = \{(\xi, \tau) : \tau = -\frac{1}{2}|\xi|^2, \xi \in \mathbb{R}^d\}$$

$$\text{so that } \Psi(t, x) = (\phi \sigma_s)^\vee(t, x)$$

$$\phi(\xi, \tau) = \hat{\Psi}_0(\xi)$$

$$\sigma_s(d\xi, d\tau) = (2\pi)^d d\xi$$

Indeed,

$$\begin{aligned} \Psi(t, x) &= (2\pi)^{-d} \int e^{i(x \cdot \xi + t\tau)} \phi(\xi, \tau) \sigma_s(d\xi, d\tau) \\ &= \int e^{i(x \cdot \xi - \frac{1}{2}t|\xi|^2)} \hat{\Psi}_0(\xi) d\xi \end{aligned}$$

( $\phi$  is the “lift” of  $\hat{\Psi}_0$  to the paraboloid)

Then we have  $\| (\phi \sigma_s)^\vee \|_{L^r(\mathbb{R}^n)} \leq C \cdot \| \phi \|_{L^2(\sigma_s)}$

$$\Rightarrow \| \Psi \|_{L^{2+4/d}_{t,x}(\mathbb{R} \times \mathbb{R}^d)} \leq C \cdot \| \hat{\Psi}_0 \|_{L^2_\xi} = C \cdot \| \Psi_0 \|_{L^2_x}$$

This is valid for  $\Psi_0$  such that  $\text{supp } \hat{\Psi} \subset B(0, 1)$ , or by scaling in  $B(0, R)$  for arbitrarily large  $R$  (the constant  $C$  is independent of  $R$ ). We conclude by density of the compactly supported functions.

**Theorem 4.** (*Kato  $\frac{1}{2}$ -smoothing estimate*)

For  $d \geq 2$ , the  $\chi \geq 0$  a smooth cutoff function such that  $\hat{\chi}$  is compactly supported. Then

$$\| \chi(x)(1 - \Delta)^{\frac{1}{4}} e^{\frac{i}{2}t\Delta} f \|_{L^2_{t,x}} \leq C \cdot \| f \|_{L^2_x} .$$

*Remark 5.* We gain 1/2 derivative provided we cut off by  $\chi$  in space. We can integrate in time because the cutoff  $\chi$  implies decay in time. It does not matter whether the spatial cutoff is done before or after taking the derivatives, as long as it is smooth.

*Proof.* Take  $\eta \in \mathcal{S}(\mathbb{R})$  so that  $\text{supp } \hat{\eta} \subset (-1, 1)$ . After passing to the limit  $\epsilon \rightarrow 0$ , the l.h.s. is

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}} |\eta(\epsilon t) \chi(x) ((1 - \Delta)^{1/4} e^{\frac{i}{2} t \Delta} f)(x)|^2 dt dx \\
&= \iint |\eta(\epsilon t) \int \hat{\chi}(\xi - \xi') (1 + |\xi|^2)^{1/4} e^{-\frac{i}{2} t |\xi'|^2} \hat{f}(\xi') d\xi'|^2 d\xi dt \\
&\quad \text{(by Plancherel in } x - \xi) \\
&= \iint \left| \iint \hat{\chi}(\xi - \xi') \frac{1}{\epsilon} \hat{\eta}\left(\frac{\tau - \tau'}{\epsilon}\right) (1 + |\xi'|^2)^{1/4} \delta\left(\tau' - \frac{1}{2} |\xi'|^2\right) \right. \\
&\quad \left. \hat{f}(\xi') d\xi' d\tau' \right|^2 d\xi d\tau \\
&\quad \text{(by Plancherel in } t - \tau) \\
&= \iint \left| \int \hat{\chi}(\xi - \xi') \frac{1}{\epsilon} \hat{\eta}\left(\frac{\tau - \frac{1}{2} |\xi'|^2}{\epsilon}\right) (1 + |\xi'|^2)^{1/4} \hat{f}(\xi') d\xi' \right|^2 d\xi d\tau \\
&\leq \iint \cdot \left( \int \left| \hat{\chi}(\xi - \xi') \frac{1}{\epsilon} \hat{\eta}\left(\frac{\tau - \frac{1}{2} |\xi'|^2}{\epsilon}\right) \right| d\xi' \right) \\
&\quad \otimes \cdot \left( \int \left| \hat{\chi}(\xi - \xi') \frac{1}{\epsilon} \hat{\eta}\left(\frac{\tau - \frac{1}{2} |\xi'|^2}{\epsilon}\right) \right| (1 + |\xi'|^2)^{1/2} |\hat{f}(\xi')|^2 d\xi' \right) d\xi d\tau \\
&\quad \text{(by Cauchy-Schwarz in } \xi')
\end{aligned}$$

The domains of integration are limited by

$$\begin{cases} -\epsilon < \tau - \frac{1}{2} |\xi'|^2 < \epsilon & (1) \\ |\xi - \xi'| < 1 & (2) \end{cases}$$

$$(2) \text{ implies that } \left| |\xi| - |\xi'| \right| = O(1)$$

$$\begin{aligned}
& \text{and also that } \left| |\xi|^2 - |\xi'|^2 \right| = \left| |\xi| - |\xi'| \right| \cdot (|\xi| + |\xi'|) \\
& = O(1 + |\xi|).
\end{aligned}$$

Since  $\tau = O(|\xi'|^2)$ , the range of values for  $\tau$  is  $O(1 + |\xi|)$ .

(1) implies that  $|\sqrt{2|\tau|} - |\xi'|| = O\left(\frac{\epsilon}{1+|\xi|}\right)$ , and on the other hand  $\xi' \in B(\xi, 1)$ , therefore the range of values taken on by  $\xi'$  is  $O\left(\frac{\epsilon}{1+|\xi|}\right)$ , the volume of the shaded area in the figure. We bound  $\otimes$  by

$$\begin{aligned}
& C \int d\xi \left( \frac{1}{\epsilon} \cdot \frac{\epsilon}{1+|\xi|} \right) \cdot \left( \frac{1}{\epsilon} \cdot \frac{\epsilon}{1+|\xi|} \right) \cdot (1 + |\xi|^2)^{1/2} \cdot (1 + |\xi|) |\hat{f}(\xi)|^2 \\
& \leq C \int |\hat{f}(\xi)|^2 d\xi = C \cdot \|f\|_{L_x^2}^2
\end{aligned}$$

This estimate is uniform as  $\epsilon \rightarrow 0$ . □

**Corollary 6.** For  $d \geq 2$ , for every  $\epsilon > 0$ ,

$$\| (1 + |x|)^{-\frac{1}{2}-\epsilon} (-\Delta)^{1/4} e^{\frac{i}{2} t \Delta} f \|_{L_{t,x}^2} \leq C_\epsilon \cdot \|f\|_{L_x^2}$$

*Proof.* Think of  $\chi$  as  $1_{B(0,1)}$ , the indicator of the unit ball. Let us consider  $(-\Delta)^{1/4}$  instead of  $(1-\Delta)^{1/4}$  in the previous estimate, in order to get a homogeneous behavior under rescaling.

$$\begin{aligned}
& \| \chi(2^{-j}x)(-\Delta)^{1/4} e^{\frac{i}{2}t\Delta} f \|_{L_{t,x}^2} \quad \text{let } x \rightarrow 2^j x \\
&= 2^{jd/2} 2^j \| \chi \cdot ((-\Delta)^{1/4} (e^{\frac{i}{2}t\Delta} f)) (2^{2j}t, 2^j x) \|_{L_{t,x}^2} \\
&= 2^{jd/2} 2^j 2^{-j/2} \| \chi \cdot (-\Delta)^{1/4} ((e^{\frac{i}{2}t\Delta} f)(2^{2j}t, 2^j x)) \|_{L_{t,x}^2} \\
&= 2^{jd/2} 2^{j/2} \| \chi \cdot (-\Delta)^{1/4} (e^{\frac{i}{2}t\Delta} f(2^j \cdot)) (t, x) \|_{L_{t,x}^2} \\
&\leq C \cdot 2^{jd/2} 2^{j/2} \| f(2^j \cdot) \|_2 \leq C \cdot 2^{j/2} \| f \|_2
\end{aligned}$$

The homogeneous Kato smoothing estimate would therefore remain valid if we replaced  $\chi(x)$  by the dyadic superposition

$$\sum_{j \geq 0} 2^{-j(\frac{1}{2} + \varepsilon)} \chi(2^{-j}x),$$

which is essentially  $(1 + |x|)^{-\frac{1}{2} - \varepsilon}$ . □

**Theorem 7.** (*1 - D Kato smoothing estimate*).

For  $d = 1$ ,

$$\sup_x \| (-\Delta)^{1/4} e^{\frac{i}{2}t\Delta} f \|_{L_t^2} \leq C \cdot \| f \|_{L_x^2}$$

*Proof.* When  $\varepsilon \rightarrow 0$ , the l.h.s. is

$$\begin{aligned}
& \| \eta(\varepsilon t) \int_{-\infty}^{\infty} e^{i(x\xi - \frac{1}{2}t|\xi|^2)} |\xi|^{1/2} \hat{f}(\xi) d\xi \|_{L_t^2}^2 \\
\textcircled{*} &= \int \left| \int_{-\infty}^{\infty} e^{ix\xi} \frac{1}{\varepsilon} \hat{\eta}\left(\frac{\tau - \frac{1}{2}|\xi|^2}{\varepsilon}\right) |\xi|^{1/2} \hat{f}(\xi) d\xi \right|^2 d\tau
\end{aligned}$$

The integration in  $\xi$  is limited by  $-\varepsilon < \tau - \frac{1}{2}\xi^2 < \varepsilon$

$$\Rightarrow |\xi - \sqrt{2\tau}| < \frac{2\varepsilon}{\sqrt{2\tau} + \xi} \simeq \frac{\varepsilon}{\sqrt{2\tau}}$$

$$\text{or } |\xi - \sqrt{2\tau}| \lesssim \frac{\varepsilon}{\sqrt{2\tau}}$$

As  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned}
\textcircled{*} &\geq C \cdot \int_{-\infty}^{\infty} d\tau \left[ \frac{\varepsilon}{\sqrt{2\tau}} \frac{1}{\varepsilon} \cdot (2\tau)^{1/4} (\hat{f}(\sqrt{2\tau}) + \hat{f}(-\sqrt{2\tau})) \right]^2 \\
&\geq C \cdot \int_{-\infty}^{\infty} \left( \frac{|\hat{f}(\sqrt{2\tau})|^2}{\sqrt{2\tau}} + \frac{|\hat{f}(-\sqrt{2\tau})|^2}{\sqrt{2\tau}} \right) d\tau \\
&\geq C \cdot \| f \|_2^2 \quad \text{uniformly in } x.
\end{aligned}$$

□

The Kato smoothing estimate is sharp, as can be seen on the example of a modulated gaussian. Consider  $f(t, x) = (e^{\frac{i}{2}t\Delta} f_0)(x)$  where  $f_0(x) = e^{-|x|^2/2} e^{-ix \cdot v}$ . Intuitively, the time spent in  $B(0, 1)$  is  $O(\frac{1}{|v|})$ , so that the  $L_t^2$  norm yields a factor  $O(\frac{1}{|v|^{1/2}})$ . This compensates the factor  $|v|^{1/2}$  coming from the half-derivative.

**Lemma.** *For every  $\delta > 0$ ,*

$$\begin{aligned} & \| \chi(-\Delta)^{1/4} e^{\frac{i}{2}t\Delta} (e^{-\frac{|x|^2}{2}} e^{ix \cdot v}) \|_{L_{x,t}^2} \sim 1 \\ & \text{and } \sup_v \| \chi(-\Delta)^{1/4+\delta} e^{\frac{i}{2}t\Delta} (e^{-\frac{|x|^2}{2}} e^{ix \cdot v}) \|_{L_{x,t}^2} = \infty \end{aligned}$$

*Proof.* Take  $\text{supp } \hat{\chi} \subset B(0, 1)$ .

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \chi(x) \eta(\varepsilon t) ((-\Delta)^{1/4} e^{\frac{i}{2}t\Delta} f_0)(x) \right|^2 dx dt \\ &= \iint \left| \int \hat{\chi}(\xi' - \xi) |\xi|^{1/2} e^{-\frac{i}{2}t|\xi|^2} \eta(t\varepsilon) \hat{f}_0(\xi) d\xi \right|^2 d\xi' d\tau \\ &= \iint \left| \int \hat{\chi}(\xi' - \xi) |\xi|^{1/2} \frac{1}{\varepsilon} \eta\left(\frac{\tau - \frac{1}{2}|\xi|^2}{\varepsilon}\right) e^{-|\xi-v|^2/2} d\xi \right|^2 d\xi' d\eta^{\otimes} \end{aligned}$$

Again,

$$\begin{aligned} & -\varepsilon < \tau - \frac{1}{2}|\xi|^2 < \varepsilon \\ & \Rightarrow ||\xi| - \sqrt{2\tau}| < \frac{2\varepsilon}{|\xi| + \sqrt{2\tau}} \end{aligned}$$

As  $\varepsilon \rightarrow 0$ ,

$$\otimes = C \cdot \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left| \int_{\sqrt{2\tau}S^{d-1}} \hat{\chi}(\xi' - \xi) e^{-|\xi-v|^2/2} |\xi|^{1/2} \frac{2}{|\xi| + \sqrt{2\tau}} d\sigma(\xi) \right|^2 d\xi' d\tau$$

The range of values for  $\xi$  is  $O(1)$  (the  $(d-1)$ -measure of the shell fragment corresponding to the shaded area is  $O(1)$ ). The range of values for  $\xi'$  is  $O(1) = |\text{supp } \hat{\chi}|$ . The range of values for  $\tau$  is

$$\begin{aligned} & \frac{1}{2}||v| - 1|^2 \lesssim \tau \lesssim \varepsilon + \frac{1}{2}||v| + 1|^2 \\ & \Rightarrow |2\tau - |v|^2| \lesssim |v|. \end{aligned}$$

Consequently,

$$\begin{aligned} \otimes &= C \cdot \int \left| \frac{(\sqrt{\tau})^{1/2}}{\sqrt{\tau}} \right|^2 d\tau = C \cdot \frac{1}{|v|} \int_{|2\tau - |v|^2| \lesssim |v|} d\tau \\ &= O(1) \quad \text{uniformly in } v. \end{aligned}$$

Had we considered  $(-\Delta)^{\frac{1}{4}+\delta}$  instead, the above  $1/|v|$  would become  $1/|v|^{1-4\delta}$  and the estimate would not be uniform in  $|v|$  anymore.  $\square$

Consider now

$$\begin{cases} i\Psi_t + \frac{1}{2}\Delta\Psi + V(t, x)\Psi = 0 \\ \Psi|_{t=0} = \Psi_0 \in H^\gamma(\mathbb{R}^d). \end{cases}$$

**Lemma.** *Let  $\Psi_0 \in H^\gamma(\mathbb{R}^d)$ , then*

$$e^{\frac{i}{2}t\Delta}\Psi_0 \in C^\circ(\mathbb{R}, H^\gamma(\mathbb{R}^d)) \cap C^1(\mathbb{R}, H^{\gamma-2}(\mathbb{R}^d))$$

*Proof.*

$$\begin{aligned} & \| e^{\frac{i}{2}t\Delta}\Psi_0 - e^{\frac{i}{2}s\Delta}\Psi_0 \|_{H^\gamma}^2 \\ &= \| e^{\frac{i}{2}(t-s)\Delta}\Psi_0 - \Psi_0 \|_{H^\gamma}^2 \\ &= \int (1 + |\xi|^2)^\gamma |e^{-\frac{i}{2}(t-s)|\xi|^2} - 1| |\hat{\Psi}_0(\xi)|^2 d\xi \end{aligned}$$

Since  $\Psi_0 \in H^\gamma$  we can apply Lebesgue's dominated convergence theorem to conclude that the integral goes to zero as  $t - s \rightarrow 0$ .

On the other hand,

$$\begin{aligned} & \| \Delta e^{\frac{i}{2}t\Delta}\Psi_0 - \Delta e^{\frac{i}{2}s\Delta}\Psi_0 \|_{H^{\gamma-2}} \\ & \leq \| e^{\frac{i}{2}t\Delta}\Psi_0 - e^{\frac{i}{2}s\Delta}\Psi_0 \|_{H^\gamma} \end{aligned}$$

□

**Theorem 8.** *Assume  $V_{\text{real}}$ ,  $\sup_t \| \partial_x^\alpha V(t, \cdot) \|_{L^\infty} < C_\alpha$  and  $\| \partial_x^\alpha V(t, \cdot) - \partial_x^\alpha V(s, \cdot) \|_{L^\infty} \rightarrow 0$  as  $s \rightarrow t$  for every multiindex  $\alpha$ . Then for every  $k \geq 0$ , if  $\Psi_0 \in H^k(\mathbb{R}^d)$ , there exists  $\Psi \in C^\circ(\mathbb{R}, H^k) \cap C^1(\mathbb{R}, H^{k-2})$  such that*

$$(DH) \quad \Psi(t) = e^{\frac{i}{2}t\Delta}\Psi_0 + i \int_0^t e^{\frac{i}{2}(t-s)\Delta} V(s) \Psi(s) ds$$

(the integral is a vector-valued Riemann integral). There is exactly one  $\Psi \in C^\circ(\mathbb{R}, H^k)$  that satisfies (DH), Moreover,

$$\| \Psi(t) \|_{H^k(\mathbb{R}^d)} \leq C_{k,V} (1 + |t|)^k$$

*Proof.*

(1) Local solvability on  $[0, T]$ . The operator

$$(A\phi)(t) = e^{\frac{i}{2}t\Delta}\Psi_0 - i \int_0^t e^{\frac{i}{2}(t-s)\Delta} V(s) \phi(s) ds$$

maps  $X = C^\circ([0, T], H^k)$  to itself, with the estimate

$$\begin{aligned} \| A\phi \|_X &= \sup_{0 \leq t \leq T} \| A\phi(t) \|_{H^k} \\ &\leq \| \Psi_0 \|_{H^k} + \int_0^T \underbrace{\| V(s)\phi(s) \|_{H^k}}_{ds} ds \end{aligned}$$

$$\begin{aligned}
&\leq C_k \cdot \sum_{|\alpha| \leq k} \|\partial^\alpha (V(s)\phi(s))\|_L^2 \\
&\leq \|\Psi_0\|_{H^k} + C_{V,k} \int_0^T \|\phi(s)\|_H^k ds \\
&\leq \|\Psi_0\|_{H^k} + C_{V,k} T \|\phi\|_X
\end{aligned}$$

Take  $R = 2 \|\Psi_0\|_{H^k}$  and  $C_{V,k}T = 1/2$ , so that  $A : B_X(OR) \circlearrowleft$ .

$\Psi$  given by (DH) belongs to  $X$  in the first place because

- of the lemma for the first term
- $t$  in the upper bound of the integral yields a continuous dependence
- Fourier transform and apply Lebesgue's dominated convergence for the dependence on  $t$  of the integrand ( $e^{\frac{i}{2}(t-s)\Delta}$ )

So we get  $\|A\phi - A\tilde{\phi}\|_X \leq \frac{1}{2} \|\phi - \tilde{\phi}\|_X$ , a contraction. By the fixed point theorem, we conclude that there exists  $\Psi \in C^0([0, T], H^k)$  so that (DH) holds for  $0 \leq t \leq T$ .

(2) Extension to a global solution.

Take  $\Psi(T)$  as initial condition and solve for  $T \leq t \leq 2T$ ,

$$\Psi(t) = e^{\frac{i}{2}(t-T)\Delta}\Psi(T) + i \int_T^t e^{\frac{i}{2}(t-T-s)\Delta}V(s)\Psi(s) ds$$

(we can apply local solvability because  $C_{V,k}$  does not depend on the initial condition). Then it is easy to see that (DH) can be composed, and that  $\Psi(t)$ ,  $T \leq t \leq 2T$  given by the above expression actually satisfies

$$\Psi(t) = e^{\frac{i}{2}t\Delta}\Psi_0 + i \int_0^t e^{\frac{i}{2}(t-s)\Delta}V(s)\Psi(s) ds ,$$

for  $T \leq t \leq 2T$ .

(3) Uniqueness

Recall the Gronwall inequality. For  $f(t) \geq 0$  and  $K(T) \geq 0$  such that  $f(t)a \leq \int_0^t K(s)f(s) ds$ , then  $f(t) \leq a \exp(\int_0^t K(s) ds)$ .

Apply this to

$$\|\Psi(t) - \tilde{\Psi}(t)\|_{H^k} \leq \int_0^t C \cdot \|\Psi(s) - \tilde{\Psi}(s)\|_{H^k} ds ,$$

so that  $\|\Psi(t) - \tilde{\Psi}(t)\|_{H^k} \leq 0$ ,  $\Psi(t) = \tilde{\Psi}(t)$ .

(4) Note that Gronwall would imply a bound on  $\|\Psi\|_{H^k}$  growing exponentially in time. For  $V_{\text{real}}$ , we actually get a bound polynomial in time, as we show next.

$$\begin{aligned}
\frac{d}{dt} \|\Psi(t)\|_2^2 &= \frac{d}{dt} \langle \Psi(t), \Psi(t) \rangle \\
&= 2\text{Re} \langle \Psi'(t), \Psi(t) \rangle \\
&= 2\text{Re} \left\langle \frac{i}{2} \Delta \Psi(t) + iV(t)\Psi(t), \Psi(t) \right\rangle
\end{aligned}$$

$$\begin{aligned}
&= 2\operatorname{Re} i \left( \int -\frac{1}{2} |\nabla \Psi|^2(t, x) dx + \int V(t, x) |\Psi(t, x)|^2 dx \right) \\
&= 0
\end{aligned}$$

Formally, let  $\phi = \partial_j \Psi(t, x)$ , so that

$$\begin{aligned}
i\phi_t + \frac{1}{2}\Delta\phi + V\phi &= -(\partial_j V)\Psi \\
\phi(t) &= U(t, 0)\partial_j\Psi_0 + i \int_0^t U(t, s)(\partial_j V)(s)\Psi(s) ds,
\end{aligned}$$

where  $U(t, s)$  is the full evolution operator.

$$\| \phi(t) \|_2 \leq \| \partial_j \Psi_0 \|_2 + \int_0^t \| \partial_j V(s) \|_\infty \| \Psi_0 \|_2 ds$$

Adding this to  $\| \Psi(t) \|_2 = \| \Psi_0 \|_2$ , we get ( $t > 0$ )

$$\| \Psi(t) \|_{H^1} \leq C_V(1+t) \| \Psi_0 \|_{H^1}.$$

Inductively, by similar arguments,

$$\| \Psi(t) \|_{H^k} \leq C_{V,k}(1+t)^k \| \Psi_0 \|_{H^k}.$$

□

*Remark 9.* The growth of the  $H^k$  norm comes from the fact that  $V$  forces  $\Psi$  to acquire higher momenta.

**Theorem 10.** (*Bourgain*). For  $d \geq 3$ ,  $V_{\text{real}}$ , assume  $\sup_t \| \partial^K V(t, x) \|_{L^\infty} < C_\alpha$ , and  $\sup_{|t-t_0| \leq 1} |V(t, x)|$  compactly supported in  $x$ , with the diameter of the support not depending on  $t_0$  (sliding potential). Then, for  $\Psi$  from the previous theorem,

$$\| \Psi(t) \|_{H^\gamma(\mathbb{R}^d)} \leq C_\varepsilon(1+|t|)^\varepsilon \| \Psi_0 \|_{H^\gamma},$$

for every  $\varepsilon > 0$ .

*Proof.* Put  $\| \| f \| \| = \inf \{ \| f_1 \|_2 + \| f_2 \|_\infty : f = f_1 + f_2 \}$

Consider the splitting

$$\begin{aligned}
\partial^\alpha \Psi(t) &= \partial^\alpha e^{\frac{i}{2}t\Delta} \Psi_0 + i \int_0^{t-A} e^{\frac{i}{2}(t-s)\Delta} \partial^\alpha (V(s)\Psi(s)) ds \\
&\quad + i \int_{t-A}^t e^{\frac{i}{2}(t-s)\Delta} \partial^\alpha (V(s)\Psi(s)) ds
\end{aligned}$$

The first and third terms will be the  $L^2$  piece and the second term will be the  $L^\infty$  piece of the combined  $\| \| \cdot \| \|$  norm.

$$\| \| \partial^\alpha \Psi(t) \| \| \leq \| \partial^\alpha \Psi_0 \|_2 + C \cdot \int_0^{t-A} |t-s|^{-d/2} \| \partial^\alpha (V(s)\Psi(s)) \|_{L^1} ds$$

(dispersive estimate)

$$+ \sup_{\|\phi\|_2=1} \int_{t-A}^t \left| \left\langle e^{-\frac{i}{2}(t-s)\Delta} \phi, \partial^\alpha (V(s)\Psi(s)) \right\rangle \right| ds$$

(note that  $|t-s|^{d/2}$  is not integrable at  $s=t$ , hence the cutoff  $t-A$ ).

$$\leq \| \partial^\alpha \Psi_0 \|_2 + C \cdot A^{1-d/2} \sup_{0 \leq s \leq t} \| \partial^\alpha (V(s)\Psi(s)) \|_{L^1}$$

$$+ \sup_{\|\phi\|_2=1} \int_{t-A}^t \left\langle (1-\Delta)^{1/4} \chi e^{-\frac{i}{2}(t-s)\Delta} \phi, (1-\Delta)^{-1/4} \partial^\alpha (V(s)\Psi(s)) \right\rangle ds$$

(take  $\chi = 1$  on  $\text{supp } V$ )

$$\leq \| \partial^\alpha \Psi_0 \|_2 + C \cdot A^{1-d/2} \sup_{\substack{0 \leq \beta \leq \alpha \\ 0 \leq s \leq t}} \| \partial^\beta V(s) \partial^{\alpha-\beta} \Psi(s) \|_1$$

$$+ \sup_{\|\phi\|_2=1} \left( \int_{t-A}^t \| (1-\Delta)^{1/4} \chi e^{-\frac{i}{2}(t-s)\Delta} \phi \|_2^2 ds \right)^{1/2}$$

$$\cdot \left( \int_{t-A}^t \| (1-\Delta)^{-1/4} \partial^\alpha (V(s)\Psi(s)) \|_2^2 ds \right)^{1/2}$$

(apply Cauchy-Schwarz once in  $x$ , once in  $s$ )

$$\leq \| \partial^\alpha \Psi_0 \|_2 + C \cdot A^{1-d/2} \sup_{\substack{0 \leq \beta \leq \alpha \\ 0 \leq s \leq t}} \| \partial^\beta \Psi(s) \|$$

(1  $\rightarrow$  2 and 1  $\rightarrow$   $\infty$  because we have a uniform control over the size of the support of  $V$ )

$$+ (C \cdot \sqrt{A}) \quad (\text{by Kato, invariant by translation})$$

$$t \rightarrow t-s$$

$$\cdot \sqrt{A} \sup_{0 \leq s \leq t} \sup_{|\alpha|=k} \| (1-\Delta)^{1/4} \partial^\alpha (V(s)\Psi(s)) \|_2 .$$

By Sobolev interpolation, there exists  $\delta = \delta(k) > 0$  such that  $\| f \|_{H^{k-1/2}} \leq C_k \cdot \| f \|_{H^k}^{1-\delta} \| f \|_{L^2}^\delta$ . So

$$\begin{aligned} & \sup_{|\alpha|=k} \| (1-\Delta)^{-1/4} \partial^\alpha (V(s)\Psi(s)) \|_2 \\ & \leq C_k \cdot \| V(s)\Psi(s) \|_2^\delta \cdot \sup_{|\beta| \leq k} \| \partial^\beta V(s)\Psi(s) \|_2^{1-\delta} \\ & \leq C_{V,k,\delta} \cdot \| \Psi_0 \|_2^\delta \cdot \sup_{|\beta| \leq k} \| \partial^\beta \Psi(s) \|^{1-\delta} \end{aligned}$$

(as before, localization by  $V$  allows us to use  $\| \cdot \|$ ). Therefore,

$$\| \partial^\alpha \Psi(t) \| \leq \| \partial^\alpha \Psi_2 \|_2 + CA^{1-d/2} \sup_{\substack{0 \leq |\beta| \leq |\alpha| \\ 0 \leq s \leq t}} \| \partial^\beta \Psi(s) \|$$

$$+ CA \|\Psi_0\|_2^\delta \sup_{\substack{0 \leq |\beta| \leq |\alpha| \\ 0 \leq s \leq t}} \|\partial^\beta \Psi(s)\|^{1-\delta}$$

Young's inequality reads  $x^{1-\delta}y^\delta \leq \frac{1}{1-\delta}x + \frac{1}{\delta}y$ , and we will use  $(\varepsilon^{\frac{1}{1-\delta}}x)^{1-\delta} \cdot (\varepsilon^{-1/\delta}y)^\delta \leq \frac{\varepsilon^{\frac{1}{1-\delta}}}{1-\delta}x + \frac{\varepsilon^{-1/\delta}}{\delta}y$ ,

$$\begin{aligned} \sup_{\substack{|\alpha| \leq k \\ 0 \leq s \leq t}} \|\partial^\alpha \Psi(s)\| &\leq \|\Psi_0\|_{H^k} + C \cdot A \cdot \varepsilon^{-1/\delta} \|\Psi_0\|_2 \\ &+ C(A^{1-\frac{d}{2}} + A\varepsilon^{\frac{1}{1-\delta}}) \sup_{|\alpha| \leq k} \|\partial^\alpha \Psi(s)\| \end{aligned}$$

We can take  $\varepsilon$  so small and  $t, A$  so large that  $C(A^{1-d/2} + A\varepsilon^{\frac{1}{1-\delta}}) \leq \frac{1}{2}$ , hence

$$\sup_{\substack{|\alpha| \leq K \\ 0 \leq s \leq t}} \|\partial^\alpha \Psi(s)\| \leq C_{V,k} \cdot \|\Psi_0\|_{H^k}$$

As a result,

$$\|\Psi(t)\|_{H^k} \leq \|\Psi_0\|_{H^k} + \int_0^t \|V(s)\Psi(s)\|_{H^k} ds$$

with

$$\begin{aligned} \|V(s)\Psi(s)\|_{H^k} &\leq C_k \sup_{|\alpha| \leq k} \|\partial^\alpha V(s)\Psi(s)\|_2 \\ &\leq C_k \sup_{|\alpha| \leq k} \|\partial^\alpha \Psi(s)\| \end{aligned}$$

(as we have seen, localization by  $V$  enables control by both  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$ ). So

$$\|\Psi(t)\|_{H^k} \leq \|\Psi_0\|_{H^k} (1 + C_{V,k}t).$$

□

## Ma 142 01/16

Take  $V = V(x)$ , real.  $H = \frac{1}{2}\Delta + V$

$$\Psi = e^{itH}\Psi_0$$

Spectral theory of  $H = -\Delta + V$  on  $L^2(\mathbb{R}^d)$ .

$\text{Dom}(H) = W^{2,2}(\mathbb{R}^d)$ ,  $V \in L^\infty(\mathbb{R}^d)$

$$Hf = -\Delta f + Vf \in L^2(\mathbb{R}^d)$$

$$\begin{aligned} f, g \in \text{Dom}H : \langle Hf, g \rangle &= \int (-\Delta f)\bar{g} dx + \int Vf\bar{g} dx \\ &= \int f - \overline{\Delta g} dx + \int fV\bar{g} = \langle f, Hg \rangle \end{aligned}$$

### Definition.

(1) a densely defined operator  $A$  on  $\mathcal{H}$  Hilbert is symmetric if

$$\langle Af, g \rangle = \langle f, Ag \rangle \quad \forall f, g \in \text{Dom}(A).$$

(2)  $A^*$  is defined as follows:

$$\begin{aligned}\text{Dom}(A^*) &= \{f \in \mathcal{H} \cdot \exists h : \langle A\phi, f \rangle \\ &= \langle \phi, h \rangle \forall \phi \in \text{Dom} A\}\end{aligned}$$

Set  $A^*f = h$ .

(3)  $A$  is closed provided the graph of  $A$  is closed as a subset of  $\mathcal{H} \times \mathcal{H}$  i.e., if

$$\left. \begin{array}{l} f_n \rightarrow f \\ Af_n \rightarrow g \end{array} \right\} \Rightarrow f \in \text{Dom} A \text{ and } Af = g$$

**Lemma.**

- (1)  $A$  symmetric  $\Rightarrow \text{Dom} A^* \supset \text{Dom} A$  ( $\Rightarrow A^*$  densely defined) and all egv of  $A$  are real
- (2) for any  $A$ ,  $A^*$  is closed
- (3)  $(\ker A)^\perp = \overline{\text{Ran} A^*}$  and  $(\ker A^*)^\perp = \overline{\text{Ran} A}$
- (4) If  $A = A^*$ , then  $\text{Spec}(A) := \mathbb{C} \setminus \{z \in \mathbb{C} : (A - z)^{-1} \text{ exists as a bdd operator}\} \subset \mathbb{R}$  and  $\|(A - z)^{-1}\| \leq |\text{Im } z|^{-1}$

*Proof.*

(1) Because if in  $\text{Dom} A$  one can integrate by parts.

(2)  $f_n \in \text{Dom} A^*$ ,  $f_n \rightarrow f$   
 $A^*f_n \rightarrow g$ .

$$\begin{array}{ccc} \langle A\phi, f_n \rangle & = & \langle \phi, A^*f_n \rangle \quad \forall \phi \in \text{Dom} A \\ \downarrow & & \downarrow \\ \langle A\phi, f \rangle & = & \langle \phi, g \rangle \end{array}$$

so  $f \in \text{Dom} A^*$  and  $g = A^*f$ .

(3)

$$\begin{array}{ccc} \langle A\phi, f \rangle = 0 & \forall \phi \in \text{Dom} A \\ \parallel & \\ \langle \phi, 0 \rangle & \Rightarrow f \in \text{Dom} A^* \text{ and } A^*f = 0 \end{array}$$

(4)  $A = A^*$

$$\begin{aligned} & \langle (A - (x + iy))f, f \rangle \\ &= \left| \underbrace{\langle (A - x)f, f \rangle}_{\text{real}} - iy \underbrace{\langle f, f \rangle}_{\text{real}} \right| \geq |y| \cdot \|f\|_2^2 \\ & \leq \|f\| \cdot \|(A - z)f\| \\ & \Rightarrow \|(A - z)f\| \geq |y| \|f\| \quad \ker \text{ is } 0. \end{aligned}$$

$$\begin{aligned} \overline{\text{Ran}(A - z)} &= (\ker(A^* - \bar{z}))^\perp \\ &= (\ker(A - \bar{z}))^\perp = \mathcal{H} \end{aligned}$$

but still have to show  $\text{Ran}(A - z)$  closed

$$(A - z)f_n \rightarrow g$$

$$\begin{aligned} \|(A - z)f_n - (A - z)f_m\| &\geq |y| \|f_n - f_m\| \\ f_n, f_m \text{ Cauchy} &\Rightarrow f_n \rightarrow f. \end{aligned}$$

and  $A = A^*$  is closed  
 $\Rightarrow A - z$  closed  
 $\Rightarrow f \in \text{Dom } A$  and  $Af = g$   
 $\Rightarrow \text{Ran } (A - z)$  closed.

(one-to-one, onto, bound OK). □

**Example**

$$A = -\frac{d^2}{dx^2} \quad \text{on} \quad L^2[0, 1].$$

$$\text{Dom } A = \left\{ f \in W^{2,2}([0, 1]) : f(0) = f(1) = 0 \right. \\ \left. f'(0) = f'(1) = 0 \right\}.$$

symmetric, not self-adjoint.

$$\text{Dom } A^* = W^{2,2}$$

**Spectral Theorem** Let  $A$  be self-adjoint on  $\mathcal{H}$ . Then  $\exists$  resolution of the identity:

$$E : \quad B(\mathbb{R}) \quad \rightarrow \quad B(\mathcal{H}, \mathcal{H}) \\ \quad \downarrow \quad \quad \quad \downarrow \\ \quad \text{Borel sets} \quad \quad \text{bdd ops}$$

- (i)  $E(I)$  is an orthogonal projection for each  $I \subset \mathbb{R}$  interval
- (ii)  $E(\mathbb{R}) = Id_{\mathcal{H}}$ ,  $E(\emptyset) = 0$
- (iii)  $E(I)E(J) = E(I \cap J) \forall I, J$
- (iv)  $E(\cup I_j) = \sum E(I_j)$

$$A = \int \lambda E(d\lambda)$$

in the following sense

$$\text{Dom } A = \left\{ f \in \mathcal{H} : \int \lambda^2 \langle E(d\lambda)f, f \rangle < \infty \right\}$$

$$\text{and} \quad \langle Af, g \rangle = \int \lambda \langle E(d\lambda)f, g \rangle$$

Def. if  $f \in C_b^\circ(\mathbb{R})$ , define  $f(A) = \int f(\lambda) E(d\lambda)$

**Example 1.**

- (1)  $\mathcal{H} = \mathbb{R}^n$ ,  $A$  hermitian matrix

$$E(I) = \sum_{j: \lambda_j \in I} P_{L_j}$$

where  $\lambda_1 \leq \dots \leq \lambda_k$  are e.v. without multiplicity and eigenspaces  $L_1, \dots, L_k$

$$E(d\lambda) = \sum_j \delta_{\lambda_j}(d\lambda) P_{L_j}$$

$$(2) e^{itA} = \int e^{it\lambda} E(d\lambda)$$

$$\begin{aligned} \| e^{itA} f \|^2 &= \int e^{it\lambda} \langle E(d\lambda) f, \int e^{it\mu} E(d\mu) f \rangle \\ &= \iint e^{it(\lambda-\mu)} \langle \underbrace{E(d\mu) E(d\lambda)}_{\mu \text{ must be } = \lambda} f, f \rangle . \end{aligned}$$

$$\delta_{\lambda=\mu}$$

$$= \int \langle E(d\lambda) f, f \rangle = \langle f, f \rangle = \| f \|^2 \rightarrow \text{isometry.}$$

$$e^{itA} e^{isA} f = e^{i(t+s)A} f$$

$$e^{-itA} e^{itA} = Id_{\mathcal{H}}$$

$$e^{itA} \text{ is unitary, and } (e^{itA})^* = (e^{itA})^{-1} = e^{-itA}$$

$$(3) \mathcal{H} = L^2(X, \mu), \mu \text{ positive probability measure.}$$

$$A = \mathcal{M}_\phi : Af = \phi f \text{ where } \phi : X \rightarrow \mathbb{R} .$$

$$\text{Dom } A = \{f \in L^2 = \phi f \in L^2\} \text{ dense in } \mathcal{H}$$

because  $\phi$  locally finite

$$\mu(|\phi| > M) \rightarrow 0 \text{ as } M \rightarrow \infty .$$

$$A = A^* , \quad \phi \text{ is real.}$$

$$A \subset A^* \text{ OK.}$$

$$\langle Af, g \rangle = \langle f, h \rangle$$

||

$$\int \phi f \bar{g} \mu(dx) = \int f(x) \overline{\phi g}(x) \mu(dx)$$

$$= \int f \bar{h}(x) \mu(dx)$$

$$\phi g = h \in L^2 \quad \text{so } g \in \text{Dom } A .$$

$$A^* \subset A \quad \text{OK.}$$

$$E(I)f = \chi_{[\phi \in I]} f$$

$$(4) A = -\Delta \text{ on } L^2(\mathbb{R}^d) \\ \text{Dom } A = W^{2,2}(\mathbb{R}^d).$$

$$\langle -\Delta f, g \rangle = \langle f, h, \rangle \quad \text{no } \int \text{ by parts.}$$

$$\mu = -\Delta g \in \mathcal{S}' | \langle \mu, f \rangle | \leq C. \| f \|_2 \text{ with } C = \| h \|_2$$

Hahn-Banach,  $\mu$  extends to a co. fnl on  $L^2$

$$\begin{aligned} & \| \mu \| \leq \| n \|_2 \\ & \Delta g \in L^2, \quad g \in W^{2,2} \\ \Rightarrow & \left. \begin{array}{l} \text{Dom } \Delta^* \subset \text{Dom } \Delta \\ + \text{ symmetric} \end{array} \right\} \Rightarrow \text{self-adj.} \end{aligned}$$

$$\begin{aligned} & \widehat{A}f = \xi^2 \widehat{f}(\xi) \\ FT(A)(FT)^{-1} &= \mathcal{M}_{\xi^2}. \quad (\Delta \text{ unit eq. to mult. } \xi^2) \\ FT : L_x^2 &\rightarrow L_\xi^2 \text{ unitary} \\ FTE_{-\Delta}(FT)^{-1} &= E_{\xi^2} \\ E_{-\Delta}(I)f &= (\chi_{[\xi^2 \in I]} \widehat{f})^\vee \\ \Rightarrow \underbrace{\widehat{e^{itA}f}}_{\substack{\text{defined} \\ \text{via spec. thm.}}} &= e^{it\xi^2} \widehat{f}(\xi) \end{aligned}$$

- (5)  $H = -\Delta + V, V \in L^\infty$ .  
**Claim:**  $H = H^*$  on  $W^{2,2}(\mathbb{R}^d)$ .  
 $H$  symmetric  $H \subset H^*$

$$\begin{aligned} \langle -\Delta f + Vf, g \rangle &= \langle f, h \rangle \\ |\langle -\Delta f, g \rangle| &\leq (\| h_2 \| + \| V_g \|_2) \| f \|_2 \\ &\leq \| V \|_\infty \| g \|_2 \end{aligned}$$

Hahn-Banach.

( $\Delta g \in \mathcal{S}'$  satisfies an  $L^2$  bound  $\Rightarrow \Delta g \in L^2$ ).

**Theorem.** If  $V \in L^\infty$  then

$$\begin{aligned} \sup_+ \| \nabla e^{itH} f \|_2 &\leq C \| f \|_{H^1} \\ \sup_+ \| \nabla^2 e^{itH} f \|_2 &\leq C \| f \|_{H^2} \end{aligned}$$

*Proof.*

$$\begin{aligned} H + C &\geq Id. \\ \langle (H + C)f, f \rangle &\geq \| f \|_2^2 \quad (\text{int. by parts, positive}) \\ C &= 1 + \| V \|_\infty \\ (H + C)^{-1} : L^2 &\rightarrow W^{2,2} \\ \| (H + C)^{-1} f \|_{W^{2,2}} &\leq C \| f \|_2 \\ \| f \|_{W^{2,2}} &\lesssim \| (H + C)f \|_2 \\ \| e^{itH} f \|_{W^{2,2}} &\lesssim \| (H + C)e^{itH} f \|_{L^2} \end{aligned}$$

$$= \| e^{itH}(H + C)f \|_{L^2} \lesssim \| f \|_{W^{2,2}} .$$

□

Calculation of  $(-\Delta - (\lambda + i\varepsilon))^{-1}$  in  $d = 1, d = 3$ .

$$\begin{aligned} (-\Delta - \widehat{(\lambda + i\varepsilon)})^{-1} f &= \frac{\hat{f}(\xi)}{\xi^2 - (\lambda + i\varepsilon)} \\ (R_0(\lambda + i\varepsilon)f)(x) &= \frac{1}{(2\pi)^d} \int \frac{e^{ix\xi} \hat{f}(\xi)}{\xi^2 - (\lambda + i\varepsilon)} d\xi . \\ d = 3. \quad R_0(\lambda + i\varepsilon)(x) &= \frac{1}{(2\pi)^3} \int \frac{e^{ix\xi}}{\xi^2 - (\lambda + i\varepsilon)} d\xi . \\ &= \frac{1}{2\pi^2} \int_0^\infty \int_{S^2} e^{irx \cdot \omega} d\sigma_{S^2}(\omega) \frac{r^2}{r^2 - (\lambda + i\varepsilon)} dr \end{aligned}$$

$$\begin{aligned} \int_{S^2} e^{ia\omega \cdot \vec{e}_3} d\sigma_{S^2}(\omega) &= 2\pi \int_0^\pi e^{ia\omega \sin\theta} \sin\theta d\theta \\ &= 2\pi \int_{-1}^1 e^{iau} du = 4\pi \frac{\sin a}{a} \\ &= \frac{1}{2\pi^2} \int_0^\infty \frac{\sin(r|x|)}{r|x|} \frac{r^2}{r^2 - (\lambda + i\varepsilon)} dr \\ &= \frac{1}{2\pi|x|} \int_0^\infty \frac{\sin(r|x|)}{r^2 - (\lambda + i\varepsilon)} r dr \\ &= \frac{1}{4\pi|x|} \int_{-\infty}^\infty \frac{\sin(r|x|)}{r^2 - (\lambda + i\varepsilon)} r dr \\ &= \frac{1}{16\pi|x|} \int_{-\infty}^\infty (e^{ir|x|} - e^{-ir|x|}) \left( \frac{1}{r - \sqrt{\lambda + i\varepsilon}} - \frac{1}{r + \sqrt{\lambda + i\varepsilon}} \right) dr \\ &= \frac{1}{4\pi|x|} e^{i|x|\sqrt{\lambda + i\varepsilon}} \\ R_0(\lambda + i0)(x) &= \lim_{\varepsilon \rightarrow 0} R_0(\lambda + i\varepsilon)(x) \\ &= \frac{e^{i|x|\sqrt{\lambda}}}{4\pi|x|} , \quad -\infty < \lambda < \infty , \\ &\quad \sqrt{\cdot} \text{ is principal branch.} \end{aligned}$$

$$\mathbf{d} = \mathbf{1} . \quad R_0(\lambda + i\varepsilon)(x, y) = \frac{f_1(\lambda + i\varepsilon, x)f_2(\lambda + i\varepsilon, y)}{W[f_1(\lambda + i\varepsilon)f_2(\lambda + i\varepsilon)]} \quad \text{if } x < y$$

otherwise reverse.

Green's fn  
 $G_{\lambda+iz}(x, y)$

**Lemma.**

$$\left( -\frac{d^2}{dx^2} - (\lambda + i\varepsilon) \right) G_{\lambda+i\varepsilon} = Id .$$

$$\left. \begin{aligned} f_1(\lambda + i\varepsilon, x) &= e^{-i\sqrt{\lambda+i\varepsilon} x} \\ f_2(\lambda + i\varepsilon, y) &= e^{+i\sqrt{\lambda+i\varepsilon} y} \end{aligned} \right\} \text{exp decaying!}$$

$$\underbrace{W[f_1, f_2]}_{\text{const. in } x} = \left| \frac{f_1 f_1'}{f_2 f_2'} \right| = -2i\sqrt{\lambda + i\varepsilon}$$

$$\Rightarrow R_0(\lambda + i\varepsilon) = \frac{-e^{+i\sqrt{\lambda+i\varepsilon}|x-y|}}{2i\sqrt{\lambda + i\varepsilon}} \quad \text{and lim as } \varepsilon \rightarrow 0$$

Finer properties of the spectral measure.

**Lemma.**  $A$  is selfadjoint on  $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{pp}$$

$$\mathcal{H}_{\substack{ac \\ sc \\ pp}} = \{f \in \mathcal{H} : \langle E(\cdot)f, f \rangle \text{ is } \begin{array}{l} ac \Rightarrow \text{absolutely continuous} \\ sc \Rightarrow \text{singular continuous} \\ pp \Rightarrow \text{pure point (atomic)} \end{array}$$

*Proof.*

(i) If  $f, g \in \mathcal{H}_{ac}$ ,  $|S| = 0$ .

$$\begin{aligned} \langle E(S)(f+g), f+g \rangle &= \left\{ \|E(S)f\|^2 + \|E(S)g\|^2 + 2\langle E(S)f, g \rangle \right\} = 0 \\ &= \left\{ \|E(S)f\|^2 + \|E(S)g\|^2 \right\} = 0 \\ &+ \langle E(S)f, g \rangle + \langle E(S)g, f \rangle = 0 \\ &= 0 \end{aligned}$$

because  $f, g$  are in a.c.

$$\begin{aligned} E(S)f_n &= 0, \quad f_n \rightarrow f, \\ \downarrow \\ E(S)f &\Rightarrow \mathcal{H}_{ac} \text{ is a closed subspace.} \end{aligned}$$

(ii)  $f, g \in \mathcal{H}_{sc}$

$$\begin{aligned} &S_f \text{ support } f; \\ &S_f, S_g \text{ s.t. } |S_f| = 0, |S_g| = 0 \\ &E(S_f^c)f = 0, \end{aligned}$$

$$E(S_g^c)f = 0 .$$

$$E(\{\lambda\})f = 0$$

$$E(\{\lambda\})g = 0$$

$$S_{f+g} = S_f \cup S_g$$

$$\begin{aligned} E(S_{f+g}^c)(f+g) &= E(S_f^c) \overbrace{E(S_g^c)g}^0 \\ &\quad + E(S_g^c) \underbrace{E(S_f^c)f}_0 = 0 . \end{aligned}$$

$$E(\{\lambda\})(f+g) = 0 .$$

$$f_n \in \mathcal{H}_{sc} \rightarrow f \quad \cup_n S f_n =: S_f$$

$$E(S_f^c)f_n = 0 \rightarrow E(S_f^c)f = 0$$

closed subspace, too.

(iii) exercise

□

$$\begin{aligned} \langle E(A)f, f \rangle &= \langle E(S_{ac} \cap A)f, f \rangle \\ &\quad + \langle E(S_{sc} \cap A)f, f \rangle \\ &\quad + \langle E(S_{pp} \cap A)f, f \rangle \end{aligned}$$

$$S_{ac} \cap S_{sc} = \emptyset \text{ etc.}$$

$$\begin{aligned} (ac) \perp (sc \text{ and } pp) : \quad f \in \mathcal{H}_{ac} , g \in \mathcal{H}_{\text{sing}} \\ g = E(S_g)g \end{aligned}$$

$$\langle f, g \rangle = \langle \underbrace{E(S_g)}_{=0} f, g \rangle = 0$$

Ma 142 01/21

**Definition**  $E_{ac} = E \circ P_{ac}$

**Ex.**

- (1)  $\mathcal{M}_\phi = f = \phi f$ .  
 $\phi : \mathbb{R} \rightarrow \mathbb{R}$ .  
 $E_{pp}$  : plateaux of  $\phi$   
 $(E(I)f)(x) = \chi_{[\phi \in I]} f(x)$ .

- (2)  $-\Delta = FT\mathcal{M}_{\xi^2}FT^{-1}$   
 $\phi(\xi) = \xi^2$   
 purely a.c. because

$$\langle E(I \setminus \overset{\text{Borel}}{[\xi^2 \in I]}) f, f \rangle = \int_{[\xi^2 \in I]} |\hat{f}(\xi)|^2 d\xi = 0 \quad \text{if } |I| = 0.$$

- (3)  $H = -\Delta - \lambda V$ ,  $V \in L^\infty_{\text{comp}}$ ,  $V \geq 0$ ,  $V \neq 0$ .

$$f \in \mathcal{S}, \langle Hf, f \rangle = \int |\nabla f|^2 - \lambda \underbrace{\int V|f|^2 dx}_{\neq 0} < 0$$

for some  $f \in \mathcal{S}$  and  $\lambda$  large.  
 $\Rightarrow \text{spec}(H) \cap (-\infty, 0) \neq \emptyset$ .

**Fact:**  $\text{ess spec}(H) = (0, \infty)$

$:= sp(H) \setminus \{z \in \mathbb{C} | z \text{ is an isolated egv of finite multiplicity}\}$ .

- (4)  $e^{itH}f = e^{itE}f$  if  $Hf = Ef$ , no time decay.

**Theorem.** ( $d = 1$ ). Take  $H = -\frac{d^2}{dx^2} + V$   
 assume  $\int (1 + |x|)^3 |V(x)| dx < \infty$  and  $V$  generic. Then

$$\| e^{itH} P_{ac}(H)f \|_\infty \leq C \cdot |t|^{-1/2} \| f \|_1$$

(contrast with bound states, here  $P_{ac} = 1 - P_{\text{bound states}}$ )

**Lemma.** If  $A$  is self-adjoint on  $\mathcal{H}$ , then for almost every  $\lambda$

$$\langle E_{ac}(d\lambda)f, g \rangle = \left\langle \frac{1}{2i\pi} (R(\lambda + i0) - R(\lambda - i0))f, g \right\rangle d\lambda$$

*Proof.* Absolutely c. measure in  $d\lambda$ , identify the densities = a.e.

$$\begin{aligned} & \left\langle \frac{1}{2i\pi} \left[ (A - (\lambda + i\varepsilon))^{-1} - (A - (\lambda - i\varepsilon))^{-1} \right] f, g \right\rangle \\ &= \int \frac{1}{2i\pi} \left( \frac{1}{t - (\lambda + i\varepsilon)} - \frac{1}{t - (\lambda - i\varepsilon)} \right) \langle E(dt)f, g \rangle \\ &= \int \frac{1}{\pi} \frac{\varepsilon}{(t - \lambda)^2 + \varepsilon^2} \langle E(dt)f, g \rangle \\ & \text{for a.e. } \lambda \quad \frac{\langle E_{ac}(d\lambda)f, g \rangle}{d\lambda} \\ & \text{as } \varepsilon \xrightarrow{+} 0_+ \end{aligned}$$

□

**Lemma.** High-energy part. Let  $\chi \in C_0^\infty(\mathbb{R}_+)$ ,

$$\chi(x) = 1 \text{ if } \lambda \geq 2\lambda_0. \quad V \in L^1$$

and  $\chi(\lambda) = 0$  if  $\lambda < \lambda_0$  (for some large but fixed  $\lambda_0$ ). (Then same estimate, but)

$$\| e^{itH} \chi(H) P_{ac}(H)f \|_\infty \leq C \cdot |t|^{-1/2} \| f \|_1$$

*Proof.*  $\langle e^{itH} \chi(H) P_{ac}(H) f, g \rangle =$

$$= \int_0^\infty e^{it\lambda} \chi(\lambda) \frac{1}{2\pi i} \left\langle (R_V(\lambda + i0) - R_V(\lambda - i0)) f, g \right\rangle d\lambda$$

$$R_V(\lambda + i0) = \lim_{\varepsilon \rightarrow 0^+} \left( -\frac{d^2}{dx^2} + V - (\lambda + i\varepsilon) \right)^{-1} \quad \text{not } L^2 \text{ op}$$

$$= \lim_{\varepsilon \rightarrow 0^+} (H_0 - (\lambda + i\varepsilon) + V)^{-1}$$

$$= R_0(\lambda + i0) - R_0(\lambda + i0) V R_0(\lambda + i0) + \dots$$

$$= \sum_{n=0}^\infty R_0(\lambda + i0) (-V R_0(\lambda + i0))^n \Rightarrow \text{Born series}$$

Neumann series for  $(H_0 - \lambda)^{-1} (1 + V(H_0 - \lambda)^{-1})^{-1}$ .

$$\left. \begin{aligned} R_0(\lambda + i0)(x) &= -\frac{1}{2i\sqrt{\lambda}} e^{i\sqrt{\lambda}|x|} \\ R_0(\lambda - i0)(x) &= \frac{1}{2i\sqrt{\lambda}} e^{-i\sqrt{\lambda}|x|} \end{aligned} \right\} \text{NB: positive and expo decaying}$$

Convergence: pick  $\lambda$  large enough.

**Claim**

$$\langle R_V(\lambda + i0) f, g \rangle = \sum_{n=0}^\infty (-1)^n \langle (R_0(\lambda + i0) (V R_0(\lambda + i0))^n f, g \rangle$$

$$\forall \lambda > \lambda_0, f, g \in \mathcal{S}$$

*Proof.*

$$\left| \langle R_0(\lambda + i\varepsilon) (V R_0(\lambda + i\varepsilon))^n f, g \rangle \right|$$

$$\leq \int \dots \int \frac{1}{|2\sqrt{\lambda + i\varepsilon}|^{n+1}} |V(x_1) \dots V(x_n)| dx_1 \dots dx_n$$

$$|f(x_0)| |g(x_{n+1})| dx_0 dx_{n+1}$$

$$\leq \left( \frac{1}{2\sqrt{\lambda}} \right)^{n+1} \|V\|_1^n \|f\|_1 \|g\|_1 \quad \text{uniform in } \varepsilon$$

uniform in  $\varepsilon$ . Conv. of the series: take  $2\sqrt{\lambda} > \|V\|_1$  so  $\lambda_0 = \|V\|_1^2$ .  $\square$

$$\therefore \left| \langle e^{itH} \chi(H) P_{ac}(H) f, g \rangle \right| \leq \sum_{n=0}^\infty \left| \frac{(-1)^n}{2\pi i} \int_0^\infty e^{it\lambda} \chi(\lambda) \dots \right.$$

$$\left. \dots \left( \langle R_0(\lambda + i0) (V R_0(\lambda + i0))^n f, g \rangle - \langle R_0(\lambda + i0) (V R_0(\lambda + i0))^n f, g \rangle \right) d\lambda \right|$$

$$\begin{aligned}
& \text{The general term is } \int_0^\infty e^{it\lambda} \chi(\lambda) \langle R_0(\lambda + i0) (V R_0(\lambda + i0))^n f, g \rangle d\lambda \\
& = \iint f(x_0) g(x_{n+1}) \int \cdots \int V(x_1) \cdots V(x_n) \int_0^\infty e^{it\lambda^2} 2\lambda \chi(\lambda) \cdot \\
& \quad \frac{e^{i\lambda(\sum_{j=1}^{n+1} |x_j - x_{j-1}|)}}{(-2i\lambda)^{n+1}} d\lambda \\
& \quad \int_0^\infty e^{it\lambda} \chi(\lambda) \langle R_0(\lambda + i0) (V R_0(\lambda + i0))^n f, g \rangle d\lambda \\
& = \iint f(x_0) g(x_{n+1}) \int \cdots \int V(x_1) \cdots V(x_n) \int_0^\infty e^{it\lambda^2} 2\lambda \chi(\lambda^2) \cdot \frac{e^{-i\lambda \sum_{j=1}^{n+1} |x_j - x_{j-1}|}}{(2i\lambda)^{n+1}} d\lambda
\end{aligned}$$

then  $\lambda \rightarrow -\lambda$  and  $\int_0^\infty - \int_0^\infty$   
 $\rightarrow \int_0^\infty + \int_{-\infty}^0$

conclusion  $|\langle - \rangle| \leq$

$$\begin{aligned}
& \sum_{n=0}^\infty \frac{1}{2\pi} \underbrace{\int dx_0 \int_{\mathbb{R}^2} dx_{n+1} |f(x_0)| |g(x_{n+1})|}_{\mathbb{R}^2} \\
& \int \cdots \int dx_1 \cdots dx_n |V(x_1) \cdots V(x_n)| \cdot \\
& \underbrace{\sup_{x_0, \dots, x_{n+1}} \left| \int_{-\infty}^\infty e^{it\lambda^2} \lambda \chi(\lambda^2) \frac{e^{i\lambda \sum_{j=1}^{n+1} |x_j - x_{j-1}|}}{(2i\lambda)^{n+1}} d\lambda \right|}_{\text{claim: } \leq \frac{C \cdot |t|^{-1/2}}{(2\lambda_0)^n} \text{ (then done)}}
\end{aligned}$$

Expression between  $|\cdot|$ .

$$\int_{-\infty}^\infty e^{i(t\lambda^2 + a\lambda)} \hat{\mu}_n(\lambda) d\lambda \quad (\text{stationary phase}).$$

$$|\quad - \quad| \leq C \cdot |t|^{-1/2} \|\mu_n\|_1 \quad (\text{disp.est.})$$

$$\text{where } \mu_n(x) = \frac{\left(\frac{i}{2}\right)^{n+1}}{2\pi} \int_{-\infty}^\infty \frac{\chi(\lambda^2)}{\lambda^n} e^{-i\lambda x} d\lambda$$

$$\|\mu_n\|_{\text{meas}} \leq C \left(\frac{1}{2\lambda_0}\right)^n$$

□

**Lemma.** (low energy) Assume  $V$  as in theorem. Let  $\chi(\lambda)$  be a smooth cutoff around 0. Then

$$\| e^{itH} \chi(H) P_{ac} f \|_\infty \leq C |t|^{-1/2} \| f \|_1$$

**First steps of the proof**

$$\langle e^{itH} \chi(H) P_{ac} f, g \rangle = \frac{1}{2\pi i} \int_0^\infty e^{it\lambda^2} 2\lambda \chi(\lambda^2)$$

$$\left\langle (R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)) f, g \right\rangle d\lambda$$

$$R_V(\lambda^2 + i0)(x, y) = \frac{f_+(\lambda, y) f_-(\lambda, x)}{W(f_+(\lambda, \cdot), f_-(\lambda, \cdot))} \chi_{\{x < y\}} + \frac{f_+(\lambda, x) f_-(\lambda, y)}{W(f_+(\lambda, \cdot), f_-(\lambda, \cdot))} \chi_{\{y < x\}}$$

( $V_{\text{generic}} \Rightarrow$  denom @ 0 is nonzero) where  $f_\pm(\lambda, \cdot)$  are the Jost solutions, i.e.,

$$\begin{cases} -f_\pm'' + V f_\pm = \lambda^2 f_\pm \\ |f_+(\lambda, x) - e^{i\lambda x}| \rightarrow 0 & \text{as } x \rightarrow \infty \\ |f_-(\lambda, x) - e^{-i\lambda x}| \rightarrow 0 & \text{as } x \rightarrow -\infty \end{cases}$$

$$R_V(\lambda^2 - i0)(x, y) = \frac{f_+(-\lambda, y) f_-(-\lambda, x)}{W[f_+(-\lambda, \cdot), f_-(-\lambda, \cdot)]} \quad y > x$$

Conclusion:

$$\begin{aligned} & \langle e^{itH} \chi(H) P_{ac}(H) f, g \rangle \\ &= \frac{1}{\pi i} \int_{-\infty}^\infty \int e^{it\lambda^2} \lambda \chi(\lambda^2) \frac{f_+(\lambda, y) f_-(\lambda, x)}{W(\lambda)} d\lambda f(x) g(y) \quad \chi_{[x < y]} dx dy \\ &+ \frac{1}{\pi i} \int_{-\infty}^\infty \int e^{it\lambda^2} \lambda \chi(\lambda^2) \frac{f_+(\lambda, x) f_-(\lambda, y)}{W(\lambda)} d\lambda f(x) g(y) \quad \chi_{[x > y]} dx dy \\ &+ \text{terms for } R_v(\lambda^2 - i0) \end{aligned}$$

then show  $\int d\lambda \leq C |t|^{-1/2}$  independent of  $x, y$ .

$\rightarrow$  show  $L^1$  bddness of  $\frac{f_+ f_-}{W}$ .

**Existence of  $f_\pm$**  (var. of const. = Duhamel)

$$-f_+'' + V f_+ = z^2 f_+, \quad \text{Im} z \geq 0$$

$$f_+(z, x) = e^{izx} + \int_x^\infty G_z(x, y) V(y) f_+(z, y) dy .$$

solves free equation, with inhom.  $\delta(x - y)$ .

$$\begin{cases} G_z(x, x) = 0 . \\ \partial_x G_z(x, x) = -1 \\ \left(-\frac{d^2}{dx^2} - z^2\right) G_z(x, y) = 0 \quad \text{if } x < y . \end{cases}$$

$$\begin{aligned} \Rightarrow G_z(x, y) &= \frac{\sin(z(x-y))}{z} \text{ if } x < y \\ \text{Solve } f_+(z, x) &= e^{izx} + \int_x^\infty \frac{\sin(z(y-x))}{z} V(y) f_+(z, y) dy \\ m_+(z, x) &= e^{-izx} f_+(z, x) \\ \textcircled{*} \quad m_+(z, x) &= 1 + \int_x^\infty \frac{e^{2iz(y-x)} - 1}{2iz} V(y) m_+(z, y) dy . \end{aligned}$$

**Lemma.** Suppose  $V \in L^1(\mathbb{R})$ . Then for every  $\Im az \geq 0$ ,  $z \neq 0$ , there exists one and only one  $m_+(z, \cdot)$  that satisfies  $\textcircled{*}$ . Moreover,  $|m_+(z, x) - 1| \rightarrow 0$  as  $x \rightarrow \infty$ .

*Proof.*

$$(Am)(x) = 1 + \int_x^\infty \frac{e^{2iz(y-x)} - 1}{2iz} V(y) m(y) dy$$

$$A : X \circlearrowleft \quad X = \{f \in C([x, \infty)) : \|f\|_{L^\infty([x, \infty))} < \infty\} .$$

$$\begin{aligned} \|Am\|_X &\leq 1 + \frac{1}{|z|} \int_{x_1}^\infty |V(y)| dy \|m\|_X \\ &\leq 1 + \underbrace{\frac{\|V\|_{L^1(x_1, \infty)}}{|z|}}_{< \frac{1}{2} \text{ if } x_1 \text{ large enough}} \|m\|_X =: R . \end{aligned}$$

$$A : B(0, R) \rightarrow B(0, R) .$$

$$\|Am - A\tilde{m}\|_X \leq \underbrace{\frac{\|V\|_{L^1(x_1, \infty)}}{|z|}}_{\text{contraction}} \|m - \tilde{m}\|_X$$

Therefore there exists one and only one  $m$  in  $B_R(X)$  that satisfies  $\textcircled{*}$  for  $x \geq x_1$ .

To extend to  $x < x_1$ , we can write

$$\begin{aligned} (A_2\phi)(x) &= 1 + \int_{x_1}^\infty \frac{e^{2iz(y-x)}}{2iz} m_+(z, y) V(y) dy \\ &\quad + \int_x^{x_1} \frac{e^{2iz(y-z)}}{2iz} \phi(y) V(y) dy . \end{aligned}$$

on some  $[x_2, x_1]$  this is a contraction:

Choose  $x_2$  s.t  $\frac{1}{|z|} \int_{x_2}^{x_1} |V(t)| dt < 1/2$ . ( $L^1 \Rightarrow$  sufficiently small on uniform intervals.)  
Extend to 0.  $\square$

## Ma 142 01/23

Low energy lemma

$$\begin{aligned} & \langle e^{itH} P_{ac}(H) \chi_{\text{small}}(H) f, g \rangle \\ &= \int_0^\infty d\lambda \cdot 2\lambda e^{it\lambda^2} \chi_{\text{small}}(\lambda^2) \\ & \quad \left\langle \frac{1}{2i\pi} (R_V(\lambda^2 + i0) - R_V(\lambda^2 - i0)) f, g \right\rangle . \\ R_V(\lambda^2 + i0) &= \chi_{[y \leq x]} \frac{f_+(\lambda, x) f_-(\lambda, y)}{W[f_+(\lambda), f_-(\lambda)]} + \chi_{[y > x]} \cdot \frac{f_+(\lambda, y) f_-(\lambda, x)}{W[f_+(\lambda), f_-(\lambda)]} . \end{aligned}$$

**Lemma.** (1) Let  $\text{Im } z > 0$ ,  $z \neq 0$ ,  $V \in L^1$ , then  $\exists! m_+(z, \cdot)$  s.t.

$$m_+(z, \cdot) = 1 + \int_x^\infty \frac{e^{2iz(y-x)} - 1}{2iz} V(y) m_+(z, y) dy .$$

(2) If  $\Im z \geq 0$ ,  $V \in L^1_1 = \{f \in L^1(\mathbb{R}) : \int (1 + |x|)|V(x)| dx < \infty\}$  then  $\exists! m_+(z, x)$  s.t.

$$\textcircled{*} \quad m_+(z, x) = 1 + \int_x^\infty D_z(y-x) V(y) m_+(z, y) dy$$

$$\text{where } D_z(y-x) = \int_0^{y-x} e^{2izt} dt \text{ when } \lambda > 0.$$

**Corollary.** Set  $f_+(z, x) = e^{izx} m_+(z, x)$

$\text{Im } z \geq 0$ , then  $f_+(z, \cdot)$  satisfies

$$\textcircled{*} \textcircled{*} \quad f_+(z, x) = e^{izx} + \int_x^\infty \frac{\sin(z(y-x))}{z} V(y) f_+(z, y) dy$$

$$|f_+(z, x) - e^{izx}| \rightarrow 0 \text{ as } x \rightarrow \infty .$$

$$|f'_+(z, x) - iz e^{izx}| \rightarrow 0 \text{ —————}$$

$$f_+(z, \cdot) \in \left\{ f \in C^1(\mathbb{R}) : f' \in AC_{\text{loc}} , \right. \\ \left. f'' \in L^1_{\text{loc}} , \right\}$$

$$\text{and } -f''_t(z, x) + V(x) f_+(z, x)$$

$$= z^2 f_+(z, x) \text{ a.e., } -x .$$

*Proof.*

(1) ok

$$\begin{aligned}
|f_+(z, x) - e^{izx}| &\leq \int_x^\infty \frac{2e^{\operatorname{Im} z(y-x)}}{|-z|} |V(y)| |f_+(z, y)| dy \\
&\leq \int_x^\infty \frac{e^{\operatorname{Im} z(y-x)}}{|z|} |V(y)| \cdot e^{-(\operatorname{Im} z)y} |m_+(z, y)| dy \\
&\leq \frac{e^{-x \operatorname{Im} z}}{|z|} \underbrace{\int_x^\infty |V(y)| dy}_{\substack{\rightarrow 0 \\ \text{as } x \rightarrow \infty}} \underbrace{\sup_{y \geq x} |m_+(z, y)|}_{\text{uniform boundedness}} \\
&\rightarrow 0.
\end{aligned}$$

$$\begin{aligned}
\textcircled{*} m_+(z, x) &= 1 + \int_x^\infty D_z(y-x) V(y) m_+(z, y) dy \\
&= 1 + \int_x^\infty D_z(y_1-x) V(y) dy + \int_x^\infty \int_{y_1}^\infty D_z(y-x) D_z(y_2-y_1) \\
&\quad V(y_1) V(y_2) dy_1 dy_2
\end{aligned}$$

+ ...

$$+ \int_x^\infty \int_{y_1}^\infty \cdots \int_{y_{n-1}}^\infty D \cdots D \cdots V \cdots V$$

$$|D_z(y-x)| \leq \frac{1}{|z|}$$

$$m_+(z, x) \leq 1 + \cdots \int_{y_n \geq y_{n-1} \geq \cdots \geq x} \frac{1}{|z|^n} \cdot |V(y_1) \cdots V(y_n)| dy_1 \cdots dy_n$$

$$= 1 + \cdots + \frac{1}{n!} \frac{1}{|z|^n} \left( \int_x^\infty |V(y)| dy \right)^n$$

$$\leq \exp\left(\frac{\|V\|_1}{|z|}\right).$$

Smoothness: Prove  $f'(z, x) = iz e^{izx} - \int_x^\infty \cos(z(y-x)) V(y) f_+(z, y) dy$   
 (...)

$$\int_{x+h} \cos \cdots \quad \text{continuous in } h$$

because int. in  $L^1$

because the exp cancel each other.

$$\begin{aligned} \text{then } & \left| \frac{1}{h} \left( \int_{x+h} \sin - \int_x \sin \right) - \int_x \cos \right| < \\ & \left| \underbrace{\int_x^{x+h} \frac{1}{h}}_{\substack{\text{OK} \\ (1)}} + \underbrace{\int_{x+h}^{\infty} \left( \frac{\sin - \sin}{hz} + \cos \right) V f_+}_{\substack{O(e^{\text{Im } zy}) \\ \text{DCT} \\ \sigma(1)}} \right| \end{aligned}$$

$$\begin{aligned} \text{and } f_+''(z, x) &= -z^2 e^{izx} + V(x) f_+(z, x) \\ &+ z^2 \int_x^{\infty} \frac{\sin(z(y-x))}{z} V(y) f_+(z, y) dy \\ &= -z^2 f_+(z, x) + V f_+(z, x) \quad \text{only a.e.} \end{aligned}$$

□

$$\begin{aligned} \mathbf{Ex. (1)} \quad V = 0 \quad f_+(\lambda, x) &= e^{i\lambda x} \\ f_-(\lambda, x) &= e^{-i\lambda x} \end{aligned}$$

$$\begin{aligned} W(\lambda) = W[f_+(\lambda), f_-(\lambda)] &= \begin{vmatrix} e^{i\lambda x} & e^{-i\lambda x} \\ i\lambda e^{i\lambda x} & -i\lambda e^{-i\lambda x} \end{vmatrix} = -2i\lambda \\ &\text{vanishes at 0. (exceptional case).} \end{aligned}$$

(2)

$$f_+(\lambda, x) = e^{i\lambda x} \quad \text{if } x > 1$$

$$\begin{aligned} 0 < x < 1: \quad -f'' + f &= \lambda^2 f \\ f''(x) &= (1 - \lambda^2) f(x) \\ f(x) &= \alpha(\lambda) e^{\sqrt{1-\lambda^2}x} \\ &+ \beta(\lambda) e^{-\sqrt{1-\lambda^2}x} \end{aligned}$$

matching  $\Rightarrow \alpha, \beta$

$$x < 0: f_+(\lambda, x) = a(\lambda) e^{i\lambda x} + b(\lambda) e^{-i\lambda x}$$

matching  $\Rightarrow a, b$ .

$$\begin{aligned} x < 0 : \quad W(\lambda) &= \begin{vmatrix} a(\lambda)e^{i\lambda x} + 0 & a(\lambda)i\lambda e^{i\lambda x} + 0 \\ e^{-i\lambda x} & -i\lambda e^{-i\lambda x} \end{vmatrix} \\ &= -2i\lambda a(\lambda) \\ W(0) &= \dots = -2(\alpha(0) - \beta(0)) \neq 0. \end{aligned}$$

$$\begin{aligned} \lambda = 0 : \quad m_+(0, x) &= f_+(0, x) \\ &= a + bx \quad b \neq 0. \\ (-f_+'' = 0) \end{aligned}$$

$$W(0) = \begin{vmatrix} f_+ & b \\ 1 & 0 \end{vmatrix} = -b \neq 0.$$

but remains bdd to the right ( $x > 0$ ) so that's why we had to  $y < x, y > x$  before ( $W(0) \neq 0$ ).

**Lemma.**  $\text{Im } z \geq 0$  and  $\int (1 + |x||V(x)|) dx < \infty$ . Then  $|m_+(z, x)| \leq K(1 + \max(0, -x))$ .

*Proof.*  $x > 0$

$$\begin{aligned} m_+(z, x) &= 1 + \dots + \int_{x \leq y_1 \leq \dots \leq y_n} D_z(y_1 - x) \cdots D_z(y_n - y_{n-1}) \\ &\quad V(y_1) \cdots V(y_n) dy_1 \cdots dy_n \end{aligned}$$

$$|m_+(z, x)| \leq 1 + \dots + \iint (y_1 - x) \cdots (y_n - x) |V(y_1) \cdots V(y_n)| dy_1 \cdots dy_n.$$

$$\leq \exp\left(\int_x^\infty (y - x) |V(y)| dy\right) \quad \text{OK when } x > 0.$$

$$\leq \exp\left(\int_0^\infty y |V(y)| dy\right) \quad \text{for } x > 0.$$

$$x < 0 : \quad |m_+(z, x)| \leq 1 + \int_x^\infty (y - x) |V(y)| |m_+(z, y)| dy$$

$$\leq 1 + \int_0^\infty y |V(y)| |m_+(z, y)| dy$$

$$+ \int_x^\infty (-x) |V(y)| \cdot |m_+(z, y)| dy.$$

$$\leq 1 + \underbrace{C}_{K} \int_x^\infty y |V(y)| + |x| \int_x^\infty |V(y)| |m_+(z, y)| dy$$

$$\begin{aligned}
M_+(z, x) &= (K_1(1 + |x|))^{-1} |m_+(z, x)| \\
&\leq 1 + \int_x^\infty |V(y)|(1 + |y|)M_+(z, y) dy .
\end{aligned}$$

has a uniformly bdd solution because Volterra

$$\sup_x M_+(z, x) \leq \exp\left(\int_{-\infty}^\infty (1 + |y|)|V(y)| dy\right) .$$

**Conclusion**  $|m_+(z, x)| \leq K_1(1 + |x|)e^{\int (1+|y|)|V(y)| dy} .$

□

Back to  $f_+, f_- \lambda \neq 0$ .

Reflexion transmission coefficient

$$\begin{aligned}
f_+(\lambda, x) &= \alpha_+(\lambda)f_-(\lambda, x) + \beta_+(\lambda)f_-(-\lambda, x) \\
f_-(\lambda, x) &= \alpha_-(\lambda)f_+(\lambda, x) + \beta_-(\lambda)f_+(-\lambda, x)
\end{aligned}$$

$$\begin{aligned}
W(\lambda) &= W[f_+(\lambda), f_-(\lambda)] = \beta_+(\lambda)W[f_-(-\lambda), f_-(\lambda)] \\
\text{as } x \rightarrow \infty, &= \beta_+(\lambda) \lim \begin{vmatrix} e^{i\lambda x} & i\lambda e^{i\lambda x} \\ e^{-i\lambda x} & -i\lambda e^{-i\lambda x} \end{vmatrix} \\
&= -2i\lambda\beta_+(\lambda) , \quad \text{same for } \beta .
\end{aligned}$$

$$\beta_+(\lambda) = \beta_-(\lambda) = \beta(\lambda)$$

$$\begin{aligned}
W[f_+(\lambda), f_-(-\lambda)] &= \alpha_+(\lambda)W[f_-(\lambda), f_-(-\lambda)] \\
&= 2i\lambda\alpha_+(\lambda)
\end{aligned}$$

$$W[f_-(\lambda), f_+(-\lambda)] = -2i\lambda\alpha_-(\lambda)$$

$$\begin{aligned}
f_+(\lambda, x) &= \alpha_+(\lambda)(\alpha_-(\lambda)f_+(\lambda, x) + \beta(\lambda)f_+(-\lambda, x)) \\
&\quad + \beta_+(\lambda)(\alpha_-(\lambda)f_+(-\lambda, x) + \beta(-\lambda)f_+(\lambda, x))
\end{aligned}$$

$$1 = \alpha_+(\lambda)\alpha_-(\lambda) + \beta(\lambda)\beta(-\lambda)$$

$$2i\lambda\beta(-\lambda) = W(-\lambda) = \overline{W(\lambda)} = 2i\lambda\overline{\beta(\lambda)}$$

$$\text{and } -2i\lambda\alpha_+(\lambda) = 2i\lambda\overline{\alpha_-(\lambda)}$$

$$1 = -|\alpha_+(\lambda)|^2 + |\beta(\lambda)|$$

$$\lambda \neq 0 \Rightarrow |\beta(\lambda)| \geq 1$$

Conclusion  $W(\lambda) \neq 0$  if  $\lambda \neq 0$  for every general  $V(x)$ .

**Lemma.**  $H = -\frac{d^2}{dx^2} + V, V \in L^\infty \cap L'_1$



*Proof.*  $f, g \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}
& \left| \langle e^{itH} P_{ac}(H) \chi(H) f, g \rangle \right| = \left| \int_1^\infty e^{it\lambda} \chi(\lambda) \langle E_{ac}(d\lambda) f, g \rangle \right| \\
& \stackrel{(\text{Stone})}{=} \left| \int_0^\infty e^{it\lambda^2} 2\lambda \chi(\lambda^2) \frac{1}{2\pi i} \langle (R_V(\lambda^2 + i0)) - (R_V(\lambda^2 - i0)) f, g \rangle \right| \\
& \quad \downarrow \text{Resolvent} = \text{Green function} \\
& \leq \frac{1}{\pi} \iint_{x < y} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda^2) \frac{f_+(\lambda, x) f_-(\lambda, y)}{W(\lambda)} d\lambda \right| |f(x)| |g(y)| dx dy \\
& \quad + \frac{1}{\pi} \iint_{x < y} \left| \int_{-\infty}^\infty e^{it\lambda^2} \lambda \chi(\lambda^2) \frac{f_+(\lambda, y) f_-(\lambda, x)}{W(\lambda)} d\lambda \right| |f(x)| |g(y)| dx dy
\end{aligned}$$

Case 1:  $y < 0 < x$

$$\begin{aligned}
& \left| \int_{-\infty}^\infty e^{it\lambda^2} e^{i\lambda(x-y)} \frac{\lambda \chi(\lambda^2)}{W(\lambda)} m_+(\lambda, x) m_-(\lambda, y) d\lambda \right| \\
& \text{disp. est. for free eq.} \\
& \leq C |t|^{-1/2} \left\| \left[ \frac{\lambda \chi(\lambda^2)}{W(\lambda)} m_+(\lambda, x) m_-(\lambda, y) \right]^\vee(\tau) \right\|_{L^1_\tau} \\
& \leq C |t|^{-1/2} \left\| \left[ \text{—————} \right]'' \right\|_{L^\infty} \\
& \text{twice diff. on compact} \rightarrow \text{decays like } 1/\tau^2
\end{aligned}$$

$$\begin{aligned}
m_+(\lambda, x) &= 1 + \int_x^\infty \int_0^{y-x} e^{2is\lambda} ds V(y) m_+(\lambda, y) dy \\
\partial_\lambda m_+(\lambda, x) &= \int_y^\infty \int_0^{y-x} 2ise^{2is\lambda} ds V(\lambda) m_+(\lambda, y) dy \\
& \quad + \int_x^\infty D_\lambda(y-x) V(y) \partial_\lambda m_+(\lambda, y) dy \\
x \geq 0, |\partial_\lambda m_+(\lambda, x)| &\leq \int_0^\infty \overbrace{(y-x)^2 |V(y)| dy}^{< \infty} \underbrace{\sup_{y \geq 0} |m_+(\lambda, y)|}_{\leq \|V\|_{L^1_\lambda}}
\end{aligned}$$

$$+ \int_x^\infty (y-x)|V(y)| |\partial_\lambda m_+(\lambda, y)| dy$$

$$\sup_{x \geq 0} \left| \frac{\partial_\lambda m_+(\lambda, x)}{\chi(\lambda^2)} \right| \leq K \exp\left(\int_0^\infty |y| |V(y)| dy\right)$$

Similarly,  $\sup_{x \geq 0} |\partial_\lambda^2 m_+(\lambda, x)| \leq \text{const.}$  since  $\int (1+|y|)^3 V(y) dy < \infty$ . Next, check that  $|W''(\lambda)| \leq C$  on the supp. of  $\chi(\lambda^2)$

$$\partial_\lambda W(\lambda) = \begin{vmatrix} \partial_\lambda f_+(\lambda, 0) & f'_+(\lambda, 0) \\ \partial_\lambda f_-(\lambda, 0) & f'_-(\lambda, 0) \end{vmatrix} + \begin{vmatrix} f_+(\lambda, 0) & \partial_\lambda \partial_x f_+(\lambda, 0) \\ f_-(\lambda, 0) & \partial_\lambda \partial_x f_-(\lambda, 0) \end{vmatrix}$$

$$f_+(\lambda, x) = e^{i\lambda x} m_+(\lambda, x)$$

$$\partial_\lambda f_+(\lambda, 0) = \partial_\lambda m_+(\lambda, 0) \rightarrow \text{OK, 2 moments.}$$

$$f'_+, f'_- \text{ are bdd in } \lambda$$

$$\partial_\lambda f_+(\lambda, 0) = i\lambda m_+(\lambda, 0) + \partial_x m_+(\lambda, 0)$$

$$\partial_\lambda \partial_x f_+(\lambda, 0) = \underbrace{im_+(\lambda, 0)}_{\text{OK}} + \underbrace{i\lambda}_{\text{OK on supp } \chi} \underbrace{\partial_\lambda m_+(\lambda, 0)}_{\text{OK}} + \partial_\lambda \partial_x m_+(\lambda, 0)$$

$$\partial_x m_+(\lambda, x) = \int_x^\infty D'_\lambda(y-x)V(y)m_+(\lambda, y) dy$$

$$\partial_\lambda \partial_x m_+(\lambda, x) = - \int_x^\infty 2i(y-x)e^{2i\lambda(y-x)}V(y)m_+(\lambda, y) dy$$

$$- \int_x^\infty e^{2i\lambda(y-x)}V(y)\partial_\lambda m_+(\lambda, y) dy .$$

$$\sup_{x \geq 0} |\partial_\lambda \partial_x m_+(\lambda, x)| < \infty .$$

second der.  $\rightarrow$  use third moment.

**Case 2:**  $y < x < 0$

$\rightarrow$  linear growth of  $f_+$  in  $x$ .

$$f_+(\lambda, x) = \alpha_+(\lambda)f_-(\lambda, x) + \beta_+(\lambda)f_(-\lambda x) \quad \text{and use asympt.}$$

$$\int_{-\infty}^\infty e^{it\lambda^2} \lambda \chi(\lambda^2) \frac{f_+(\lambda, x)f_-(\lambda, y)}{W(\lambda)} d\lambda = \int_{-\infty}^\infty e^{it\lambda^2} e^{-i\lambda(x+y)}$$

$$\cdot \frac{\lambda \chi(\lambda)}{W(\lambda)} \alpha_+(\lambda) m_-(\lambda, x) m_-(\lambda, y) d\lambda$$

$$+ \int_{-\infty}^\infty e^{it\lambda^2} e^{i\lambda(x-y)} \frac{\lambda \chi(\lambda^2)}{W(\lambda)} \beta_+(\lambda) m_(-\lambda, x) m_-(\lambda, y) d\lambda$$

$$\begin{aligned}
\text{use } 2i\lambda\beta_+(\lambda) &= W(\lambda) \\
2i\lambda\alpha_+(\lambda) &= W[f_+(\lambda), f_-(-\lambda)] \\
\frac{\lambda\alpha_+(\lambda)}{W(\lambda)} &= \frac{W[f_+(\lambda), f_-(-\lambda)]}{2iW(\lambda)} \quad \text{has second der. in } \lambda.
\end{aligned}$$

need uniform bound of second der. in  $x, y$ , etc. □

if  $V$  is not generic, you lose 1 moment. (expand  $W(\lambda)$  about 0).

Wiener algebra:  $f, \hat{f}$  in  $L^1$

stable under multiplication.

*Remark 1.* If  $|V(x)| < C \cdot (1 + |x|)^{-2-\varepsilon}$  in 1-D.

- Then  $\exists$  only finitely many negative eigv.
- $\text{spec}(H) \cap (0, \infty)$  is purely abs. cont.

$$P_{ac} = I - \sum_{j=1}^N P_{\lambda_j} \quad \text{where } \lambda_N < \dots < \lambda_1 < \lambda_0 \leq 0$$

$$|V(x)| < (1 + |x|)^{-1-\varepsilon}$$

$$N \rightarrow \infty$$

(Cluster about the origin)

$$Hf = Ef, \quad E > 0? \quad \text{no.}$$

*Remark 2.* Supp.  $H = -\frac{d^2}{dx^2} + V$  does not have *any* egv. Then  $P_{ac} = Id$ .

$$\Rightarrow \| e^{itH} \|_{1 \rightarrow \infty} \lesssim |t|^{-1/2}$$

$$\text{Claim.} \quad \begin{cases} i\partial_t \Psi - \partial_x^2 \psi + V\Psi = F \\ \Psi|_{t=0} = f \end{cases}$$

$$\text{then} \quad \| \Psi \|_{L_{t,x}^6} \lesssim \| F \|_{L_{t,x}^{6/5}} + \| f \|_{L_x^2} .$$

proof: identical (interp.,  $TT^*$ )

replace  $e^{it\Delta}$  by  $e^{itH}$

*Remark 3.* If  $V$  is singular, what matters is whether  $H$  is self-adjoint. Properties of egf depends on long time  $\rightarrow$  decay.

**Def. and lemma**  $A : \mathcal{H} \rightarrow \mathcal{H}$  TFAE

- (1)  $A$  takes bdd sets to rel. comp. sts.
- (2)  $A$  takes bdd sets to sets that have finite  $\varepsilon$ -nets for every  $\varepsilon > 0$ .
- (3) if  $\{u_n\}$  is a bdd sequence in  $\mathcal{H}$ , then  $Au_n$  has a strongly conv. subsequence.
- (4)  $A$  takes weakly conv. seq. to strongly conv. seq.

$\Rightarrow A$  compact.

**Theorem 4.** *Spectral thm for compact s.a. operators. Let  $A : \mathcal{H} \rightarrow \mathcal{H}$  be compact and  $A = A^*$ . Then  $\exists$  ONB  $\{\phi_n\}_{n=1}^\infty$  of  $\mathcal{H}$  s.t.*

$$A\phi_n = \lambda_n\phi_n \quad \text{where } \lambda_n \in \mathbb{R}$$

and they can only accumulate at zero. (no ac., no sc.).

**Thm:** *Weyl's criterion. Let  $A, B$  be s.a. and bdd below. Assume  $(A+M)^{-1} - (B+M)^{-1} =: K$  is compact for some numerical constant  $M$ . Then  $\text{ess spec}(A) = \text{ess spec}(B)$   
 $\text{ess spec}(A) = \text{spec}(A) \setminus \{\text{isolated egv with finite mult}\}$*

*Proof.*  $\lambda \in \text{ess spec}(A)$ ,  $\varepsilon > 0$ .

claim:  $\exists$   $\infty$ -dim subspace  $\mathcal{L} \subset \mathcal{H}$  so that

$$\| (A + M)^{-1}f - (\lambda + M)^{-1}f \| < \varepsilon \| f \|$$

$\forall f \in \mathcal{L}$ .

Because

$$\begin{aligned} \| (A + M)^{-1}f - (\lambda + M)^{-1}f \|_2^2 &= \int \left| \frac{1}{t + M} - \frac{1}{\lambda + M} \right|^2 \langle E(dt)ff \rangle \\ &< \varepsilon^2 \| f \|^2 \end{aligned}$$

$$\text{take } \mathcal{L} = \text{Ran} \underbrace{E(\lambda - \delta, \lambda + \delta)}_P$$

Since  $\lambda \in \text{ess spec} A \Rightarrow \dim \mathcal{L} = \infty$ , (otherwise purely atomic in the interval).

$$\| \underbrace{PK^2Pg}_{\substack{\text{compact} \\ \& \text{ s.a.}}} \| < \varepsilon^2 \| g \| \text{ for all } g \in \tilde{\mathcal{L}} \subset \mathcal{L}, \dim \tilde{\mathcal{L}} = \infty$$

(use spectral theorem)

$$\| Kg \|^2 = \langle PK^2Pg, g \rangle < \varepsilon^2 \| g \|^2 \quad \forall g \in \tilde{\mathcal{L}}$$

$$\forall g \in \tilde{\mathcal{L}} \Rightarrow \| Kg \| < \varepsilon \| g \| \text{ on } \mathcal{L}.$$

$$\text{switch to } \| (B + M)^{-1}g - (\lambda + M)^{-1}g \| < 2\varepsilon \| g \| \quad \forall g \in \tilde{\mathcal{L}}$$

□

**Example**  $A = -\Delta$ ,  $B = -\Delta + V$

$$\begin{aligned} &(-\Delta + M)^{-1} - (-\Delta + V + M)^{-1} \\ &= \sum_{n=1}^{\infty} (-\Delta + M)^{-1} (-V(-\Delta + M))^{-n} \quad \text{op. conv. in } L^2 \text{ if } M \text{ large} \end{aligned}$$

$$\| \underbrace{(-\Delta)}_{\substack{\text{strictly} \\ \text{post spec}}} + M)^{-1} \|_{2 \rightarrow 2} \leq \frac{1}{M}$$

$$\| V(-\Delta + M)^{-1} \|_{2 \rightarrow 2} \leq \frac{\| V \|_\infty}{M}$$

If  $V \rightarrow 0$  at  $\infty$ , the  $V(-\Delta + M)^{-1}$  is compact  $\rightarrow$  then the series is compact, as is shown below.

**Theorem.**  $V \in L^\infty$  If  $V \rightarrow 0$  at  $\infty$ , then  $V(-\Delta + 1)^{-1} =: K$  is compact.

*Proof.*  $B \subset L^2(\mathbb{R}^n)$  bdd

show  $\forall \varepsilon > 0$ ,  $K(B)$  has a finite  $\varepsilon$ -net.

$$\begin{aligned} & \| V(-\Delta + 1)^{-1}u - V(-\Delta + 1)^{-1}v \|_2 \quad u, v \in B_1 \\ & < 2 \underbrace{\| V \|_{L^\infty(\mathbb{R}^n \setminus B_R)}}_{\substack{< \varepsilon/2 \\ \text{(take } R \text{ large)} \\ \text{use } V \rightarrow 0 \text{ as } x \rightarrow \infty}} + \underbrace{\| V(-\Delta + 1)^{-1}u - V(-\Delta + 1)^{-1}v \|_{L^2(B_R)}}_{\substack{\text{find finite collection} \\ \text{of } v' \text{ s.t. small.}}} \\ & \leq \| V \|_\infty \| (-\Delta + 1)^{-1}u - (-\Delta + 1)^{-1}v \|_{L^2 B_R} \end{aligned}$$

(Arzela-Ascoli later).

$$W := \{w = (-\Delta + 1)^{-1}v \mid \|v\|_2 \leq 1\}$$

$$w_\varepsilon = \chi_\varepsilon \star w$$

$$\begin{aligned} \|w_\varepsilon(x) - w(x)\|_{L^2(B_R)}^2 &= \int_{B_R} \chi_\varepsilon^2(y) (w(x-y) - w(x))^2 dy \\ &= \int_{B_R} dy \chi_\varepsilon^2(y) \left( \int_0^1 \nabla w(x-ty) y dt \right)^2 \\ &\leq \int_{B_R} dy \chi_\varepsilon^2(y) |y|^2 \underbrace{\left| \sup_t \nabla w(x-ty) \right|^2}_{\leq \|\nabla w\|_{L^2} \leq C} \\ &\leq C\varepsilon \quad \text{good approximants} \end{aligned}$$

$\rightarrow$  work with mollified family, does it admit  $\varepsilon$ -net?  $W_\varepsilon = \{\chi_\varepsilon \star w \mid w \in W\}$ .

$$\text{bddness} \quad |w_\varepsilon(x)| = |(\chi_\varepsilon \star w)(x)| \leq C \cdot \varepsilon^{-n/2} \|w\|_2 \leq C \cdot \varepsilon^{-n/2}$$

$$\begin{aligned} \text{equicontinuity} \quad |w_\varepsilon(x) - w_\varepsilon(y)| &= \left| \int (\chi_\varepsilon(x-z) - \chi_\varepsilon(y-z)) w(z) dz \right| \\ &\leq C \cdot \varepsilon^{-n/2} \varepsilon^{-1} |x-y| \end{aligned}$$

Hence  $W_\varepsilon$  unif. bdd. and equico.

Arzela Ascoli  $\Rightarrow W_\varepsilon$  is rel. comp. in  $C(B_R)$ .

$$\rightarrow L^\infty \varepsilon\text{-net}$$

$$\rightarrow L^2 \varepsilon\text{-net}$$

□

**Ma 142** 01/30

$H = -\Delta + V$ ,  $V \in L^\infty$ ,  $V(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

spec( $H$ ) looks like

**Lemma.** Suppose  $V(x) \rightarrow 0$  as  $x \rightarrow \infty$ . TFAE

- (1)  $\exists \lambda < 0$  eig of  $H = -\Delta + V$
- (2)  $\exists f \in W^{2,2}$  s.t.  $\langle Hf, f \rangle < 0$ .

*Proof.*

- (1)  $\Rightarrow$  (2), take  $f$  the egf
- (2)  $\Rightarrow$  (1), spec must extend below zero (by spectral theorem), but another theorem says this must be an egv. □

**Theorem.** Birman-Schwinger. Let  $V \subset \mathbb{R}^3$  satisfy

$$\iint_{\mathbb{R}^3 \mathbb{R}^3} \frac{|V^-(x)| |V^-(y)|}{|x-y|^2} dx dy < (4\pi)^2 .$$

Then  $-\Delta + V$  does not have any negative egv.

*Proof.* Assume  $\exists \lambda < 0$  egv  $\Rightarrow$  (lemma)  $\exists f \in W^{2,2}$  s.t.  $\langle Hf, f \rangle < 0$ . Since  $V = V^+ + V^-$ ,

$$\langle (-\Delta + V^-)f, f \rangle < 0 \quad (\text{only makes it worse})$$

lemma  $\Rightarrow \tilde{H} = -\Delta + V^-$  has negative bound state.

$$(-\Delta + V^-)\phi = \lambda\phi \quad \text{for some } \phi \neq 0.$$

$$\underbrace{(-\Delta - \lambda)}_{\substack{\text{positive} \\ \text{invertible}}} \phi = -V^- \phi \in L^2$$

$$\begin{aligned} \phi &= -(-\Delta - \lambda)^{-1} V^- \phi \\ \sqrt{|V^-|} \phi &= \underbrace{\sqrt{|V^-|} (-\Delta - \lambda)^{-1} \sqrt{|V^-|}}_A \underbrace{\sqrt{|V^-|} \phi}_{\neq 0} \end{aligned}$$

$A$  has egv 1, s.a.  $\Rightarrow \|A\| \geq 1$

$$\begin{aligned} \text{but } \|A\| &\leq \|A\|_{HS} = \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\text{kernel}(A)|^2 dx dy \right)^{1/2} \\ &= \left( \iint_{\mathbb{R}^6} |V^-(y)| |(-\Delta - \lambda)^{-1}(y, x)|^2 |V^-(x)| dx dy \right)^{1/2} \\ &= \left( \iint_{\mathbb{R}^6} |V^-(y)| |V^-(x)| \frac{e^{-i\sqrt{\lambda}|x-y|}}{(4\pi|x-y|)^2} dx dy \right)^{1/2} < 1 . \end{aligned}$$

□

Theorem (B-S) actually says

$$\#\{\lambda_j | \lambda_j \leq 0 \text{ is an egv of } H\}$$

$$\leq \frac{1}{16\pi^2} \iint_{\mathbb{R}^6} \frac{|V^-(x)| |V^-(y)|}{|x-y|^2} dx dy$$

look into Reed-Simon, IV.

**Lemma.** Suppose  $V(x) \in L^1(\mathbb{R})$  and assume  $\int_{-\infty}^{\infty} V(x) dx < 0$ . Then  $H = -\frac{d^2}{dx^2} + V$  has a negative bound state (in 1-D!!).

*Proof.* Find  $f$  s.t.  $\langle Hf, f \rangle < 0$ . Take  $f(x) = \phi(\varepsilon x)$  where

$$\begin{aligned} \langle Hf, f \rangle &= \varepsilon^2 \int |\phi'(\varepsilon x)|^2 dx + \int V(x) \phi(\varepsilon x)^2 dx \\ &= \varepsilon \|\phi'\|_2^2 + \underbrace{\int V(x) \phi(\varepsilon x)^2 dx}_{\rightarrow fV < 0} < 0 \quad \text{for } \varepsilon \rightarrow 0 \end{aligned}$$

□

**NB:** unitary equivalence of  $-\Delta$  and  $-\Delta + V$  for  $V$  small, i.e.,

$$\int \frac{|V(x)| |V(y)|}{|x-y|^2} dx dy < (4\pi)^2.$$

**NB:**  $\|V\|_{\text{Rollnik}} = \left( \int_{\mathbb{R}^6} \frac{|V(x)| |V(y)|}{|x-y|^2} \right)^{1/2}$  upper bound for  $\sqrt{|V|} R_0(\lambda) \sqrt{|V|}$ . (Hilbert-Schmidt norm).

**Theorem.**

$$\text{Suppose } \|V\|_K = \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} dy < 4\pi$$

$$\text{Then } \|e^{it(-\Delta+V)}(P_{ac})\|_{1 \rightarrow \infty} \leq C \cdot |t|^{-3/2}$$

*Proof.*

$$\langle e^{itH} P_{ac} f, g \rangle = \lim_{L \rightarrow \infty} \left\langle e^{itH} \chi\left(\frac{H}{L}\right) P_{ac} f, g \right\rangle$$

$$= \lim_{L \rightarrow \infty} \int_0^\infty e^{it\lambda} \chi\left(\frac{\lambda}{L}\right) \frac{1}{2\pi i} \langle (R_V(\lambda + i0) - R_V(\lambda - i0)) f, g \rangle d\lambda$$

$$\text{Formally, } R_V(\lambda + i0) = \sum_{n=0}^{\infty} R_0(\lambda + i0) (-V R_0(\lambda - i0))^n$$

$$\begin{aligned}
R_0(\lambda \pm i0)(x, y) &= \frac{e^{\pm i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} \\
\langle (R_V(\lambda + i0) - R_V(\lambda - i0))f, g \rangle &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^3} dx_0 \int dx_{n+1} \cdot \\
\frac{1}{\pi} \int_{\mathbb{R}^{3n}} \frac{\sin(\sqrt{\lambda} \sum_{j=1}^{n+1} |x_j - x_{j-1}|)}{\prod_{j=1}^{n+1} (4\pi|x_j - x_{j-1}|)} \prod_{j=1}^n V(x_j) dx_1 \cdots dx_n f(x_0) g(x_{n+1}) \\
&\leq \sup_{L \geq 1} \frac{1}{\pi} \sum_{n=0}^{\infty} \left| \int_0^{\infty} 2\lambda e^{it\lambda^2} \chi\left(\frac{\lambda^2}{L}\right) \frac{\sin(\lambda \sum_{j=1}^n |x_j - x_{j-1}|)}{\prod_{j=1}^n 4\pi|x_j - x_{j-1}|} d\lambda \right| \\
&\quad \prod_{j=1}^n |V(x_j)| \cdot |f(x_0)| |g(x_{n+1})| dx_0 dx_{n+1}
\end{aligned}$$

integrate by parts. gen  $\sum | \cdot |$  stationary phase or interpret as solution of the  $1 - D$  free eq. check init data has FT in  $L^1$

$$\begin{aligned}
&\leq C \cdot \sum_{n=0}^{\infty} \int_{\mathbb{R}^6} \int_{\mathbb{R}^{3n}} t^{-3/2} \sum_{j=1}^{n+1} |x_j - x_{j-1}| \frac{\prod |V(x_j)|}{\prod_{j=1}^n 4\pi|x_j - x_{j-1}|} \\
&\quad |f(x_0)| |g(x_{n+1})| dx_0 \cdots dx_{n+1} \\
n = 0 &\quad \text{OK} \leq \|f\|_1 \|g\|_1 \\
n = 1 &\quad \int \frac{|V(x_1)|}{|x_2 - x_1| |x_1 - x_0|} (|x_0 - x_1| + |x_1 - x_2|) |f(x_0)| |g(x_2)| \\
&\leq 2 \|V\|_K \|f\|_1 \|g\|_1 \\
n = 2 &\quad \int \frac{|V(x_2)| |V(x_1)|}{|x_2 - x_3| |x_0 - x_1|} |f(x_0)| |g(x_3)| \\
&\leq 3 \|V\|_K^2 \|f\|_1 \|g\|_1 \\
&\quad \text{take sup } x_3 \quad \text{take sup } x_0 \\
&\quad \text{(free } x_3) \\
&\leq C \cdot t^{-3/2} \cdot \sum_{n=0}^{\infty} (n+1) \left( \frac{\|V\|_K}{4\pi} \right)^n \|f\|_1 \|g\|_1 \quad (\text{exercise}).
\end{aligned}$$

□

Conv. of Born series? Does if Rollnik norm  $< \infty$ .

If Rollnik  $< 4\pi$ , unit. eq  $\rightarrow$  can remove  $P_{ac}$

Schlag-Bourgain. Scale invariance of the theory

$$f_R(x) = f(Rx)$$

$$\begin{aligned}
((\Delta + V)f_R)(x) &= -R^2(\Delta f)(Rx) + (V(x)f(Rx)) \\
&= R^2(-\Delta f + \underline{R^{-2}V(R^{-1}\cdot)}f)(Rx)
\end{aligned}$$

all conditions of the previous theorem must be invariant under  $V \rightarrow R^{-2}V(R^{-1})$ , e.g., Kato and Rollnik is OK.  $L^{3/2}$  is OK and close to these conditions.

*Remark 1.*

$$\begin{aligned} & \| R^{-2}V(R^{-1}\cdot) \|_{L^p(\mathbb{R}^n)} \\ &= R^{-2}R^{n/p} \| V \|_{L^p} \quad p = n/2 \end{aligned}$$

e.g.,  $\| V \|_R \leq C \cdot \| V \|_{L^{3/2}(\mathbb{R}^3)}$  when  $V \leq 0$  or  $V \geq 0$ .

*Proof.*

$$(TV)(y) = \int_{\mathbb{R}^3} \frac{V(y)}{|x-y|^2} dy \quad T : L^p \rightarrow L^q \text{ provided}$$

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{2}{3}$$

$$\begin{aligned} V \geq 0, \quad \| V \|_R^2 &= \langle TV, V \rangle \quad L^{3/2} \rightarrow L^3 \\ &\leq C \cdot \| V \|_{L^{3/2}}^2 \end{aligned}$$

□

**Theorem.** *Perturbation argument (Schlag-Bourgain):*

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi + V(t)\psi = 0 \\ \psi|_{t=0} = \psi_0 \end{cases}$$

$$\text{Assume } \sup_t (\| V(t) \|_1 + \| V(t) \|_\infty) < \varepsilon .$$

$$\text{Then } \| \psi(t) \|_{L_x^\infty + L_x^2} \leq C \cdot \langle t \rangle^{-n/2} \| \psi(0) \|_{L^1 \cap L^2}$$

$$\text{Here } \langle t \rangle = (1 + t^2)^{1/2}$$

$$\| f \|_{L^2 + L^\infty} = \inf \{ \| f_1 \|_{L^2} + \| f_2 \|_\infty : f = f_1 + f_2 \} .$$

*Proof.* Duhamel

$$\| \psi \|_{2+\infty} \leq \text{free part OK (use } L^2 \text{ and dispersive)}$$

$$\begin{aligned} & C \langle t \rangle^{-n/2} \| \psi_0 \|_{1 \cap 2} \\ &+ \left\| \int_0^{t-1} e^{\frac{i}{2}(t-s)\Delta} V(s)\psi(s) ds \right\|_\infty \\ & \left\| \int_{t-1}^t e^{\frac{i}{2}(t-s)\Delta} V(s)\psi(s) ds \right\|_2 \\ & \leq \underbrace{C \cdot \langle t \rangle^{-n/2} \| \psi_0 \|_{1 \cap 2}}_\alpha + \int_0^{t-1} \underbrace{C \cdot \langle t-s \rangle^{-n/2} \| V(s)\psi(s) \|_1}_{(1)} ds \end{aligned}$$

$$+ \int_{t-1}^t \|V(s)\psi(s)\|_2 ds \quad (2)$$

$$J(t) = \langle t \rangle^{n/2} \|\psi(t)\|_{2+\infty} \quad \text{show bdd by const.}$$

$$\|V(s)\psi(s)\|_{\binom{1}{2}} \leq \left( \|V(s)\|_{\binom{2}{\infty}} + \|V(s)\|_{\binom{1}{2}} \right) \|\psi(s)\|_{\binom{2+\infty}{2}}$$

$$(1) \leq \int_0^{t-1} C \langle t-s \rangle^{-n/2} \langle s \rangle^{-n/2} J(s) ds \sup_s (\|V(s)\|_2 + \|V(s)\|_1)$$

$$(2) \leq \int_{t-1}^t \langle s \rangle^{-n/2} J(s) ds \sup_s (\|V(s)\|_2 + \|V(s)\|_\infty)$$

$$\sup_{0 \leq t \leq T} J(t) \leq C \cdot \|\psi_0\|_{1 \cap 2} + \sup_{0 \leq s \leq T} J(s) \cdot \varepsilon$$

□

## Agmon bound

$$(-\Delta + V)\psi = E\psi$$

$$\psi \in W^{2,2}, \quad E < 0.$$

**Theorem.** Assume  $V \in C(\mathbb{R}^n)$ , and  $V(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Then

$$\int_{\mathbb{R}^n} e^{\alpha|x|} |\psi(x)|^2 dx < \infty. \quad \text{for some } \alpha > 0$$

If, in addition,  $\sup \|\partial^\gamma V\|_\infty < \infty$ , then  $|\psi(x)| \leq C e^{-\frac{\alpha}{2}|x|}$

$$|\gamma| \leq \frac{n+1}{2}$$

*Some motivation.*  $-\psi'' + V(x)\psi = E\psi.$

$$\psi(x) \simeq \exp\left(-\int_0^x \sqrt{(V(y) - E)_+} dy\right)$$

$$\psi'(x) = -\sqrt{V(x) - E} \psi(x)$$

$$\psi'' = \underbrace{(V(x) - E)\psi(x)}_{\text{equation}} - \underbrace{V'(x)}_{\text{small}} (V(x) - E)^{-1/2} \psi$$

*Proof.* Define Agmon metric

$$P_E(x) = \inf_{\gamma: 0 \leftrightarrow x} \int_0^1 \sqrt{(V(\gamma(t)) - E)_+} |\dot{\gamma}(t)| dt$$

Long excursions of the curve  $\rightarrow$  makes  $f$  big.

$$|\nabla P_E(x)| \leq \sqrt{(V(x) - E)_+} \quad (\text{steepest descent})$$

$$\text{and } P_E(x) \leq (\|V\|_\infty + E)^{1/2} |x|$$

$$\text{introd. } \omega(x) = \exp[\min[2(1 - \varepsilon)P_E(x), N]]$$

Let  $\phi \in C^\infty$  s.t.  $\phi = 1$  for large  $x$  and  $\text{supp } \phi \subset \{x : V(x) - E > 0\}$  □

**Claim:**

$$\int \omega(x) |\psi(x)|^2 \phi^2(x) dx \leq C_{\varepsilon, M, E} \|\psi\|_2^2$$

**Ma 142 02/04**

**Theorem.** (Agmon). Let  $V \in L^\infty(\mathbb{R}^n) \cap C^\circ(\mathbb{R}^n)$   $V(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Suppose  $\psi \in W^{2,2}$ ,  $-\Delta\psi + V\psi = E\psi$ ,  $E < 0$ . Then  $\exists \alpha > 0$ ,  $\int e^{\alpha|x|} |\psi(x)|^2 dx < \infty$ . Moreover, if  $V \in W^{k,\infty}(\mathbb{R}^n)$  where  $k > \frac{n}{2}$ , then

$$|\psi(x)| \leq C e^{-\alpha|x|/2} \quad \forall x \in \mathbb{R}^n.$$

*Proof.* (1-D). Agmon's metric.

$$P_E(x) = \inf_{\gamma: 0 \leftrightarrow x} \int (V(\gamma(t)) - E)_+^{1/2} |\dot{\gamma}(t)| dt \quad \text{Lipschitz}$$

$$|\nabla P_E(x)| \leq (V(x) - E)_+^{1/2} \quad \text{a.e.}$$

Let  $N > 0$ , set  $P_E^{(N)}(x) = \min(P_E(x), N) \rightarrow$  still have  $|\nabla P_E^{(N)}(x)| \leq (V(x) - E)_+^{1/2}$

Take  $\phi \in C^\infty$ ,  $\phi(x) = 1$  if  $x$  large.  $\text{supp } \phi \subset \{x : V(x) - E > 0\}$ .

$V(x) - E > \delta$  on  $\text{supp } \phi$  cfr  $\delta$  later.

Fix  $\varepsilon > 0$ .

$$\omega(x) = \exp(2(1 - \varepsilon)P_E^{(N)}(x)).$$

**Claim.**

$$I_N = \int \omega(x) |\psi(x)|^2 \phi(x)^2 dx \leq C \text{ independent of } N.$$

*Remark 2.* Can take  $\alpha < 2\sqrt{-E}$  (dist. from bound state to origin).

$$\begin{aligned} \delta > 0. \quad \delta I_N &\leq \int \underbrace{(V(x) - E)}_{\Delta\psi} \psi(x) \omega(x) \bar{\psi} \phi^2 dx \\ &\leq - \int |\nabla\psi|^2 \omega(x) \phi^2 dx \\ &\quad + 2(1 - \varepsilon) \int |\nabla\psi| |\psi| \omega \sqrt{V(x) - E} \phi^2(x) dx \\ &\quad + 2 \int |\nabla\psi| |\psi| \omega \phi |\nabla\phi| dx \end{aligned}$$

and use estimate on  $|\nabla P_E|$ .

$$\begin{aligned}
&\leq - \int |\nabla \psi|^2 \omega(x) \phi^2(x) dx \\
&\quad + (1 - \varepsilon) \int |\nabla \psi|^2 \omega(x) \phi^2(x) dx \\
&\quad + (1 - \varepsilon) \int |\psi|^2 (V(x) - E) \omega(x) \phi^2(x) dx \quad \text{cfr. l.h.s!} \\
&\quad + \varepsilon \int |\nabla \psi|^2 \omega \phi^2(x) dx \\
&\quad + 1/\varepsilon \int |\psi|^2 \omega |\nabla \phi|^2 dx
\end{aligned}$$

$$\begin{aligned}
\text{Cond.} \quad \delta I_n \cdot \varepsilon &\leq \frac{1}{\varepsilon} \int |\psi(x)|^2 \underbrace{\omega(x)}_{\substack{\text{bdd} \\ \text{dep on } N}} \underbrace{|\nabla \phi(x)|^2}_{\substack{\text{bdd} \\ \text{comp. supp.}}} dx \\
&\leq \frac{1}{\varepsilon} \int_{|x| \leq R_{\delta, V, E}} e^{2C_{V, E}|x|} |\psi(x)|^2 dx < \infty \text{ independent of } N
\end{aligned}$$

$\delta, \varepsilon$  fixed small, so the claim holds true.

Claim  $\Rightarrow$  concl. because  $\omega(x)$  grows as  $x \rightarrow \infty$ .

Pointwise bound: elliptic regularity.

$$\begin{aligned}
(-\Delta + V)\psi &= E\psi & \psi &\in W^{2,2} \\
\rightarrow \int |\nabla \psi|^2 + V|\psi|^2 &= \int E|\psi|^2 < 0. \\
\int |\nabla \psi|^2 dx &\leq \|V\|_\infty \int |\psi|^2. \\
-\Delta(\partial_\alpha \psi) + \partial_\alpha(V\psi) &= E\partial_\alpha \psi \\
\int |\nabla \partial_\alpha \psi|^2 dx &\leq \int |\partial_\alpha(V\psi)| |\partial_\alpha \psi| dx
\end{aligned}$$

$$\begin{aligned}
\text{Inductively,} \quad \int |\nabla \partial^\gamma \psi|^2 &\leq C \cdot \|V\|_{W^{k, \infty}} \cdot \|\psi\|_2^2 \\
&k = |\gamma|
\end{aligned}$$

$$\text{So} \quad \int_{B_R(x_0)} e^{\alpha|x|} |\psi(x)|^2 dx \sim e^{\alpha|x_0|} \int_{B_R(x_0)} |\psi(x)|^2 dx$$

majorize  $|\psi(x_0)|^2$  by  $\int_{B_R(x_0)} |\psi(x)|^2 dx$ . Sobolev embedding.

$$\begin{aligned}
\text{Localize} \quad \int (-\Delta \psi + V\psi)(x) \bar{\psi}(x) \phi^2(x-y) dx \\
= E \int |\psi(x)|^2 \phi^2(y-x) dx
\end{aligned}$$

$$\begin{aligned}
&= \int |\nabla\psi|^2(x)\phi^2(x-y) dx \\
\text{c.s. } &+ \int \nabla\psi(x)\bar{\psi}2\phi(x-y)\nabla\phi(x-y) dx \\
&+ \int V(x)|\psi|^2(x)\phi^2(x-y) dx \\
\frac{1}{2} \int |\nabla\psi|^2(x)\phi^2(x-y) dx &\leq 4 \int |\psi(x)|^2|\nabla\phi(x-y)|^2 dx \\
&+ \|V\|_\infty \int |\psi|^2(x)\phi^2(x-y) dx \\
&+ E \int |\psi|^2(x)\phi^2(x-y) dx \\
\int_{|x-y|\leq 1} |\nabla\psi(x)|^2 dx &\leq C \cdot (\|V\|_\infty + |E| + 1) \int_{|x-y|\leq 2} |\psi(x)|^2 dx \\
\text{more generally, } \int_{|x-y|\leq 1} |\nabla\partial^\alpha\psi(x)|^2 dx &\leq C_\alpha (\|V\|_{W^{|\alpha|\infty}} + |E| + 1) \cdot \\
&\int_{|x-y|\leq 2} |\psi(x)|^2 dx
\end{aligned}$$

To finish Agmon, use Sobolev embedding

$$\sup_{|x-y|\leq 1} |\psi(x)| \leq C_n \sum_{|\alpha|\leq k} \int_{|x-y|\leq 1} |\partial^\alpha\psi(y)|^2 dy .$$

Sobolev embedding is false for  $L^\infty$ , you end up in BMO.

$$\begin{aligned}
k > n/2 \quad : \quad W^{\alpha,p} &\hookrightarrow L^q \\
\frac{1}{p} - \frac{1}{q} = \frac{|\alpha|}{n} \quad \frac{1}{2} - \frac{1}{\infty} = \frac{|\alpha|}{n} &\Rightarrow |\alpha| = \frac{n}{2} \\
\leq C_{n,V,E} \int_{|x-y|\leq 2} |\psi(y)|^2 dy \\
\leq C_{nVE} \int_{|x-y|\leq 2} e^{\alpha|y|} |\psi(y)|^2 dy e^{-\alpha(|x|-2)} \\
\leq e^{-\alpha|x|} \cdot C_{nVE} \|\psi\|_2^2
\end{aligned}$$

□

## Short-range scattering theory

Basic object  $\Omega_\mp = s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}$

$$\begin{cases} H_0 = -\Delta \\ H = -\Delta + V \end{cases} \quad \| e^{-itH_0} f - e^{-itH} g \|_2 \rightarrow 0 .$$

$s$ -lim means  $\forall f \in L^2, \exists g \in L^2$  s.t.  $\| e^{itH} e^{-itH_0} f - g \|_2 \rightarrow 0$ ?

**Theorem.**  $\sup_x |V(x)|(1+|x|)^{1+\varepsilon} < \infty$ , then  $\Omega_{\mp}$  exist and are isometries and  $\Omega_- e^{-itH_0} = e^{-itH} \Omega_-$  (intertwining property).

$$\Omega_- H_0 f = H \Omega_- f \quad \forall f \in W^{2,2}$$

$$H \circ P_{\text{Ran } \Omega_-} = \Omega_- H_0 \Omega_-^* \quad \text{on } W^{2,2}$$

$$\mathcal{H}_{ac} = L_{ac}^2(H) \supset \text{Ran } \Omega_-$$

*Proof.* (Cook's Method)

Let's show  $\int_1^\infty \| \frac{d}{dt} e^{itH} e^{-itH_0} f \|_2 dt < \infty$ .

$\forall f$  in a dense subspace of  $L^2$ .

(\*)

$$\text{Then okay: } e^{isH} e^{-isH_0} f - e^{itH} e^{-itH_0} f = \int_t^s \frac{d}{d\tau} e^{i\tau H} e^{-i\tau H_0} f d\tau$$

$$\| e^{isH} e^{-isH_0} f - e^{itH} e^{-itH_0} f \|_2 \leq \int_t^s \| \frac{d}{d\tau} e^{i\tau H} e^{-i\tau H_0} f \|_2 d\tau$$

Cauchy seq.

$\rightarrow 0$

$\rightarrow 0$  as  $s, t \rightarrow \infty$

dense subset  $\rightarrow L^2$  since  $\sup_t \| e^{itH} e^{-itH_0} \|_{2 \rightarrow 2} = 1$

$$\begin{aligned} (*) \quad \| \frac{d}{dt} e^{itH} e^{-itH_0} f \|_2 &\leq \| e^{itH} (H - H_0) e^{-itH_0} f \|_2 \\ &= \| V e^{-itH_0} f \|_2 \end{aligned}$$

$$\mathcal{C} = \{ f \in \mathcal{S}(\mathbb{R}^n) : \hat{f}(\xi) = 0 \text{ if } |\xi| < a \\ \text{or } |\xi| > b \text{ for some } a, b \}$$
 is dense.

if  $f \in \mathcal{C}$ , then  $(e^{-itH_0} f)(x) = \int e^{-i(t|\xi|^2 + x \cdot \xi)} \hat{f}(\xi) d\xi$

Stationary phase:

Critical point is:  $2t\xi + x = 0, \xi = -\frac{x}{2t}$

Rapid decay if critical point  $\notin [a, b]$ .

$$|e^{-itH_0} f(x)| \leq C f |t|^{-n/2}$$

Moreover,  $|e^{-itH_0} f(x)| \leq C_N |t|^{-N}$  if  $|x| < \frac{a}{2}|t|$  or  $> 4b|t|$ . (non-stat. phase).

$$\begin{aligned} \therefore \| V e^{-itH_0} f \|_2 &\leq \| V e^{-itH_0} f \|_{L_x^2(\frac{at}{2} < |x| < 4bt)} \\ &+ \| V e^{-itH_0} f \|_{L^2(|x| > 4bt \text{ or } |x| < \frac{a}{2}t)} \\ &\leq C t^{-1-\varepsilon} \underbrace{t^{n/2} t^{-n/2}}_{\substack{\text{volume shell} \\ \times \text{ size in } L^\infty}} + C (at)^{n/2} t^{-N} + C t^{-1-\varepsilon} \cdot 1 \end{aligned}$$

$$\leq C.t^{-1-\varepsilon} \quad \text{integrable.}$$

- Isometries.  $\checkmark$  (OK  $\forall t$ , take  $s$ -lim).
- $\Omega_- e^{-itH_0} f = O_{L^2}(\varepsilon) + e^{isH} e^{-isH_0} e^{-itH_0} f$ 

$$= e^{-itH} e^{i(s+t)H} e^{-i(s+t)H_0} f + O_{L^2}(\varepsilon)$$

$$= e^{-itH} \Omega_- f + O_{L^2}(\varepsilon)$$

$O_{L^2}$  not increased because unitary.

- Laplace transform

$$\Omega_- \int_0^\infty e^{-itH_0} e^{-t\varepsilon} dt = \int_0^\infty e^{-t\varepsilon} e^{-itH} dt \Omega_-$$

$$\Omega_- (iH_0 + \varepsilon)^{-1} = \underbrace{(iH + \varepsilon)^{-1}}_{\text{bdd from } L^2 \text{ to } W^{2,2}} \Omega_-$$

$$\Omega_- : W^{2,2} \rightarrow W^{2,2}$$

$$\Omega_-^* : W^{2,2} \rightarrow W^{2,2}$$

$$\text{apply } (iH + \varepsilon)\Omega_- (iH_0 + \varepsilon)^{-1} = \Omega_- \quad \text{on } L^2$$

$$(iH + \varepsilon)\Omega_- = \Omega_- (iH_0 + \varepsilon) \quad \text{on } W^{2,2}$$

- $H \circ P_{\text{Ran}\Omega_-} \stackrel{?}{=} \Omega_- H_0 \Omega_-^* \quad \text{on } W^{2,2}$

Since  $\Omega_-$  is an isometry,

$$Q := \Omega_-^* \Omega_- = Id_{L^2}, \quad \Omega_- \Omega_-^* = P_{\text{Ran}\Omega_-}$$

$$Q^* = Q \geq 0$$

$$\langle Qf, f \rangle = \|\Omega_- f\|^2 = \|f\|^2 = \langle f, f \rangle$$

$$\langle (Q - I)f, f \rangle = 0 \quad \forall f, \quad Q \text{ s.a} \Rightarrow Q = I$$

↓  
spectral  
theorem

$$P = \Omega_- \Omega_-^*$$

$$P^2 = \Omega_- \underbrace{\Omega_-^* \Omega_-}_{=I} \Omega_-^* = P \quad P^* = P \geq 0.$$

$P$  is an orthog. proj.  
→ an identify on its range  
→  $\text{Ker } P = (\text{Ran } P)^\perp$

$$\text{Ker } P = \{f : \langle Pf, f \rangle = 0\}$$

$$= \{f : \|\Omega_-^* f\|^2 = 0\} = \text{Ker } \Omega_-^*$$

⇒  $P$  is proj. onto  $\text{Ran } \Omega_-$ . (There is nothing more in  $\text{Ker } P$  than  $\text{Ker } \Omega_-$ ).

then take  $\Omega_- H_0 f = H \Omega_- f$  and mult right by  $\Omega_-^*$ .

$$\begin{aligned} E(d\lambda) P_{\text{Ran}\Omega_-} &= \Omega_- E_0(d\lambda) \Omega_-^* \\ P_{\text{Ran}\Omega_-} E(d\lambda) P_{\text{Ran}\Omega_-} &= \Omega_- \overbrace{\Omega_-^* \Omega_-}^I E_0(d\lambda) \Omega_-^* \\ \langle E(d\lambda) P f, P f \rangle &= \underbrace{\langle E_0(d\lambda) \Omega_-^* f, \Omega_-^* f \rangle}_{\text{always a.c.}} \\ &\Rightarrow P f \in L_{ac}^2 \text{ rel. to } H \end{aligned}$$

□

2/6/2003

Last time we showed under SR (short range condition)

$$|V(x)| \lesssim (1 + |x|)^{-1-\varepsilon} \quad (\text{SR})$$

that  $\Omega_- = s\text{-}\lim_{t \rightarrow \infty} e^{itH} e^{-itH_0}$  exists and  $\text{Ran } \Omega_- \subset L_{ac}^2(\mathbb{R}^n)$

**Q:** When is  $\text{Ran } \Omega_- = L_{ac}^2(\mathbb{R}^n)$ ?

**Prop:** Assume  $n \geq 3$ , (SR)  $\|V\|_{L^2(\mathbb{R}^n)} < \infty$ ,  $\|e^{-itH} P_{ac}(H)\|_{1 \rightarrow \infty} \lesssim |t|^{-n/2}$  Then  $\text{Ran } \Omega_- = L_{ac}^2$ .

*Proof.*  $\tilde{\Omega} = s\text{-}\lim e^{itH_0} e^{-itH}$  doesn't exist. So define  $\tilde{\Omega} = s\text{-}\lim e^{itH_0} e^{-itH} P_{ac}(H)$ . By Cook's method, it suffices to show

$$\int_1^\infty \underbrace{\left\| \frac{d}{dt} e^{itH_0} e^{-itH} P_{ac}(H) f \right\|_2}_{\|V e^{-itH} P_{ac}(H) f\|_2} dt < \infty$$

(for  $f \in L^1 \cap L^2$ ).

$$\begin{aligned} \int_1^\infty \|V e^{-itH} P_{ac}(H) f\|_2 dt &\leq \int_1^\infty \|V\|_2 \|e^{-itH} P_{ac}(H) f\|_\infty dt \\ &\leq \|V\|_2 \int_1^\infty ct^{-\frac{n}{2}} \|f\|_1 dt < \infty \end{aligned}$$

So  $\tilde{\Omega}$  exists. Then  $\Omega_- \tilde{\Omega} f = e^{itH} e^{-itH_0} \tilde{\Omega} f + o_{L^2}(1) \quad \forall t \geq T$ .

Hence  $\Omega_- \tilde{\Omega} f = e^{itH} e^{-itH_0} e^{itH_0} e^{-itH} P_{ac}(H) f + o_{L^2}(1) = P_{ac}(H) f + o_{L^2}(1)$

So  $\Omega_- \tilde{\Omega} f = P_{ac}(H) f \quad \forall f \in L^2 \Rightarrow \text{Ran } \Omega_- = \text{Ran } P_{ac}(H) = L_{ac}^2$

□

*Remark 1.* Instead of  $V \in L^2$  take  $V \in L^p$  so that

$$\begin{aligned} \|V e^{-itH} P_{ac}(H) f\|_2 &\leq \|V\|_p \|e^{-itH} P_{ac}(H) f\|_q \quad \frac{1}{2} = \frac{1}{p} + \frac{1}{q} \\ &\leq Ct^{-1-\varepsilon} \quad \text{if } 1 + \varepsilon = \frac{n}{2} \left( \frac{1}{q'} - \frac{1}{q} \right) \end{aligned}$$

(because  $\| e^{-itH} P_{ac}(H) f \|_q \leq C |t|^{-\frac{n}{2}(\frac{1}{q'} - \frac{1}{q})} \| f \|_{q'}$ )

so if  $\varepsilon = 0 \Rightarrow n = p$ , i.e.,  $V \in L^{n+\varepsilon}$  is enough. But (SR)  $\subset L^{n+\varepsilon} \Rightarrow$  don't need  $V \in L^2$ .

*Remark 2.* When does the Proposition apply? We did case  $n = 3$ ,  $|V(x)| \leq C_0(1 + |x|)^{-2-\varepsilon}$  and  $c_0$  small so that  $\sup_x \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} dy < 4\pi$ .

## 2/6/2003

Last time we showed under SR (short range condition)

$$|V(x)| \lesssim (1 + |x|)^{-1-\varepsilon} \quad (\text{SR})$$

that  $\Omega_- = \text{s-lim}_{t \rightarrow \infty} e^{itH} e^{-itH_0}$  exists and  $\text{Ran } \Omega_- \subset L_{ac}^2(\mathbb{R}^n)$ .

**Question:** When is  $\text{Ran } \Omega_- = L_{ac}^2(\mathbb{R}^n)$ ?

**Proposition.** Assume  $n \geq 3$ , (SR),  $\| V \|_{L^2(\mathbb{R}^n)} < \infty$ ,  $\| e^{-itH} P_{ac}(H) \|_{1 \rightarrow \infty} \lesssim |t|^{-n/2}$ . Then  $\text{Ran } \Omega_- = L_{ac}^2$ .

*Proof.*  $\tilde{\Omega} = \text{s-lim}_{t \rightarrow \infty} e^{itH_0} e^{-itH}$  doesn't exist. So define  $\tilde{\Omega} = \text{s-lim}_{t \rightarrow \infty} e^{itH_0} e^{-itH} P_{ac}(H)$ . By Cook's method, it suffices to show (for  $f \in L^1 \cap L^2$ )

$$\begin{aligned} & \int_1^\infty \underbrace{\left\| \frac{d}{dt} e^{itH_0} e^{-itH} P_{ac}(H) f \right\|_2}_{\| V e^{-itH} P_{ac}(H) f \|_2} dt < \infty \\ & \int_1^\infty \| V e^{-itH} P_{ac}(H) f \|_2 dt \leq \int_1^\infty \| V \|_2 \| e^{-itH} P_{ac}(H) f \|_\infty dt \\ & \leq \| V \|_2 \int_1^\infty ct^{-\frac{n}{2}} \| f \|_1 dt < \infty \end{aligned}$$

So  $\tilde{\Omega}$  exists. Then  $\Omega_- \tilde{\Omega} f = e^{itH} e^{-itH_0} \tilde{\Omega} f + O_{L^2}(1) \quad \forall t \geq T$ .

Hence  $\Omega_- \tilde{\Omega} f = e^{itH} e^{-itH_0} e^{itH_0} e^{-itH} f + O_{L^2}(1) = P_{ac}(H) f + O_{L^2}(1)$

So  $\Omega_- \tilde{\Omega} f = P_{ac}(H) f$  for all  $f \in L^2 \Rightarrow \text{Ran } \Omega_- = \text{Ran } P_{ac}(H) = L_{ac}^2$ .  $\square$

*Remark 1.* Instead of  $V \in L^2$  take  $V \in L^p$  so that

$$\begin{aligned} \| V e^{-itH} P_{ac}(H) f \|_2 & \leq \| V \|_p \| e^{-itH} P_{ac}(H) f \|_q \quad \frac{1}{2} = \frac{1}{p} + \frac{1}{q} \\ & \leq C t^{-1-\varepsilon} \quad \text{if } 1 + \varepsilon = \frac{n}{2} \left( \frac{1}{q'} - \frac{1}{q} \right) \end{aligned}$$

(because  $\| e^{-itH} P_{ac}(H) f \|_q \leq C |t|^{-\frac{n}{2}(\frac{1}{q'} - \frac{1}{q})} \| f \|_{q'}$ )

so if  $\varepsilon = 0 \Rightarrow n = p$ , i.e.,  $V \in L^{n+\varepsilon}$  is enough. But (SR)  $\subset L^{n+\varepsilon} \Rightarrow$  don't need  $V \in L^2$ .

*Remark 2.* When does the Proposition apply? We did case  $n = 3$ ,  $|V(x)| \leq C_0(1 + |x|)^{-2-\varepsilon}$  and  $C_0$  small so that  $\sup_x \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} dy < 4\pi$ .

**Classical (SR) scattering:**  $\dot{x} = \xi$   
 $\dot{\xi} = -\nabla V(x)$

$$\Sigma_{\pm} = \left\{ (y, \eta) \in \mathbb{R}^{2n} : V(y) + \frac{1}{2}|\eta|^2 > 0 \text{ and } \overline{\lim}_{t \rightarrow \pm\infty} |x(t; y, \eta)| = \infty \right\}$$

**Note:**  $\frac{d}{dt} \left( V(x(t)) + \frac{1}{2}|\xi(t)|^2 \right) = \nabla V(x(t)) \cdot \xi - \xi \cdot \nabla Vx(t) = 0$

$$\Sigma_{\text{scat}} = \Sigma_+ \cap \Sigma_-.$$

**Lemma.** Assume  $|\nabla V(x)| \leq C_0(1 + |x|)^{-1-\varepsilon}$  and  $V(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Then:  $(y, \eta) \in \Sigma_+ \Leftrightarrow \lim_{t \rightarrow \infty} \xi(t; y, \eta)$  exists and is  $\neq 0$ .

**Note** This is long-range.

If  $\lim_{t \rightarrow \infty} \xi(t; y, \eta)$  exists and  $\neq 0$ , then  $|x(t; y, \eta)| \rightarrow \infty$ .

By conservation of energy  $V(x(t)) + \frac{1}{2}|\eta|^2 > 0$ .

Conversely, suppose  $(y, \eta) \in \Sigma_+$ . Want to show  $|x(t)| \geq at$  for some  $a > 0$ .

If so, done:  $|\xi(t) - \xi(s)| \leq \int_s^t |\nabla V(x(\tau))| d\tau \leq \int_s^\infty C_0 a^{-1-\varepsilon} \tau^{-1-\varepsilon} d\tau \rightarrow 0$  as  $s \rightarrow \infty$ .

$t \geq t_0 \Rightarrow$  particle stays inside the cone

**Claim**  $|x(t)| \geq \max\left(\frac{R}{2}, C_1(t - t_0)\right)$ , ( $C_1$  not depending on  $R$ )

*Proof.* Assume the above formula holds on  $t \in [t_0, T]$  and show we can extend it

$$|\xi(t) - \xi(t_0)| \leq \int_{t_0}^{t_0 + \frac{R}{2C_1}} C_0 \left(\frac{R}{2}\right)^{-1-\varepsilon} dt + \int_{t_0 + \frac{R}{2C_1}}^{\infty} C_0 (C_1(t - t_0))^{-1-\varepsilon} dt$$

$$\leq C_0 \frac{2^\varepsilon R^{-\varepsilon}}{C_1} + \frac{C_0 C_1^{-1-\varepsilon}}{\varepsilon} \left(\frac{R}{2C_1}\right)^{-\varepsilon} \leq \frac{C_0}{C_1} 2^\varepsilon \left(1 + \frac{1}{\varepsilon}\right) R^{-\varepsilon}$$

$$x(t) = x(t_0) + \int_{t_0}^t \xi(\tau) d\tau = x(t_0) + (t - t_0)\xi(t_0) + \int_{t_0}^t \xi(\tau) - \xi(t_0) d\tau$$

$$|x(t)| \geq \sqrt{R^2 + (t - t_0)^2 |\xi(t_0)|^2} - \int_{t_0}^t \frac{C_0}{C_1} \frac{2^{1+\varepsilon}}{\varepsilon} R^{-\varepsilon} d\tau$$

because  $\alpha \geq 90^\circ$

$$\geq \frac{R}{\sqrt{2}} + \frac{(t - t_0)|\xi(t_0)|}{\sqrt{2}} - (t - t_0) \frac{C_0}{C_1} \frac{2^{1+\varepsilon}}{\varepsilon} R^{-\varepsilon}$$

$$\geq \max\left(\frac{R}{\sqrt{2}}, (t - t_0) \left( \underbrace{\frac{|\xi(t_0)|}{2} - \frac{C_0 2^{1+\varepsilon}}{C_1 \varepsilon} R^{-\varepsilon}}_{\text{less than } C_1, \text{ by choosing small } C_1 \text{ and then huge } R} \right) \right) \Rightarrow \text{can extend beyond } T$$

□

**Proposition:** Assume (SR):  $|\nabla V(x)| \lesssim (1 + |x|)^{-2-\varepsilon}$  and  $|D^2V(x)| \lesssim (1 + |x|)^{-2-\varepsilon}$ . Then  $\forall (y, \eta) \in \Sigma_+$ ,  $\exists (a, p) \in \mathbb{R}^{2n}$ ,  $p \neq 0$  so that for  $t \rightarrow \infty$

$$\begin{aligned} |a + tp - x(t; y, \eta)| &\rightarrow 0 \\ |p - \xi(t; y, \eta)| &\rightarrow 0 \end{aligned} \quad (\text{and conversely})$$

*Proof.*  $(y, \eta) \in \Sigma_+ \Rightarrow \lim_{t \rightarrow \infty} \xi(t) = p \neq 0$  (we know)

$$\begin{aligned} x(t) &= y + \int_0^t \xi(s) ds = y + tp + \int_0^t (\xi(s) - p) ds \\ &= y + tp + \int_0^t \int_s^\infty \nabla V(x(\tau)) d\tau ds \\ &= y + tp + \underbrace{\int_0^\infty \int_s^\infty \nabla V(x(\tau)) d\tau ds}_{=a-y} - \underbrace{\int_t^\infty \int_s^\infty \nabla V(x(\tau)) d\tau ds}_{\rightarrow 0} \quad \text{because (SR)} \end{aligned}$$

Conversely:  $x(t) = a + tp + \Delta x(t)$ —want to find  $x$

$$a + tp + \Delta x(t) = a + tp - \int_t^\infty ds \int_s^\infty d\tau \nabla V(a + \tau p + \nabla x(\tau))$$

Contract in the unit ball in  $C([T, \infty), \mathbb{R}^n)$

$$(Az)(t) = - \int_t^\infty ds \int_s^\infty d\tau \nabla V(a + \tau p + z(\tau))$$

$$|a + tp + z(\tau)| \geq |a + tp| - 1 \geq \frac{1}{2}t|p| \text{ if } t \geq T \text{ large}$$

$$\text{So } |Az(t)| \leq C \int_t^\infty \int_s^\infty \left(\frac{1}{2}\tau|p|\right)^{-2-\varepsilon} d\tau ds \leq C_\varepsilon |p|^{-2-\varepsilon} T^{-\varepsilon} \leq 1$$

$$|Az(t) - A\tilde{z}(t)| \leq C \int_t^\infty \int_s^\infty \left(\frac{1}{2}\tau|p|\right)^{-2-\varepsilon} \|z - \tilde{z}\| d\tau ds \leq C|p|^{-2-\varepsilon} T^{-\varepsilon} \|z - \tilde{z}\|$$

So  $A$  is a contraction and such an  $x$  exists.

□

$$\Omega : \sum_{\text{free}} = \{(a, p) \in \mathbb{R}^{2n} | p \neq 0\} \rightarrow \sum_+ \{(y, \eta) | \dots\}$$

Hence,  $\Omega$  is one-to-one and onto

Moreover,  $V(t)\Omega = \Omega V_0(t)$ , since

$$\begin{aligned} V(s)V(t)\Omega(a, p) &= V(s+t)\Omega(a, p) \rightarrow \binom{a+(s+t)p}{p} \\ V(s)\Omega V_0(t)(a, p) &= V(s)\Omega(a+tp, p) \rightarrow \binom{a+tp+sp}{p} \end{aligned}$$

2/11/2003

### Easy propagation facts

**Lemma.** *Let  $A \in \mathcal{B}(\mathcal{H})$ . Then  $A$  is compact  $\Leftrightarrow \forall \varepsilon > 0 \exists \tilde{A}$  of finite rank such that  $\|A - \tilde{A}\| < \varepsilon$ .*

*Proof.* By definition  $A$  is compact  $\Leftrightarrow \forall \varepsilon > 0$   $A(B)$  has a finite  $\varepsilon$ -net where  $B =$  unit ball in  $\mathcal{H}$ .

$$\Leftrightarrow \tilde{A}(B) \subset \underbrace{\text{Ran } \tilde{A}}_{\substack{\text{finite dimensional} \\ \text{subspace of } \mathcal{H}}} \text{ and } \tilde{A}(B) \text{ bounded} \Rightarrow \tilde{A}(B) \text{ compact} \Rightarrow$$

$$\left. \begin{array}{l} \tilde{A}(B) \text{ has a finite } \varepsilon\text{-net} \\ \text{But } \|A - \tilde{A}\| < \varepsilon \end{array} \right\} \Rightarrow A(B) \text{ has a finite } 2\varepsilon \text{ net}$$

$\Rightarrow$  Let  $\{\phi_j\}$  be an orthonormal basis in  $\mathcal{H}$ . Let  $P_N$  be the projection on span  $\{\phi_1, \phi_2, \dots, \phi_n\}$  i.e.,  $P_N : \mathcal{H} \rightarrow \mathcal{H}$ ,  $P_N f = \sum_{j=1}^N \langle f, \phi_j \rangle \phi_j$ .

Obviously,  $P_N A$  is of finite rank.

It suffices to prove that  $\forall \varepsilon > 0$ ,  $\|A - P_N A\| < \varepsilon$  if  $N$  is large. Since  $A$  is compact, it suffices to prove that  $\forall K \subset \mathcal{H}$ ,  $K$  compact we have

$$\sup_{f \in K} \|f - P_N f\| < \varepsilon \text{ if } N \text{ is large}$$

This follows from the fact that we can consider a finite  $\frac{\varepsilon}{2}$ -net for  $K$  and  $N$  large enough such that  $\|f - P_N f\| < \frac{\varepsilon}{2} \forall f$  in the  $\frac{\varepsilon}{2}$  net.  $\square$

**Lemma.** *Let  $H = -\Delta + V$ . Then for  $\forall f \in L_{ac}^2$*

$$\|\chi_{|x| \leq R} e^{-itH} f\|_2 \rightarrow 0 \text{ as } t \rightarrow \infty.$$

*Proof.* It suffices to check this for  $f = (H + i)^{-1}g$  so show

$$\|\chi_{|x| \leq R} (H + i)^{-1} e^{-itH} g\|_2 \rightarrow 0 \text{ (or } (H + \mu)^{-1})$$

By the previous lemma, it suffices to show that

$$\|C e^{-itH} g\|_2 \rightarrow 0 \text{ as } t \rightarrow \infty$$

$C$  has finite rank, without loss of generality, we can assume rank  $C = 1$

$$\begin{aligned} \textcircled{*} \text{ becomes } & \|\psi_1 \langle e^{-itH} g, \psi_2 \rangle\|_2 = \|\psi_1\| \cdot |\langle e^{-itH} g, \psi_2 \rangle| = \\ & = \|\psi_1\| \int_{-\infty}^{\infty} e^{-it\lambda} \underbrace{\langle E(d\lambda)g, \Psi_2 \rangle}_{\text{a.c. measure}} \xrightarrow{t \rightarrow \infty} 0 \text{ by Riemann-Lebesgue} \end{aligned}$$

$\square$

**RAGE (Ruelle, Amrein, Georgescu, Enss) Theorem** For any  $f \in L^2_{\text{cont}} = L^2_{ac} \oplus L^2_{sc}$ :

$$\frac{1}{T} \int_{-T}^T \|\chi_{\{|x| \leq R\}} e^{-itH} f\|_2^2 dt \rightarrow 0 \text{ as } T \rightarrow \infty$$

*Proof.* As before, let  $f = (H + \mu)^{-1}g$ . Then, it suffices to show that

$$\frac{1}{T} \int_{-T}^T |\langle e^{-itH} g, \Psi_2 \rangle|^2 dt \xrightarrow{T \rightarrow \infty} 0 \Leftrightarrow \frac{1}{T} \int_{-T}^T |\hat{\mu}_{g, \Psi_2}(t)|^2 dt \xrightarrow{T \rightarrow \infty} 0$$

Apply Wiener's theorem

$$\frac{1}{T} \int_{-T}^T |\hat{\mu}_{g, \Psi_2}(t)|^2 dt \rightarrow 2 \sum_{\tau \in \mathbb{R}} |\mu_{g, \Psi_2}(\{\tau\})|^2 = 0$$

□

**Theorem.** (Wiener) Let  $\mu \in \mathcal{M}(R)$ . Then

$$\frac{1}{2T} \int_{-T}^T |\hat{\mu}(t)|^2 dt \rightarrow \sum_{\tau \in \mathbb{R}} |\mu(\{\tau\})|^2$$

*Proof.*

$$(\mu * \nu)(E) = \int_R \mu(E - t) \nu dt \quad \text{Let } \mu^\#(E) = \overline{\mu(-E)} \text{ so that } \widehat{\mu^\#}(\xi) = \overline{\hat{\mu}(\xi)}$$

$$E < R(\text{usual})$$

$$\mu * \mu^\#(\{0\}) = \int_{\mathbb{R}} \mu(\{-t\}) \mu^\#(dt) = \sum_{\substack{-t \text{ atom} \\ \text{of } \mu}} \mu(\{-t\}) \cdot \mu^\#(\{-t\}) = \sum_{\substack{t \text{ atom} \\ \text{of } \mu}} |\mu(\{t\})|^2$$

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T e^{it\tau} \hat{\mu}(t) dt &= \frac{1}{2T} \int_{-T}^T \int_{-\infty}^{\infty} e^{-it(s-\tau)} \mu(ds) dt \stackrel{\text{Fubini}}{=} \int_{-\infty}^{\infty} \frac{1}{2T} D_T(s-\tau) \mu(ds) dt = \\ &= \int_{-\infty}^{\infty} \frac{1}{2T} D_T(s-\tau) [\mu(ds) - \mu(\tau) \delta_{\{\tau\}}(ds)] dt + \mu(\tau) \xrightarrow{T \rightarrow \infty} \mu(\tau) \end{aligned}$$

(where  $D_T$  is the Dirichlet kernel).

$$\text{Apply for } \mu * \mu^\#, \tau = 0 \text{ and get } \frac{1}{2T} \int_{-T}^T |\hat{\mu}(t)|^2 dt \mapsto \sum_{\tau \in \mathbb{R}} |\mu(\{\tau\})|^2 \quad \square$$

**Theorem.** (Enss) Let  $V$  be a SR potential (i.e.,  $|V(x)| \lesssim (1 + |x|)^{-1-\varepsilon}$ ). Then we have asymptotic completeness, i.e.,

$$\text{ran } \Omega_{\mp} = L^2_{ac} \text{ and } L^2_{sc} = \{0\}.$$

**Corollary.** If  $V$  is SR, then  $\forall f \in L^2(\mathbb{R}^n) \exists \{f_j\}_{j=0}^N \in L^2$  such that  $Hf_j = \lambda_j f_j$  for  $j \geq 1$ .

$$\lambda_j \leq 0 \text{ and } e^{-itH} f = e^{-itH_0} f_0 + \sum_{j=1}^N e^{-it\lambda_j} f_j + O_{L^2}(1) \text{ as } t \rightarrow \infty.$$

*Proof of the corollary.* From the theorem  $L^2 = L_{ac}^2 + L_{pp}^2 \Rightarrow$

$$\Rightarrow f = g + \sum_{j=1}^N f_j \quad g \in L_{ac}^2 = \text{ran } \Omega_- \quad f_j \in L_{pp}^2$$

By Kato there cannot be positive eigenvalues. Thus  $\mathcal{H}f_j = \lambda_j f_j$ ,  $\lambda_j \leq 0$ . Furthermore,  $g_0 = \Omega_- f_0$

$$e^{-itH} f = e^{-itH} \Omega_- f_0 + \sum_{j=1}^N e^{-it\lambda_j} f_j$$

Since  $\| e^{-itH} \Omega_- f_0 - e^{-itH_0} f_0 \|_2 \xrightarrow{t \rightarrow \infty} 0$  we are done.  $\square$

*Proof of Enss's Theorem.*

**I. Heuristic sketch:** Take  $\Psi \in L_{\text{cont}}^2$  and  $\Psi \perp \text{ran } \Omega_-$ . Want to prove that  $\Psi = 0$

Pick  $\tau_n \rightarrow \infty$  increasing, such that  $e^{-i\tau_n H} \Psi$  gets far away from the origin.

$$\begin{aligned} & \| \chi_{[|x| \leq n]} e^{-i\tau_n H} \Psi \|_2 \rightarrow 0 \\ \Psi_n &= \chi_{[|x| \leq n]} \Psi_n + \Psi_n^{(\text{out})} + \Psi_n^{(\text{in})} \end{aligned}$$

Let  $\chi_j : \chi_j(\xi) + \chi_j(-\xi) = 1$  if  $0 < a < |\xi| < b$

$$\begin{aligned} \chi_{[|x| > n]} \Psi_n &= \sum_j \chi_{j,n} \Psi_n = \sum_j \chi_j(p) \chi_{j,n} \Psi_n + \sum_j \chi_j(-p) \chi_{j,n} \Psi_n \\ \chi_j(p) f &= (\chi_j(\xi) \hat{f})^\vee \end{aligned}$$

- $\| (\Omega_- - \mathbf{I}) \Psi_n^{(\text{out})} \|_2 \rightarrow 0$  as  $n \rightarrow \infty$  (therefore  $\| (\Omega_+ - \mathbf{I}) \Psi_n^{(\text{in})} \|_2 \rightarrow 0$ )  $\circledast$

$$\begin{aligned} \| (\Omega_- - \mathbf{I}) \Psi_n^{(\text{out})} \|_2 &\leq \int_0^\infty \left\| \frac{d}{dt} e^{itH} e^{-itH} \Psi_n^{(\text{out})} \right\| dt = \\ &= \int_0^\infty \| V e^{-itH_0} \Psi_n^{(\text{out})} \| dt \rightarrow 0 \text{ as } n \rightarrow \infty \\ &\lesssim (n+t)^{-1-\varepsilon} \end{aligned}$$

- $\langle \Psi_n^{(\text{in})}, \Psi_n \rangle = \langle \Omega_+ \Psi_n^{(\text{in})}, e^{-i\tau_n H} \Psi_n \rangle + O_{L^2}(1)$   
 $\uparrow$  see  $\circledast$   
 $= \langle e^{i\tau_n H} \Omega_+ \Psi_n^{(\text{in})}, \Psi_n \rangle + O_{L^2}(1) = \underbrace{\langle e^{i\tau_n H_0} \Psi_n^{(\text{in})}, \Omega_+^* \Psi_n \rangle}_{O_{L^2}(1)} + O_{L^2}(1)$

$\square$

**II. Rigorous proof.** Let  $\tilde{\Psi} \in L_{\text{cont}}^2$ ,  $\tilde{\Psi} \perp \text{ran } \Omega_-$

$$\| \tilde{\Psi} - \Psi \|_2 \text{ so that } \Psi = \chi_{(a,b)}(H) \tilde{\Psi}$$

By RAGE, pick  $\tau_n \rightarrow \infty$  increasing such that 
$$\begin{cases} \|\chi_{[|y| \leq 2C_0 n]} e^{-i\tau_n H} \Psi\|_2 \rightarrow 0 \\ \int_{-n}^n \|\chi_{[|x| \leq n]} e^{-i(t+\tau_n)H} \Psi\|_2 dt \rightarrow 0 \end{cases}$$

Let  $\Psi_n := e^{-i\tau_n H} \psi$ ,  $\Phi_n = \chi_{(a,b)}(H_0) \Psi_n$

Claim  $\|\Psi_n - \Phi_n\|_2 = \|\chi_{(a,b)}(H) - \chi_{(a,b)}(H_0)\| \Psi_n\|_2$

**Lemma.** Let  $\phi \in L^1(\mathbb{R})$ . Then  $\|(\hat{\phi}(H) - \hat{\phi}(H_0))\Psi_n\|_2 \mapsto 0$ .

*Proof.*

$$\begin{aligned} & \left\| \left( \int_{-\infty}^{\infty} \phi(x) (e^{-ixH} - e^{-ixH_0}) dx \right) \Psi_n \right\|_2 \\ & \leq \int_{-n}^n |\phi(x)| \| (e^{ixH_0} e^{-ixH} - I) \Psi_n \|_2 dx + \int_{|x|>n} |\phi(x)| dx \cdot 2 \|\Psi_n\| \\ & \leq \int_{-n}^n |\phi(x)| \left| \int_0^x \| V e^{-iyH} \Psi_n \|_2 dy \right| dx + O(1) \text{ as } n \rightarrow \infty \\ & \leq \|\phi\|_1 \|V\|_{\infty} \int_{-n}^n \|\chi_{[|x| \leq n]} e^{-i(t+\tau_n)H} \psi\|_2 dt + \\ & + \|\phi\|_1 \int_{-n}^n \|V \chi_{[|x| > n]}\|_{\infty} dt \|\psi\|_2 + O(1) = O(1) \end{aligned}$$

□

**2/16/2003**

We are in the middle of the proof of Enss's theorem:

If  $V$  is SR (i.e.,  $|V|(x) \lesssim (1+|x|)^{-1-\epsilon}$ ), then  $\text{ran } \Omega_- = L^2_{\text{cont}}$  (in particular,  $\text{ran } \Omega_- = L^2_{ac}$ ,  $L^2_{sc} = \{0\}$ ).

*Proof.* Let  $\tilde{\Psi} \in L^2_{\text{cont}}$ ,  $\tilde{\Psi} \perp \text{ran } \Omega_-$ .

Pick  $\Psi$ :

Now switch to  $\Psi \in L^2_{\text{cont}}$  and  $\chi_{(a,b)}(H)\Psi = \Psi$ . Define  $\Psi_n$  via RAGE:

$$\|\chi_{[|x| \leq 2c_0 n]}(H) \underbrace{e^{-i\tau_n H} \Psi}_{\Psi_n}\|_2 \rightarrow 0$$

and

$$\int_{-n}^n \|\chi_{\{|x| \leq n\}}(H) e^{-i(\tau_n+t)H} \Psi\|_2 dt \rightarrow 0.$$

We let  $\Phi_n = \chi_{(a,b)}(H_0) \Psi_k$ . □

**Lemma.** *From last time  $\Rightarrow \|\Phi_n - \Psi_n\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ .*

$$\text{supp } \hat{\Phi}_n \subset \{\xi \in \mathbb{R}^d \mid \sqrt{200}a \leq |\xi| \leq b-a\}.$$

Phase space decomposition:

Pick  $\{\vec{e}_j\}_{j=1}^\tau \in \mathcal{S}^{d-1}$  (we are in  $\mathbb{R}^d$ ).

Fines decomposition:  $\rho \in \mathcal{S}$ ,  $\rho \geq 0$ ,  $\int \rho = 1$ ,  $\text{supp } \hat{\rho} \subset \{\xi \mid |\xi| \leq a\}$

$$1 = \sum_{\nu \in \mathbb{Z}^d} \chi_{[0,1]^d}(\cdot + \nu)$$

$$1 = \sum_{\nu \in \mathbb{Z}^d} \underbrace{\chi_{[0,1]^d}(\cdot + \nu) * \rho}_{\chi_{\nu \geq 0}}$$

$$\text{supp } \hat{\chi}_\nu \subset \{|\xi| - |\xi| \leq a\}.$$

$$\begin{aligned} \text{So } \Phi_n &= \sum_{\nu \in \mathbb{Z}^d \setminus \cup_j C_{j,n}} \chi_\nu \Phi_n + \sum_{j=1}^\tau \sum_{\nu \in C_{j,n} \cap \mathbb{Z}^d} \chi_\nu \Phi_n \\ &= \sum_{\nu \in \mathbb{Z}^d \setminus \cup_j C_{j,n}} \chi_\nu \Phi_n + \sum_{j=1}^\tau \sum_{\nu \in C_{j,n} \cap \mathbb{Z}^d} \chi_j(p) \chi_\nu \Phi_n + \sum_{j=1}^\tau \sum_{\nu \in C_{j,n} \cap \mathbb{Z}^d} \chi_j(-p) \chi_\nu \Phi_n \\ &\Rightarrow \Phi_n = \underbrace{\sum_{\nu \in \mathbb{Z}^d \setminus \cup_j C_{j,n}} \chi_\nu \Phi_n}_{\Phi_n^{(\text{trash})}} + \Phi_n^{(\text{out})} + \Phi_n^{(\text{in})} \end{aligned}$$

$$p = -i\nabla$$

$$\chi_j(p)f = (\chi_j(\xi)\hat{f}(\xi))^\vee$$

We need  $\chi_j(\xi) + \chi_j(-\xi) = 1$  for

$$(\sqrt{200}-1)a \leq |\xi| \leq (b-a) + a$$

$$\text{and } \text{supp } \chi_j \subset \{12a \leq |\xi| \leq 2b\}$$

We need  $x \cdot \xi \geq -\frac{1}{2}|x| |\xi|$  for  $\forall x \in C_{n,j}$ ,  $\forall \xi \in \text{supp } \chi_j$ . ( $\cos \angle(x_1 \xi) \geq -\frac{1}{2}$ ).

**First observation:**

$$\begin{aligned} &\left\| \sum_{\nu \in \mathbb{Z}^d \setminus \cup_{j=1}^\tau C_{j,n}} \chi_\nu \Phi_n \right\|_2 \leq \left\| \left( \sum_{\nu \in \mathbb{Z}^d \setminus \cup_{j=1}^\tau C_{j,n}} \chi_{[0,1]^d}(\cdot + \nu) * \rho \right) \Phi_n \right\|_2 \leq \\ &\leq \|\underbrace{|\Phi_n|(\chi_{\{|x| \leq c_0 n + |d|\}} * \rho)}_{O(1)}\|_2 \leq \|\underbrace{\Phi_n \chi_{\{|x| \leq 2c_0 n\}}}_{O(1)}\|_2. \end{aligned}$$

$$\cdot \underbrace{\| \chi_{(|x| \leq c_0 n + |d|)} * \rho \|_\infty}_{\leq 1} + \underbrace{\| \Phi_n \|_2}_{\|\Psi\|_2} \underbrace{\| \chi_{(|x| \leq c_0 n + \sqrt{d})} * \rho \|_{L^\infty(|x| > 2cn)}}_{O(1)}$$

$\Rightarrow \rightarrow 0$  as  $n \rightarrow \infty$

**Claim:**

$$\| \chi_{(|x| \leq n+at)} e^{-itH_0} \Phi_{n,j}^{(\text{out})} \|_2 \leq C_{a,b} (n+t)^{-N}, \forall N.$$

Side calculation:

Take:  $\nu \in C_{j,n}$ ,  $\xi \in \text{supp } \chi_j$

$$\begin{aligned} \text{Then } |\nu + t\xi|^2 &= |\nu|^2 + t^2|\xi|^2 + 2t\nu \cdot \xi \geq |\nu|^2 + t^2|\xi|^2 - t|\nu| |\xi| \\ &\geq \frac{1}{2}(|\nu|^2 + t^2|\xi|^2) \\ &\geq \frac{1}{4}(|\nu| + t|\xi|)^2 \end{aligned}$$

$$\begin{aligned} |x - \nu - t\xi| &\geq \frac{1}{2}(|\nu| + t|\xi|) - |x| \geq \frac{1}{2}(|\nu| + t|\xi|) - n - at \\ &\geq \frac{1}{2}|\nu| - \frac{|\nu|}{C_0} + 6at - at \geq \frac{1}{4}|\nu| + 5at \end{aligned}$$

$$\begin{aligned} &\| e^{-itH_0} \chi_j(p) \chi_\nu \Phi_n \|_{L^2(|x| \leq n+at)} \\ &= \left\| \int e^{i \underbrace{x \cdot \xi}_{(x-\nu)\xi + \nu \cdot \xi} - \frac{1}{2}t|\xi|^2} \chi_j(\xi) \widehat{\chi_\nu \Phi_n}(\xi) d\xi \right\|_{L^2(|x| \leq n+at)} = (1) \end{aligned}$$

Introduce

$$L = \frac{x - \nu - t\xi}{|x - \nu - t\xi|^2} \cdot \nabla_\xi \Rightarrow L(e^{i((x-\nu)\xi - \frac{1}{2}t\xi^2)}) = e^{i((x-\nu)\xi - \frac{1}{2}t\xi^2)}$$

Then

$$\begin{aligned} (1) &= \left\| \int |(L^*)^N \chi_j(\xi) (\chi_\nu \widehat{\Phi_n})(\nu + \cdot)(\xi)| d\xi \right\|_{L^2(|x| \leq n+at)} \\ &\leq C_{N,a,d} (|\nu| + t)^{-N} \sum_{|\gamma| \leq N} \left\| \partial^\gamma \chi_j(\xi) \chi_\nu \widehat{\Phi_n}(\nu + \cdot)(\xi) \right\|_{L_\xi^1} (h+t)^{\frac{d}{2}} \\ &\leq C_{N,a,d_p} (|\nu| + t)^{-N + \frac{d}{2}} \left( \sum_{|\gamma| \leq N} \left\| \partial^\gamma \chi_\nu \widehat{\Phi_n}(\nu + \cdot)(\xi) \right\|_{L_\xi^2}^2 \right)^{\frac{1}{2}} \\ &\uparrow \text{Leibnitz \& Schwartz} \\ &\leq C_{N,a,d,b} (|\nu| + t)^{-N + \frac{d}{2}} \left\| \underbrace{(1 + |x - \nu|)^N \chi_\nu \Phi_n}_{\|\cdot\|_\infty \leq \text{const. indep. of } \nu, n} \right\|_{L_x^2} \\ &\leq C_{N,a,b,d,\rho} (|\nu| + t)^{-N + \frac{d}{2}} \underbrace{\left\| \Phi_n \right\|_{L^2}}_{\leq \|\Psi\|_{L^2}} \end{aligned}$$

To get claim, sum in  $|\nu| \geq n : \sum_{|\nu| \geq n} (|\nu| + t)^{\underbrace{-N + \frac{d}{2}}_{< -d}}$ .

Important consequence:

$$\begin{aligned}
& \int_0^\infty \frac{d}{dt} (e^{itH} e^{-itH_0}) dt \\
& \left\| \left( \overbrace{\Omega_- - 1} \quad \Phi_n^{(\text{out})} \right) \right\|_2 \leq \\
& \leq \int_0^\infty \left\| V e^{-itH_0} \Phi_n^{(\text{out})} \right\|_2 dt \\
& \leq \int_0^\infty \left\| V \chi_{\{|x| \leq n+at\}} e^{-itH_0} \Phi_n^{(\text{out})} \right\|_2 dt \\
& \quad + \int_0^\infty \left\| V \chi_{\{|x| \geq n+at\}} e^{-itH_0} \Phi_n^{(\text{out})} \right\|_2 dt \\
& \leq \|V\|_\infty \int_0^\infty (n+t)^{-2} dt + \|\Psi\|_2 \int_0^\infty (n+at)^{-1-\varepsilon} dt \rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}$$

Dually:

$$\begin{aligned}
& \left\| (\Omega_+ - 1) \Phi_n^{(\text{in})} \right\|_2 \rightarrow 0 \text{ as } n \rightarrow \infty \\
& \langle \Phi_n^{(\text{in})}, \Psi_n \rangle = \langle \Omega_+ \Phi_n^{(\text{in})}, \Psi_n \rangle + o(1) \text{ as } n \rightarrow \infty \\
& \quad \parallel \\
& \quad e^{-i\tau_n H} \Psi \\
& = \langle \Omega_+ e^{i\tau_n H_0} \Phi_n^{(\text{in})}, \Psi \rangle + o(1) \\
& = \left\langle \underbrace{\chi_{\{|x| \leq n+a\tau_n\}} e^{-i(\tau_n)H_0} \Phi_n^{(\text{in})}}_{\parallel \rightarrow 0, \text{ by similar arg to "Claim"}}, \Omega_+^* \Psi \right\rangle + \\
& \quad + \left\langle \underbrace{\chi_{\{|x| \geq n+a\tau_n\}} e^{i\tau_n H_0} \Phi_n^{(\text{in})}}_{\parallel e^{i\tau_n H_0} \Phi_n^{(\text{in})} \parallel_2 \cdot \parallel \chi_{\{|x| > n+a\tau_n\}} \parallel_2 \rightarrow 0 \text{ as } n \rightarrow \infty}, \Omega_+^* \Psi \right\rangle + o(1) = o(1) \\
& \quad \parallel e^{i\tau_n H_0} \Phi_n^{(\text{in})} \parallel_2 \cdot \parallel \chi_{\{|x| > n+a\tau_n\}} \parallel_2 \rightarrow 0 \text{ as } n \rightarrow \infty \\
& \langle \Phi_n, \Psi_n \rangle = \|\Psi_n\|_2 + o(1) \\
& \langle \Phi_n^{(\text{trash})}, \Psi_n \rangle + \langle \Phi_n^{(\text{out})}, \Psi_n \rangle + \langle \Phi_n^{(\text{in})}, \Psi_n \rangle \\
& = o(1) + \langle \Omega_- \Phi_n^{(\text{out})}, \Psi_n \rangle \\
& = o(1) + \langle \Omega_- e^{i\tau_n H_0} \Phi_n^{(\text{out})}, \Psi - \tilde{\Psi} \rangle \quad (\text{as } \tilde{\Psi} \perp \text{Ran } \Omega). \\
& \Rightarrow \|\Psi_n\|^2 \leq o(1) + \|\Psi\| \cdot \|\Psi - \tilde{\Psi}\| \lesssim \varepsilon \text{ for large } n. \\
& \quad \parallel \\
& \quad \|\Psi\|^2 \\
& \Rightarrow \|\tilde{\Psi}\|_2 \leq \sqrt{\varepsilon} + \varepsilon \lesssim \sqrt{\varepsilon}.
\end{aligned}$$

Thus  $\tilde{\Psi} = 0$ .

**Long range case:**

$$|V(x)| \leq C(1 + |x|)^{-\alpha}$$

(no longer require  $\alpha > 1$ ) and :

$$|\partial^\alpha V(x)| \leq C(1 + |x|)^{-\alpha-|\gamma|}$$

for all  $|\gamma| \leq k$ .

**Theorem 3.** (*Buslaev-Matveev*). Assume  $(LR)_k$  for  $k = \lfloor \frac{d}{2} \rfloor + 2$  and  $\alpha > \frac{1}{2}$ . Then

$$W_-(H_1 H_0) := \text{s-lim}_{t \rightarrow \infty} e^{itH} \exp(-itH_0 - i \int_0^t V(p\tau) d\tau)$$

exists : modified wave operator

### Remarks

- (1)  $\exp(-i \int_0^t V(p\tau) d\tau) f(\xi) = \exp(-i \int_0^t V(\xi\tau) d\tau) \hat{f}(\xi)$
- (2) If  $\alpha > 1$ ; then  $W_-(H_1 H_0) = \Omega_-(H_1 H_0) \exp(-i \int_0^\infty V(p\tau) d\tau)$  where the final operator is well-defined:

$$\int_0^\infty V(\xi\tau) d\tau = \frac{1}{|\xi|} \int_0^\infty \left( V \frac{\xi}{|\xi|} \tau \right) d\tau$$

- (3) Clearly:  $\| W_-(H_1 H_0) f \|_2 = \| f \|_2$

$$\begin{aligned} &= e^{isH} W_-(H_1 H_0) = \text{s-lim}_{t \rightarrow \infty} e^{i(t+s)H} \exp(-i(t+s)H_0 - i \int_0^{t+s} V(p\tau) d\tau) \cdot \\ &\quad \cdot \exp(+isH_0 + i \int_t^{t+s} V(p\tau) d\tau) \end{aligned}$$

Clearly:

$$\begin{aligned} &\| \exp(i \int_t^{t+s} V(p\tau) d\tau) f - f \|_2 = \\ &= \| (e^{i \int_t^{t+s} V(\xi\tau) d\tau} - 1) \hat{f}(\xi) \|_2 \rightarrow 0 \text{ by Dominate conv. theorem} \\ &\Rightarrow e^{isH} W_-(H_1 H_0) = W_-(H_1 H_0) e^{isH_0} \end{aligned}$$

□

2/20/2003

Last time: long range potentials

$$\begin{aligned} (L - R)_k \quad & |\partial^\gamma V(x)| \leq C_\gamma (1 + |x|)^{-\alpha-|\gamma|} \\ & 0 \leq |\gamma| \leq k \end{aligned}$$

**Theorem 4.** (*Buslaev-Matveev*)

Assume  $(L - R)_k$  for large enough  $k$  depending on the dimension  $d$ , and  $\alpha > 1/2$ . Then the modified wave operators

$$W_-(H_1 H_0) = \text{s-lim}_{t \rightarrow \infty} e^{itH} e^{-itH_0 - i \int_0^t V(\tau p) d\tau}$$

exists. Moreover, they are isometries and have the intertwining property for  $e^{-itH}$  and  $e^{-itH_0}$ .

**Remarks**

- (1) if  $\alpha > 1$  nothing new here.
- (2) motivation, classically

$$\begin{aligned} H_{cl} &= \frac{1}{2}p^2 + V(x) \approx \frac{1}{2}p^2 + V(tp) & x &\approx tp \\ H &\approx -\frac{1}{2}\Delta + V(tp) & p &= -i\nabla \\ U(t) &= e^{i(-\frac{1}{2}\Delta t + \int_0^t V(\tau p) d\tau)} \\ &= e^{i(tH_0 + \int_0^t V(\tau p) d\tau)} \end{aligned}$$

- (3) Why  $\alpha > \frac{1}{2}$ ? Take  $V(x) = (1 + |x|)^{-\alpha}$   
 $0 < \alpha < 1$

$$\begin{cases} \dot{x} = \xi \\ \dot{\xi} = -\nabla V(x) \end{cases}$$

$$\begin{aligned} \xi(t) &= \eta + \int_0^t -\nabla V(x(\tau)) d\tau \\ &= \eta - \int_0^\infty \nabla V(x(\tau)) d\tau + \underbrace{\int_t^\infty \nabla V(x(\tau)) d\tau}_{o(t^{-\alpha})} \end{aligned}$$

$$\begin{aligned} x(t) &= t\xi_\infty + o(t^{1-\alpha}) \\ &= t\xi(t) + o(t^{1-\alpha}) \end{aligned}$$

$$V(x) - V(t\xi) = o(t^{-1-\alpha})t^{1-\alpha} = o(t^{-2\alpha})$$

this is integrable if  $\alpha > 1/2$

- (4) More generally: What if  $\alpha \leq \frac{1}{2}$ ? Ansatz:

$$W_-(H_1 H_0) = \text{s-lim}_{t \rightarrow \infty} e^{itH} e^{-iW(t,p)} \quad \text{is isometry}$$

$$W(t, \xi) \in \mathbb{R}$$

$$\begin{aligned} e^{isH} W_-(H_1 H_0) e^{-isH_0} &= \text{s-lim}_{t \rightarrow \infty} e^{i(s+t)H} e^{-iW(t+s,p)} e^{i[W(t+s,p) - W(t,p) - s\frac{p^2}{2}]} \\ &= W_-(H_1 H_0) \quad \text{provided we have} \end{aligned}$$

$$\lim_{t \rightarrow \infty} W(t+s, \xi) - W(t, \xi) - s \frac{\xi^2}{2} = 0$$

How to choose the function  $W$ ?

$$\begin{aligned} \text{Cook's method} \quad & \int_1^\infty \left\| \frac{d}{dt} e^{itH} e^{-iW(t,p)} f \right\|_{L^2} dt \stackrel{?}{<} \infty \\ & = \int_1^\infty \left\| (V(x) - \partial_t W(t,p)) e^{-iW(t,p)} f \right\|_{L_x^2} dt \\ & = \int_1^\infty \left\| \int_{\mathbb{R}^d} e^{i(x\xi - W(t,\xi))} (V(x) - \partial_t W(t,\xi)) \hat{f}(\xi) d\xi \right\|_{L_x^2} dt \end{aligned}$$

stationary phase should vanish at critical point:  $x - \nabla_\xi W(t, \xi) = 0$

leads to: Hamilton-Jacobi equation

$$\frac{1}{2} \xi^2 + V(\nabla_\xi W(t, \xi)) - \partial_t W(t, \xi) = 0$$

$$\text{Characteristic ODE: } \begin{cases} \dot{x} = \xi \\ \dot{\xi} = -\nabla V(x) \end{cases}$$

$$\text{first approx: } W^{(0)}(t, \xi) = \frac{1}{2} + \xi^2$$

$$W^{(1)}(t, \xi) = W^{(0)}(t, \xi) + \int_0^t \underbrace{V(\nabla_\xi W^{(0)}(\tau, \xi))}_{\tau\xi} dt$$

We are working with the  $W^{(1)}$  approx.

- (5) Let  $\alpha = 1$ ,  $V(x) = (1 + |x|)^{-1}$ . Assume  $W_-(H_1 H_0)$  exists. Then  $(*) = \text{s-lim}_{t \rightarrow -\infty} e^{-itH_0} e^{itH} P_{ac}$  does not exist.

*Proof.* Let  $M(t) = e^{-i \int_0^t V(\tau p) d\tau}$ . Assume  $(*)$  exists.

$$\begin{aligned} g &= P_{ac} A, \langle e^{itH} e^{-itH_0} M(t) f, g \rangle \\ &= \langle M(t) f, \underbrace{e^{itH_0} e^{-itH} P_{ac} g}_{\xrightarrow{\text{strongly } h}} \rangle \\ &= \langle M(t) f, h \rangle + o(1) \text{ as } t \rightarrow \infty \\ &= \int e^{-i \int_0^t V(\tau\xi) d\tau} \hat{f}(\xi) \overline{\hat{h}(\xi)} d\xi \\ &= \int_{\mathbb{R}^d} \exp\left(-i \frac{\log(1+t|\xi|)}{|\xi|}\right) \hat{f}(\xi) \overline{\hat{h}(\xi)} d\xi \\ &\longrightarrow 0 \end{aligned}$$

claim

It suffices to check that:

$$\int_0^{\infty} \exp\left(-i \frac{\log(1+t\tau)}{\tau}\right) \phi(\tau) d\tau \longrightarrow 0$$

for  $\phi \in C_0^\infty(0, \infty)$

$$= \int_0^{\infty} \exp\left(-i \frac{\log t}{\tau}\right) e^{-i \frac{\log \tau}{\tau}} \exp\left(-\frac{i}{\tau} \log\left(1 + \frac{1}{t\tau}\right)\right) \phi(\tau) d\tau$$

this term tends to 0, using  $\tau \rightarrow \frac{1}{\tau}$  and integration by parts.  $\square$

*Proof.* of Theorem: Apply Cook's method

$$\Phi(t, x, \xi) = x \cdot \xi - \frac{1}{2} t \xi^2 - \int_0^t V(\tau \xi) d\tau$$

We need to check that:

$$\infty > \int_1^{\infty} \left\| \int_{\mathbb{R}^d} e^{i\Phi(t,x,\xi)} (V(x) - \int_0^t V(\tau \xi) d\tau) \hat{f}(\xi) d\xi \right\|_{L_x^2} dt$$

Here  $f \in \mathcal{S}_1$ ,  $\text{supp } \hat{f} \subset \{0 < a < |\xi| < b\}$

Classically forbidden region.

Note:

$$\nabla_\xi \Phi = x - t\xi - \int_0^t \nabla V(\tau \xi) \tau d\tau$$

$$(1) \quad |x| < \frac{a}{2} t$$

$$\begin{aligned} |\nabla_\xi \Phi(t, x, \xi)| &\geq at - \frac{a}{2} t - \int_0^t |\nabla V(\tau \xi)| \tau d\tau \\ &\geq \frac{a}{2} t - C \int_0^t (1 + \tau a)^{-\alpha-1} \tau d\tau \\ &\geq \frac{at}{2} - \frac{C}{a^2} (1 + ta)^{1-\alpha} \\ &\geq \frac{a}{4} t \quad \text{if } t \geq t_0(f, V) \end{aligned}$$

integrate by parts  $N$  times  $\Rightarrow$

$$L_\xi = \frac{\text{diff. operator}}{|\nabla\Phi|^2} \cdot \nabla_\xi$$

$$\left| \int_{\mathbb{R}^d} e^{i\Phi(t,x,\xi)} \left( V(x) - \int_0^t V(\tau\xi) d\tau \right) \hat{f}(\xi) d\xi \right| \geq Ct^{-N} t^{1-\alpha}$$

Side remark

$$\left| \int_0^t \nabla V(\tau\xi) \tau d\tau \right| \lesssim t^{1-a}$$

$$\left| \partial_\xi^\gamma \int_0^t \nabla V(\tau\xi) \tau d\tau \right| \lesssim t^{1-\alpha}$$

$L$  chosen in such a way:

$$Le^{i\phi} = e^{i\phi}$$

Aside. Go back to

$$\| \dots \|_{L^2(|x| < \frac{a}{2}t)} \lesssim t^{\frac{d}{2}} t^{-N+1-\alpha}$$

You need  $N \geq \lfloor \frac{d}{2} \rfloor + 2$

Forbidden region (2)  $|x| > 2bt$

$$|\nabla\Phi(t,x,\xi)| \geq |x| - tb - Ct^{1-a} \geq \frac{1}{4}|x|$$

$$\left| \int_{\mathbb{R}^d} e^{i\Phi(t,x,\xi)} \left( V(x) - \int_0^t V(\xi) \right) \hat{f}(\xi) d\xi \right|$$

$$\leq \int_{a < |\xi| < b} \left| (L_\xi^*)^N \left( V(x) - \int_0^t V(\tau\xi) d\tau \right) \hat{f}(\xi) d\xi \right|$$

$$\geq (1 + |x|)^{-N} t^{1-\alpha}$$

$$\| \dots \|_{L^2(|x| > 2bt)} \lesssim t^{1-\alpha} \left( \int_{|x| > 2bt} (1 + |x|)^{-2N} dx \right)^{\frac{1}{2}}$$

$$\lesssim t^{1-\alpha} t^{-N + \frac{d}{2}} \begin{cases} t^{-1-\alpha} \\ t^{-\frac{1}{2}-\alpha} \end{cases}$$

Note:

$$N = \left\lfloor \frac{d}{2} \right\rfloor + 2 \begin{cases} \frac{d}{2} + 2 & \text{if even} \\ \frac{d}{2} + 2 - \frac{1}{2} & \text{if odd} \end{cases}$$

(Classically allowed region (3) important case

$$\int_1^\infty \left\| \int_{\mathbb{R}^d} e^{i\Phi(t,x,\xi)} \left( V(x) - \int_0^t V(\tau\xi) d\tau \right) \hat{f}(\xi) d\xi \right\|_{L^2(|x|\sim t)} dt$$

*Intuition* critical point  $\xi_0(t, x)$

$$x - t\xi_0 - \int_0^t \nabla V(\tau\xi_0) \tau d\tau = 0$$

$$\left| V(x) - \int_0^t V(\tau\xi_0) d\tau \right| \lesssim t^{-2a}$$

$$\text{stationary phase} \Rightarrow \int_1^\infty t^{\frac{d}{2}} t^{-\frac{d}{2}} t^{-2a} dt < \infty$$

Aside: Stationary phase:

$$\left| \int e^{it\Psi(\xi)} a(t, \xi) d\xi \right| \lesssim$$

$$\left( \begin{array}{l} \text{we are in a non-linear} \\ \text{case/never mind} \end{array} \right) \lesssim t^{-\frac{d}{2}} |a(t, \xi_0)|$$

where  $\xi_0$  is the critical point.

Pick a smooth characteristic function  $\phi$  around 0

$$\delta = t^{-\frac{1}{2}}$$

$$\int_{\mathbb{R}^d} e^{i\Phi} \underbrace{\left( V(x) - \int_0^t V(\tau\xi) d\tau \right)}_{\omega} \hat{f}(\xi) d\xi =$$

$$\int_{\mathbb{R}^d} e^{i\Phi} \phi\left(\frac{\xi - \xi_0(t, x)}{\delta}\right) \omega(x, t, \xi) d\xi + \int_{\mathbb{R}^d} e^{i\Phi} \left( 1 - \phi\left(\frac{\xi - \xi_0(t, x)}{\delta}\right) \right) \omega d\xi$$

Justify the existence of the critical point:

$$x - t\xi - \int_0^t \nabla V(\tau\xi) \tau d\tau = 0$$

solve this for fixed  $t, |x| \sim t$

$$x - t\psi(t\xi) = 0 \text{ for } a < |\xi| < b$$

$$D\psi(t, \xi) = Id + \underbrace{\frac{1}{t} \int_0^t D^2V(\tau\xi)\tau^2 d\tau}_{o(t^{-a})}$$

$$\psi(t, \xi) = \xi + \frac{1}{t} \int_0^t \nabla V(t\xi)\tau d\tau$$

□

PDE (Schlag) 02/25/03

From last time we need to prove that

$$\int_1^\infty \left\| \int e^{i\Phi(t,x,\xi)} (V(x) - V(t\xi)) d\xi \right\|_{L_x^2} dt < \infty$$

we did the “classically forbidden” region

- (1)  $|x| < \frac{at}{2}$
- (2)  $|x| > 2bt$

$$f \in \xi \sup \hat{f} \in \{\xi : 0 < |a| < |\xi| \subset b\}$$

We had  $L = \frac{-i\nabla_\xi \Phi}{|\nabla_\xi \Phi|^2} \cdot \nabla$ ,  $L \cdot e^{i\Phi(t,x,\xi)} = e^{i\Phi(t,x,\xi)}$  then do integral by part.

$$L^* = i \operatorname{div} \left( \frac{\nabla_\xi \Phi}{|\nabla_\xi \Phi|^2} \cdot \right)$$

- (3) critical point in  $\frac{at}{2} < |x| < 2bt$

$$\nabla_\xi \Phi = 0 \quad \Phi = x\xi - \frac{t \cdot |\xi|^2}{2} - \int_0^t V(\xi\tau) d\tau$$

$$\nabla_\xi \Phi = x - t\xi - \int_0^t \nabla V(\xi\tau)\tau d\tau = 0$$

$$D_\xi^2 \Phi = -tI_d - \int_0^t D^2V(\xi\tau)\tau^2 d\tau$$

 $t$  fixed as large number. $B_j$  = radius independent on  $t$ .

Consider the rescaled map

$$\Phi_t : \xi \rightarrow \frac{x}{t}$$



*Proof.*

$$N = 1 \quad \partial_j \left( \frac{\partial_j \Phi}{|\nabla \Phi|^2} f \right) = \frac{\Delta \Phi}{|\nabla \Phi|^2} g + \frac{\partial_j \Phi \cdot \partial_j g}{|\nabla \Phi|^2} - 2 \frac{\partial_j \Phi \partial_{ji} \Phi}{|\nabla \Phi|^4} \cdot \partial_i \Phi g$$

$$\begin{array}{ccc} \beta = 0 & m = 1 & \beta = 1 \\ m = 0 & \beta = 0 & m = 0 \\ 2 = 2 & 1 + 2 + 1 = 4 & 1 + 1 = 2 \end{array}$$

by induction

$$\begin{aligned} & \partial_j \left( \frac{\partial_j \Phi}{|\nabla \Phi|^2} \cdot \frac{\partial^{\gamma_1} \Phi \dots \partial^{\gamma_{N+2m}} \Phi}{|\nabla \Phi|^{2N+2m}} \partial^\beta g \right) \\ &= \left( \frac{\Delta \Phi}{|\nabla \Phi|^2} - \frac{2 \partial_j \Phi \partial_{ji} \Phi \partial_i \Phi}{|\nabla \Phi|^4} \right) \frac{\partial^{\gamma_1} \Phi \dots \partial^{\gamma_{N+2m}} \Phi}{|\nabla \Phi|^{2N+2m}} \partial^\beta g \\ &+ \frac{\partial_j \Phi}{|\nabla \Phi|^2} \frac{\partial^{\gamma_1 + e_j} \Phi \dots \partial^{\gamma_{N+2m}} \Phi}{|\nabla \Phi|^{2N+2m}} \partial^\beta g \\ &+ \frac{\partial_j \Phi}{|\nabla \Phi|^2} \frac{\partial^{\gamma_1} \Phi \dots \partial^{\gamma_{N+2m}} \Phi}{|\nabla \Phi|^{2N+2m}} \partial^{\beta+1} g \\ &+ \frac{\partial_j \Phi}{|\nabla \Phi|^2} \frac{\partial^{\gamma_1} \Phi \dots \partial^{\gamma_{N+2m}} \cdot \partial_i \Phi \partial_{ji} \Phi}{|\nabla \Phi|^{2N+2(m+1)}} \partial_g^\beta \end{aligned}$$

check the  $|\partial_1| + \dots + |\gamma_{N+2m}| + |\beta| = 2N + 2m$  □

Let  $\lambda = \#\{1 \leq j \leq N + 2m = |\gamma_j| : 1\}$

then  $\lambda + 2(N + 2m - \lambda) + |\beta| \leq 2N + 2m \Rightarrow 2m + |\beta| \leq \lambda$

First case  $\beta = 0$   $x - t\xi = x - t\xi_0 + t(\xi_0 - \xi)$

$$\begin{aligned} & \int_{|\xi - \xi_0| > \delta} \frac{t^{N+2m-\lambda}}{|\nabla \Phi|^{2N+2m-\lambda}} t^{-1\alpha} (t^{-1-\alpha} + t(\xi_0 - \xi)) d\xi \\ & \lesssim \int_{|\xi - \xi_0| < \delta} \frac{t^{N+2m-\lambda} \cdot t^{-\alpha}}{t^{N+2m-\lambda} |\xi - \xi_0|^{2N+2m-\lambda-1}} d\xi \lesssim t^{-N} \cdot t^{-\alpha} \delta^{-2N-2m+\lambda+1} \cdot \delta^d \\ & \leq t^{-N} \delta^d \delta^{-2N+1} \cdot t^{-\alpha} \leq t^{-\frac{d}{2}} t^{-\alpha - \frac{1}{2}} \end{aligned}$$

similar for other term

Commutator methods

Proposition

$$F(x) = \sum_{|\alpha| \leq N} a_\alpha x^d \quad \text{polynomial}$$

Then

$$\begin{aligned} e^{-i\frac{t}{2}\Delta} F(x) e^{i\frac{t}{2}\Delta} &= e^{-i\frac{|x|^2}{2t}} F(tx) e^{i\frac{|x|^2}{2t}} \\ &= F(x + tx) \end{aligned}$$

*Proof.* Suffice to consider  $\dim d = 1$

$$e^{-i\frac{t}{2}\Delta} x_1^{\alpha_1} \cdot x_d^{\alpha_d} e^{i\frac{t}{2}\Delta} = e^{-i\frac{t}{2}\partial_1^2} x_1^{\alpha_1} e^{\frac{it}{2}\partial_1^2} e^{-i\frac{t}{2}\partial_2^2} \dots$$

$$\underbrace{e^{-i\frac{t^2}{2}\partial_d^2} x_d \partial_d \cdot e^{i\frac{t^2}{2}\partial_d^2}}_{(x_d+tp_d)^{\alpha_d}}$$

$$= \pi(x_j + tp_j)^{\alpha_j}$$

$$(*) = (e^{-i\frac{t}{2}\Delta} F(x) e^{i\frac{t}{2}\Delta} f)(x)$$

$$= \iint \frac{1}{(-2\pi it)^{\frac{1}{2}}} e^{-i\frac{|x-u|^2}{2t}} F(u) \frac{e^{\frac{i}{2t}|u-y|^2}}{(2\pi it)^{\frac{d}{2}}} f(y) dy du$$

$$= (2\pi)^{-d} t^{-d} e^{-i\frac{|x|^2}{2t}} \iint \underbrace{e^{-\frac{i}{t}u(-x+y)} F(u) du}_{\hat{F}\left(\frac{y-x}{t}\right)} e^{\frac{i}{2t}|y|^2} f(y) dy$$

$$\left( \begin{array}{l} \text{if } F(x) = x_1^{\alpha_1} \cdots x_d^{\alpha_d} = x^\alpha \\ \hat{F}(\xi) = (2\pi)^d (i d_\xi)^\alpha \delta_0 \end{array} \right)$$

$$t^{-d} \int_{R^d} \hat{F}\left(\frac{y-x}{t}\right) \psi(y) dy = \int_{R^d} \hat{F}\left(y - \frac{x}{t}\right) \psi_{(ty)} dy$$

$$= (2\pi)^d \int \delta_0\left(y - \frac{x}{t}\right) ((-it\partial_y)^\alpha \psi)_{(ty)} dy$$

$$= (2\pi)^d (tp)^\alpha \psi(x)$$

$$\text{So } (*) = (e^{-\frac{i|x|^2}{2t}} F(tp) e^{-i\frac{|x|^2}{2t}} f)(x)$$

Check:

$$e^{-i\frac{t}{2}\frac{d^2}{dx^2}} x^\lambda e^{i\frac{t}{2}\frac{d^2}{dx^2}} = (x+tp)^\lambda$$

Induction:  $\lambda = 0 \quad \checkmark$

$$i \frac{d}{dt} (\lambda h s) = e^{-\frac{i}{2}t\Delta} \frac{1}{2} [\Delta, x^\lambda] e^{i\frac{t}{2}\Delta}$$

$$= e^{-i\frac{t}{2}\Delta} \left( \frac{\lambda(\lambda-1)}{2} x^{\lambda-2} + \lambda x^{\lambda-1} \frac{d}{dx} e^{i\frac{t}{2}\Delta} \right)$$

$$= \frac{\lambda(\lambda-1)}{2} (x+tp)^{\lambda-2} + \lambda(x+tp)^{\lambda-1} \frac{d}{dx}$$

$$i \frac{d}{dt} (\gamma h s) = \frac{d}{dx} (x+tp)^{\lambda-1} + (x+tp) \frac{d}{dx} (x+tp)^{\lambda-2} + \cdots + (x+tp)^{\lambda-1} \frac{d}{dx}$$

$$\frac{d}{dx} (x+tp)^m = m(x+tp)^{m-1} + (x+tp)^m \frac{d}{dx}$$

By induction

$$\frac{d}{dx} (x+tp)^{m+1} = \left( it(x+tp) \frac{d}{dx} \right) (x+tp)^m$$

$$= (x+tp)^m + m(x+tp)^m + (x+tp)^{m+1} \frac{d}{dx}$$

□

Plug in

$$i \frac{d}{dt}(\text{rhs}) = \frac{l(\lambda - 1)}{2}(x + tp)^{l-2} + l(x + tp)^{\lambda-1} \frac{d}{dx}$$

Corollary

$$\| |x - tp| e^{it\frac{\Delta}{2}} \phi \|_2^2 = \| |x| \phi \|_2^2$$

*Proof.*

$$\langle |x - tp|^2 e^{it\frac{\Delta}{2}} \phi, e^{it\frac{\Delta}{2}} \phi \rangle = \langle |x|^2 \phi, \phi \rangle$$

□

Corollary

- (1) Suppose  $(1 + |x|)\psi_0 \in L^2$  and  $\nabla\psi_0 \in L^2$   
then

$$\| (1 + |x|) e^{\frac{it}{2}\Delta} \psi_0 \|_2 \leq (1 + |t|)$$

- (2) Suppose  $\sum_{|\alpha|+|\beta|\leq m} \| x^\alpha \partial^\beta \psi_0 \|_2 < \infty$   
then

$$\| |x|^m e^{\frac{it}{2}\Delta} \psi_0 \|_2 \lesssim (1 + |t|)^m$$

**2/27/2003**

Last time: Pseudo-conformal identity

$$\begin{aligned} \langle |x - tp|^2 \psi(t), \psi(t) \rangle &= \text{const} \\ &= \langle |x|^2 \psi(0), \psi(0) \rangle \\ \text{if } i\partial_t \psi + \frac{1}{2}\Delta\psi &= 0 \end{aligned}$$

**Corollary.**

- (i) Suppose  $(1 + |x|)\psi \in L^2$  and  $\psi \in H^1$ . Then

$$\| (1 + |x|) e^{\frac{it}{2}\Delta} \psi \|_2 \lesssim (1 + |t|).$$

- (ii) Suppose  $\sum_{|\alpha|+|\beta|\leq m} \| x^\alpha \partial^\beta \psi \|_2 < \infty$ . Then

$$\| |x|^m e^{\frac{it}{2}\Delta} \psi \|_2 \lesssim (1 + |t|)^m$$

*Proof.*

$$\begin{aligned} \| x^\alpha e^{\frac{it}{2}\Delta} \psi \|_2 &= \| e^{-\frac{it}{2}\Delta} x^\alpha e^{\frac{it}{2}\Delta} \psi \|_2 \\ &= \| (x + tp)^\alpha \psi \|_2 \\ &\leq (1 + |t|)^{|\alpha|} \sum_{|\beta|+|\gamma|\leq|\alpha|} \| x^\beta \partial^\gamma \psi \|_2 \end{aligned}$$

□

**Proposition** Suppose  $x^\alpha \phi \in L^2 \forall |\alpha| \leq k$ . Then  $\partial^\alpha e^{-i\frac{t}{2}\Delta} \phi \in L^2$  for  $\forall |\alpha| \leq k$  and  $t \neq 0$ . (Gain of derivatives, cfr. comp. supp. init. cond. becomes analytic when  $t > 0$ ).

*Proof.*

$$\begin{aligned}
e^{-\frac{it}{2}\Delta} x^\alpha e^{\frac{it}{2}\Delta} &= e^{-i|x|^2/2t} (tp)^\alpha e^{i|x|^2t/2t} \\
\infty > \|x^\alpha \phi\|_2 &= \|e^{i\frac{t}{2}\Delta} e^{-i\frac{|x|^2}{2t}} (tp)^\alpha e^{\frac{i|x|^2}{2t}} e^{-i\frac{t}{2}\Delta} \phi\|_2 \\
&= \|(tp)^\alpha e^{\frac{i|x|^2}{2t}} e^{-\frac{it}{2}\Delta} \phi\|_2 \\
&\Rightarrow \|\partial^\beta e^{-\frac{it}{2}\Delta} \phi\|_2 < \infty \text{ if } t \neq 0, |\beta| \leq k
\end{aligned}$$

□

Now  $H = -\frac{1}{2}\Delta + V$ .

Make operators commute through the evolution.

$$\underbrace{Ae^{-itH} = e^{-itH}A}_{\substack{\text{globally} \\ \text{(Lie Group)}}} + i \int_0^t e^{-i(t-s)H} \underbrace{[H, A]}_{\substack{\text{infinitesimally} \\ \text{(Lie Algebra)}}} e^{-isH} ds.$$

$$\begin{aligned}
\text{Because } \frac{d}{dt} e^{itH} A e^{-itH} &= e^{itH} i(HA - AH) e^{-itH} \\
&= i e^{itH} [H, A] e^{-itH} \\
\Rightarrow e^{iTH} A e^{-iTH} - A &= i \int_0^T e^{itH} [H, A] e^{-itH} dt.
\end{aligned}$$

**Lemma.** Suppose  $V \in L^\infty$ ,  $H = -\frac{1}{2}\Delta + V$

- (i) If  $(1 + |x|)\phi \in L^2$  and  $\phi \in H^1$  then
$$\|(1 + |x|)e^{-itH}\phi\|_2 \lesssim (1 + |t|).$$
- (ii) If  $\sum_{|\alpha|+|\beta|\leq 2} \|x^\alpha \partial^\beta \psi\|_2 < \infty$ , then
$$\|(1 + |x|)^2 e^{-itH}\psi\|_2 \lesssim (1 + |t|)^2.$$

*Proof.*

$$(i) \quad x_j e^{-itH} \phi = e^{-itH} x_j \phi + i \int_0^t e^{-i(t-s)H} [H, x_j] e^{-isH} \phi ds$$

$$\text{with } [H, x_j] = [-\frac{1}{2}\Delta + V_j, x_j] = [-\frac{1}{2}\Delta, x_j] = -\partial_j$$

$$(1) \quad \|x_j e^{-itH} \phi\|_2 \leq \|x_j \phi\|_2 + \int_0^t \|\partial_j e^{-isH} \phi\| ds$$

Take  $M$  large enough,  $\|\partial_j(H + M)^{-\frac{1}{2}}\|_{2 \rightarrow 2} < \infty$  because obviously  $\|\Delta(H + M)^{-1}\|_{2 \rightarrow 2} < \infty$  (resolvent  $\rightarrow$  gains 2 derivatives) then apply complex interpolation: we have

$$\sup_\sigma \|(-\Delta)^{1+i\sigma} (H + M)^{-1-i\sigma}\|_{2 \rightarrow 2} \leq C_M$$

$$\sup_{\sigma} \| (-\Delta)^{i\sigma} (H + M)^{-i\sigma} \|_{2 \rightarrow 2} \leq C_M .$$

*Proof.*

$$\begin{aligned} & \| \underbrace{(-\Delta)^{i\sigma}}_{\text{bdd from } 2 \rightarrow 2 \text{ (multipliers)}} \underbrace{(-\Delta)(H + M)^{-1}}_{\text{OK}} \underbrace{(H + M)^{-i\sigma}}_{\text{bdd from } 2 \rightarrow 2 \text{ (multipliers)}} \|_{2 \rightarrow 2} \\ & \leq \| (-\Delta)(H + M)^{-1} \|_{2 \rightarrow 2} < \infty . \end{aligned}$$

□

then  $T_z = (-\Delta)^z (H + M)^{-z}$ ,  $T_{\frac{1}{2}} : L^2 \rightarrow L^2$ .

$$\begin{aligned} \therefore (1) & \leq \| |x|\phi \|_2 + \int_0^t \| \partial_j (H + M)^{-\frac{1}{2}} \|_{2 \rightarrow 2} \| e^{-iHs} (H + M)^{\frac{1}{2}} \phi \|_2 ds \\ & \leq \| |x|\phi \|_2 + C.t. \| \phi \|_{H^1} \end{aligned}$$

(ii)  $|\alpha| = 2$

$$\begin{aligned} [H, x^\alpha] &= \left[ -\frac{1}{2} \Delta, x^\alpha \right] \\ &= -\frac{1}{2} \Delta(x^\alpha) - \nabla(x^\alpha) \cdot \nabla \end{aligned}$$

$$\begin{aligned} \| x^\alpha e^{-itH} \phi \|_2 &\leq \| |x|^2 \phi \|_2 + C \int_0^t \| \phi \|_2 ds \\ &\quad + C \max_{j,l} \int_0^t \| x_j \partial_e^{-isH} \phi \|_2 ds \end{aligned}$$

$$\begin{aligned} \text{Use } & \| [(H + M)^{\frac{1}{2}}, x_j] \|_{2 \rightarrow 2} < \infty . \\ & \| [\partial_e, x_j] \|_{2 \rightarrow 2} < \infty \end{aligned}$$

because  $[(H + M)^{\frac{1}{2}}, x_j] = ?$

$$(H + M)^{-\frac{1}{2}} \stackrel{?}{=} C \int_0^\infty (H + M + \lambda)^{-1} \frac{d\lambda}{\sqrt{\lambda}}$$

(integr. at zero)  $\left(\frac{1}{\sqrt{\lambda}}\right)$  and at  $\infty$   $(\lambda^{-\frac{3}{2}})$

$$= C \int_0^\infty \int_{-\frac{M}{2}}^\infty \frac{1}{\tau + M + \lambda} E(d\tau) \frac{d\lambda}{\sqrt{\lambda}}$$



*Proof.*

$$\begin{aligned}
 \frac{d^2}{dt^2} \int |x|^2 |\psi(t, x)|^2 dx &= \frac{d^2}{dt^2} \langle |x|^2 e^{\frac{i}{2}t\Delta} \psi_0, e^{\frac{i}{2}t\Delta} \psi_0 \rangle \\
 &= \frac{d^2}{dt^2} \langle e^{-\frac{it}{2}\Delta} |x|^2 e^{\frac{i}{2}t\Delta} \psi_0, \psi_0 \rangle \\
 &= \frac{d}{dt} \left( -\frac{i}{2} \right) \langle e^{-i\frac{t}{2}\Delta} [\Delta, |x|^2] \cdot e^{\frac{i}{2}t\Delta} \psi_0, \psi_0 \rangle \\
 (1) \quad &= \left( -\frac{i}{2} \right)^2 \langle [\Delta, [\Delta, |x|^2]] \psi_0, \psi_0 \rangle
 \end{aligned}$$

but

$$\begin{aligned}
 [\Delta, |x|^2] &= \Delta(|x|^2) + 2\nabla(|x|^2) \cdot \nabla \\
 &= 2d + 4 \underbrace{x \cdot \nabla}_{\substack{\text{generator} \\ \text{of dilations}}} \\
 [\Delta, [\Delta, |x|^2]] &= 4[\Delta, x \cdot \nabla] \\
 &= 8\nabla(x \cdot \nabla) \cdot \nabla = 8\Delta.
 \end{aligned}$$

$$(1) = -2\langle \Delta\psi, \psi \rangle = 2 \|\nabla\psi\|_2^2 \quad \square$$

Semilinear equations:

$$\begin{cases} \psi|_{t=\infty} = \psi_0 \\ i\psi_t + \frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi = 0 \end{cases}$$

Questions:

- well posedness: existence, uniqueness, continuous dependence on initial data? ( $H^s$  spaces)
- local vs. global well-posedness
- conservation laws (symmetries of the equation) (e.g.,  $\psi \rightarrow e^{i\beta}\psi$ )
- blow up
- special solutions—solitons and stability around special solutions

Conservation laws and symmetries

$$1) \quad \frac{d}{dt} \|\psi(t)\|_2^2 = 0$$

$$\begin{aligned}
 \frac{d}{dt} \langle \psi(t), \psi(t) \rangle &= 2\operatorname{Re} i \langle -i\psi_t, \psi \rangle \\
 &= 2\operatorname{Re} i \left\langle \frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi, \psi \right\rangle \\
 &= 2\operatorname{Re} i \left\langle \frac{1}{2}\Delta\psi, \psi \right\rangle = 0 \quad (V \text{ real})
 \end{aligned}$$

$$2) \quad H(t) = \int \left[ \frac{1}{2} |\nabla\psi(t, x)|^2 - V|\psi|^2 - F(|\psi|^2) \right] dx$$

where  $F'(u) = f(u)$ .

Then  $\frac{d}{dt} H(t) = 0$ .

$$\begin{aligned}\dot{H}(t) &= 2\text{Re} \int \left( \frac{1}{2} \nabla \psi_t \nabla \bar{\psi} - V \psi_t \bar{\psi} - f(|\psi|^2) \psi_t \bar{\psi} \right) dx \\ &= 2\text{Re} \int \psi_t \underbrace{\left( -\frac{1}{2} \Delta \bar{\psi} - V \bar{\psi} - f(|\psi|^2) \right)}_{-i\bar{\psi}_t} dx = 0\end{aligned}$$

$$3) \quad X = \int x |\psi|^2 dx / \int |\psi|^2 dx \quad \text{center of mass}$$

$$NX = \int x |\psi|^2 dx \quad N = \text{charge}$$

$$\begin{aligned}N\dot{X} &= 2\text{Re} \int x \psi_t \bar{\psi} \\ &= 2\text{Re} i \int x \left( \frac{1}{2} \Delta \psi + V \psi + f(|\psi|^2) \psi \right) \bar{\psi} dx \\ &= 2\text{Re} i \int x \Delta \psi \bar{\psi} dx \\ &= \text{Re} i \int x \text{div}(\nabla \psi \cdot \bar{\psi}) dx \\ &= -\text{Re} i \int \nabla \psi \cdot \bar{\psi} dx = \langle p\psi, \psi \rangle = \vec{P}\end{aligned}$$

where  $p = -i\nabla$  s.a.

$$N\dot{X} = \vec{P} \text{.without potential we will have } \dot{\vec{P}} = 0$$

### 3/4/2003

Last time we started doing some formal calculations concerning properties of solution of semilinear Schrödinger equation

$$\begin{cases} i\partial_t \psi + \frac{1}{2} \Delta \psi + V \psi + f(|\psi|^2) \psi = 0, \\ \psi|_{t=0} = \psi_0, \end{cases}$$

where  $V$  and  $f$  are real.

(1)  $L^2$  conservation:

$$\frac{d}{dt} \int |\psi|^2 = 0.$$

(2) Let  $\mathcal{H}(t) = \int \left[ \frac{1}{2} |\nabla \psi|^2 - V |\psi|^2 - F(|\psi|^2) \right] (t) dx$ , where  $F' = f$ .

It holds  $\dot{\mathcal{H}}(t) = 0$ .

(3)  $N\vec{x} = \int x |\psi|^2 dx$ .

$$N\dot{\vec{x}} = \int x \psi_t \bar{\psi} dx + \int x \psi \bar{\psi}_t dx = 2\text{Re} \int x \psi_t \bar{\psi} dx$$

$$\begin{aligned}
&= 2\operatorname{Re} i \int x(-i\psi_t)\bar{\psi} dx \\
&= 2\operatorname{Re} i \int x \left( \frac{1}{2}\Delta\psi + \tilde{V}\psi + \underbrace{f(|\psi|^2)\psi}_{\in\mathbb{R}} \right) \overbrace{\bar{\psi}}^{\in\mathbb{R}} dx \\
&= \operatorname{Re} i \int x\Delta\psi\bar{\psi} dx \\
&= \operatorname{Re} i \int x \operatorname{div}(\nabla\psi\bar{\psi}) dx \\
&= \operatorname{Re} i \int \nabla\psi \cdot \bar{\psi} dx = \operatorname{Re}\langle p\psi, \psi \rangle = \frac{\langle p\psi, \psi \rangle + \overline{\langle p\psi, \psi \rangle}}{2} \\
&\stackrel{p^*=p}{=} \langle p\psi, \psi \rangle =: \vec{P}, \text{ the linear momentum.}
\end{aligned}$$

Is  $\vec{P}$  preserved?

$$\begin{aligned}
\dot{\vec{P}} &= \langle p\psi_t, \psi \rangle + \langle p\psi, \psi_t \rangle = 2\operatorname{Re}\langle p\psi_t, \psi \rangle = 2\operatorname{Re} i \langle p(-i\psi_t), \psi \rangle \\
&= 2\operatorname{Re} i \left\langle \frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi, p\psi \right\rangle \\
&= \operatorname{Re} i \underbrace{\langle \Delta\psi, p\psi \rangle}_{\in\mathbb{R}} + 2\operatorname{Re} i \langle V\psi, p\psi \rangle + 2\operatorname{Re} i \langle f(|\psi|^2)\psi, p\psi \rangle \\
&= -2\operatorname{Re}\langle V\psi, \nabla\psi \rangle - 2\operatorname{Re} \int f(|\psi|^2)\psi \nabla\bar{\psi} \\
&= -\langle V\psi, \nabla\psi \rangle - \langle \nabla\psi, V\psi \rangle - \int f(|\psi|^2) \underbrace{(\psi \nabla\bar{\psi} + \bar{\psi} \nabla\psi)}_{\nabla|\psi|^2} \\
&= \langle (\nabla V - V\nabla)\psi, \psi \rangle - \int \nabla(F(|\psi|^2)) \\
&= 0
\end{aligned}$$

Hence,  $N\ddot{X} = \dot{\vec{P}} = \langle (\nabla V)\psi, \psi \rangle$ .

**Side remark:**

If  $f = 0$ , then  $\psi(t) = e^{itH}\psi_0$ , where  $H = \frac{1}{2}\Delta + V$  and

$$\begin{aligned}
\frac{d}{dt}\langle A\psi, \psi \rangle &= \langle AiH\psi, \psi \rangle + \langle A\psi, iH\psi \rangle \\
&= i\langle [A, H]\psi, \psi \rangle.
\end{aligned}$$

In particular, if  $A = -i\nabla$ , then  $i\left[-i\nabla, \frac{1}{2}\Delta + V\right] = 0 + [\nabla, V] + (\nabla V)$ .

(4) Angular momentum (we will be in  $\mathbb{R}^3$ )

$$\begin{aligned}\vec{M} &= \frac{1}{2}i \int \vec{x} \times (\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}) dx \\ &= \frac{1}{2} \int \psi i \vec{x} \times \vec{\nabla} \psi dx + \frac{1}{2} \int \psi \overline{i \vec{x} \times \vec{\nabla} \psi} dx .\end{aligned}$$

If we put  $\vec{\rho} = i \vec{x} \times \vec{\nabla}$ , then  $\vec{M} = \text{Re} \langle \psi, \vec{\rho} \psi \rangle$ .

Clearly,  $\vec{\rho}^* = \vec{\rho}$  (by integration by parts:  $\int i x_k \partial_e \psi \bar{\phi} = \int \psi \overline{i x_k \partial_e \phi} \dots$ ). So,  $\vec{M} = \langle \vec{\rho} \psi, \psi \rangle$  and we have

$$\begin{array}{ll}\text{linear contribution} & \text{nonlinear contribution} \\ \dot{\vec{M}} = i \langle [\rho, H] \psi, \psi \rangle + & 2 \text{Re } i \langle f(|\psi|^2) \psi, \rho \psi \rangle .\end{array}$$

$$\begin{aligned}\text{Re } i \int f(|\psi|^2) \psi \overline{i (\vec{x} \times \vec{\nabla})_j \psi} dx \\ &= \text{Re} \int f(|\psi|^2) \psi \sum_{k,l=1}^3 \varepsilon_{jkl} x_k \partial_l |\psi|^2 dx \\ &= \frac{1}{2} \int f(|\psi|^2) \sum_{k,l=1}^3 \varepsilon_{jkl} x_k \partial_l |\psi|^2 dx \\ &= \frac{1}{2} \int \sum_{k,l=1}^3 \varepsilon_{jkl} x_k \partial_l (F(|\psi|^2)) dx \\ &= -\frac{1}{2} \int F(|\psi|^2) \varepsilon_{jkk} dx = 0\end{aligned}$$

where  $\varepsilon$  is the Levi-Civita symbol.

$$[\rho, H] \phi = [i \vec{x} \times \vec{\nabla}, \frac{1}{2} \Delta + V] \phi .$$

Since

$$\begin{aligned}[(\vec{x} \times \vec{\nabla})_j, \Delta] \phi &= \sum_{k,l,m} \varepsilon_{jkl} (x_k \partial_{lmm} f - \partial_{lmm} (x_k \partial_l f)) \\ &= - \sum_{k,l} \varepsilon_{jkl} \partial_{kl} f \stackrel{\varepsilon_{jkl} + \varepsilon_{jlk} = 0}{=} 0 ,\end{aligned}$$

we have  $[\rho, H] \phi = i((\vec{x} \times \vec{\nabla}) V) f$  and  $\dot{\vec{M}} = -\langle ((\vec{x} \times \vec{\nabla}) V) \psi, \psi \rangle$ .

(5) Scaling

Let  $V = 0$  and  $f(u) = u^{2\sigma}$ .

If  $\partial_t \psi + \frac{1}{2} \Delta \psi + |\psi|^{2\sigma} \psi = 0$ ,

then  $\psi_\lambda(t, x) := \lambda^{\frac{1}{\sigma}} \psi(\lambda^2 t, \lambda x)$  and satisfies (\*).

(\*)

$$\| \psi_\lambda(t) \|_{\dot{H}_x^s} = \lambda^{\frac{1}{\sigma}} \lambda^s \lambda^{-\frac{d}{2}} \stackrel{?}{=} \lambda^0 .$$

$$\sigma_{\text{critical}} = \frac{1}{-s + \frac{d}{2}} = \frac{2}{d - 2s} .$$

Subcritical case ( $\sigma < \sigma_{\text{critical}}$ ) — well understood

critical case — many open questions

supercritical case — young person gets Fields medal for solving.

(6) Galilei invariance

$$\begin{array}{ccc} \mathfrak{g}_{\vec{v},y}(t) & = e^{-i\frac{|v|^2}{2}t} & e^{-i\vec{x}\cdot\vec{v}} & e^{i(y+t\vec{v})\cdot\vec{p}} \\ & & \uparrow & \uparrow \\ \text{we will see:} & & \text{translation in} & \text{translation in} \\ & & \text{momentum space} & \text{physical space} \end{array}$$

Operator  $e^{i\vec{a}\cdot\vec{p}}$ , where  $\vec{a} \in \mathbb{R}^n$  (and  $\vec{p} = -\vec{\nabla}$ ) is well defined and is unitary.

Let  $U(s) = e^{si\vec{a}\cdot\vec{p}}$ .

Then  $U(0) = id$  and  $\frac{d}{ds}(U|s|f)(x) = (U(s)i\vec{a}\cdot\vec{p}f)(x)$ .

So,  $\partial_s g - \vec{a}\cdot\vec{\nabla}g = 0$ ,  $g(0, x) = f(x)$ , where we denoted  $g(s, x) = U(s)f(x)$ .

It follows that  $g(s, x) = f(x + s\vec{a})$ , i.e.,  $\boxed{e^{i\vec{a}\cdot\vec{p}}f(x) = f(x + \vec{a})}$ .

Another way of seeing this:

$$[e^{i\vec{a}\cdot\vec{p}}]^\wedge f(\xi) = e^{i\vec{a}\cdot\xi} \hat{f}(\xi), \text{ so } (e^{i\vec{a}\cdot\vec{p}}f)(x) = f(x + \vec{a})$$

What does this tell us?

Take  $f \in \xi$ .

$$\mathfrak{g}_{\vec{v},y}(t)f = e^{-i\frac{|v|^2}{2}t} e^{-i\vec{x}\cdot\vec{v}} f(x + y + t\vec{v}) .$$

classically:  $x \mapsto x - y - t\vec{v}$ .

And what happens in momentum space?

$$\begin{aligned} [\mathfrak{g}_{\vec{v},y}(t)f]^\wedge(\xi) &= e^{-i\frac{|v|^2}{2}t} [e^{-i\vec{x}\cdot\vec{v}} f(x + y + t\vec{v})]^\wedge(\xi) \\ &= e^{-i\frac{|v|^2}{2}t} [f(x + y + t\vec{v})]^\wedge(\xi + \vec{v}) \\ &= e^{-i\frac{|v|^2}{2}t} + e^{i(y+t\vec{v})\cdot(\xi+\vec{v})} \vec{f}(\xi + \vec{v}) \end{aligned}$$

so  $p \mapsto p - \vec{v}$

Finally, we will prove

$$\boxed{\mathfrak{g}_{\vec{v},y}(t)e^{it\frac{\Delta}{2}} = e^{it\frac{\Delta}{2}} \mathfrak{g}_{\vec{v},y}(0)}$$

by proving

$$\begin{aligned} (e^{-it\frac{\Delta}{2}} \mathfrak{g}_{\vec{v},y}(t)e^{it\frac{\Delta}{2}})^\cdot &= \\ &= \dot{\mathfrak{g}}_{\vec{v},y}(t) + i \mathfrak{g}_{\vec{v},y}(t) \frac{\Delta}{2} - i \frac{\Delta}{2} \mathfrak{g}_{\vec{v},y}(t) = 0 (*) \end{aligned}$$

$$(1) \quad \dot{\mathfrak{g}}_{\vec{v},y}(t) = -i\frac{|v|^2}{2} \mathfrak{g}_{\vec{v},y}(t) + i\vec{v}\cdot\vec{p}\mathfrak{g}_{\vec{v},y}(t)$$

$$\begin{aligned}
-i\frac{\Delta}{2}(\mathfrak{g}_{\vec{v},y}(t)f(x)) &= i\frac{|v|^2}{2}\mathfrak{g}_{\vec{v},y}(t)f(x) \\
&+ 2\frac{-i}{2}(-\vec{v} \cdot \vec{\nabla})\mathfrak{g}_{\vec{v},y}(t)f(x) \\
&+ \mathfrak{g}_{\vec{v},y}(t)\Delta f(x)
\end{aligned}
\tag{2}$$

Summing (1) and (2) gives (\*).

**Local well posedness** in  $H^1$  of

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi + F(\psi) = 0 \\ \psi(0) = \phi \in H^1(\mathbb{R}^d) \end{cases}$$

with  $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ ,  $F(0) = 0$ ,  $|F^1(z)| \leq C|z|^{p-1}$  if  $|z| \leq 1$ , where  $1 < p < \frac{d+2}{d-2}$  if  $d \geq 3$  and  $1 < p < \infty$  if  $d = 1, 2$ .

**Theorem.** (Kato, '87)

(1) If  $\phi \in H^1$ , then there exists  $T > 0$  and a unique

$$\psi \in C([0, T], H^1) \cap C^1([0, T], H^{-1}) \text{ solving (**)}$$

Moreover,  $\psi$  depends continuously on  $\phi$  in  $H^1$ .

$$\psi(t) = e^{i\frac{t}{2}\Delta}\phi + i \int_0^t e^{i\frac{t-s}{2}\Delta} \Delta F(\psi(s)) ds . \tag{**}$$

We will prove that this is meaningful.

regularity: if  $\phi \in H^2$ , then  $\psi \in C([0, T], H^2) \cap C^1([0, T], L^2)$

and one has continuous dependence in  $H^2$  norm.

**Remarks:**

Let  $F(z) = |z|^{2\sigma}z$ .

Then  $\frac{\partial F}{\partial z} = \sigma(z\bar{z})^{\sigma-1}\bar{z}z + (z\bar{z})^\sigma$  and  $\frac{\partial F}{\partial \bar{z}} = \sigma(z\bar{z})^{\sigma-1}z^2$  are continuous at  $z = 0$  provided  $\sigma > 0$ .

What about subcriticality?

$$2\sigma = p - 1 < \frac{d+2}{d-2} \Rightarrow \sigma < \frac{1}{2} \left( \frac{d+2}{d-2} - 1 \right) = \frac{2}{d-2} .$$

We will need more general Strichartz estimates:

**Lemma.** (Strichartz estimates):

Let  $G_0(t) = e^{it\frac{\Delta}{2}}$  and

$$(Gv)(t) = \int_0^t e^{i(t-s)\frac{\Delta}{2}} v(s) ds .$$

Then

(i)  $G_0 : L^2 \rightarrow L_t^r(L_x^q)$  provided  $\frac{2}{r} + \frac{d}{q} = \frac{d}{2}$  and  $2 < r < \infty$ . (Note: this includes  $r = 0$ ,  $g = \Omega$ ;  $r = 2$  is also okay, but we won't prove it.)

(ii)  $G = L_t^1(L_x^q) \rightarrow L_t^r(L_x^q)$  and  
 $G : L_t^{p'}(L_x^{q'}) \rightarrow L_t^r(L_x^q)$ .

(iii)  $G : L_t^{r'}(L_x^{q'}) \rightarrow L_t^r(L_x^q)$ .

*Proof.* Necessary conditions on  $r$  and  $q$  we get by scaling:

If  $\psi_t(t) := \psi(\lambda^2, \lambda x)$ , where  $\psi(t, x) = (e^{-it\frac{\Delta}{2}}\psi(0, \cdot))(x)$ , then  $\|\psi_\lambda\|_{L_t^r(L_x^q)} = \lambda^{-\frac{2}{r}}\lambda^{-\frac{d}{q}}\|\psi\|_{L_t^r(L_x^q)}$  and  $\|\psi_\lambda(0, x)\|_{L^2} = \lambda^{-\frac{d}{q}}\|\psi(0, x)\|_{L^2}$ .

Hence  $-\frac{2}{r} - \frac{d}{q} = -\frac{d}{2}$ .

It holds

$$\begin{aligned} (G_0^*v)(x) &= \int (e^{-i\frac{s}{2}\Delta}v(s))(x) dx, \text{ so} \\ (G_0G_0^*v)(t, x) &= \int e^{i(t-s)\frac{\Delta}{2}}v(s) \text{ and} \\ &\left( \int (\| (G_0G_0^*v)(t) \|_{L_x^q})^r dt \right)^{\frac{1}{r}} \leq \\ &\leq c \left( \int \left( \int |t-s|^{-\frac{d}{2}(\frac{1}{q'}-\frac{1}{q})} \|v(s)\|_{L_x^{q'}} \right)^{r'} dt \right)^{\frac{1}{r'}} \end{aligned}$$

Now, if  $1 + \frac{1}{r} = \frac{d}{2}(\frac{1}{q'} - \frac{1}{q}) + \frac{1}{r'}$ , and  $0 \leq \frac{d}{2}(\frac{1}{q'} - \frac{1}{q})$ , then apply fractional integration and complete the proof as we did when we first proved Strichartz estimates.

We have

$$\begin{aligned} (Gv, w) &= \int \left\langle \int_{\delta}^t e^{i(t+s)v(s)} ds, w(t) \right\rangle dt \\ &\quad - \int \left\langle v(s), \chi \int_{[0<s<t]} \overbrace{e^{-i(t-s)\frac{\Delta}{2}} w(t)}^{G^*r} ds \right\rangle dt \\ &\| \int_0^t e^{i(t-s)\frac{\Delta}{2}} v(s) ds \|_{L_x^2} \leq C \int_{-\infty}^{\infty} |t-s|^{-\frac{d}{2}(\frac{1}{q'}-\frac{1}{q})} \|v(s)\|_{L^q}^{q'} dx \end{aligned}$$

and the claim follows.  $\square$

### PDE (Schlag) 03/06/03

$$(SE) \quad i\partial_t\psi + \frac{1}{2}\Delta\psi + F(\psi) = 0$$

$F$  is  $C^1(\mathbb{R}^2, \mathbb{R}^2)$

$\frac{\partial F}{\partial z}, \frac{\partial F}{\partial \bar{z}}$  exist and are continuous

$$F(0) = 0$$

$$|F'(z)| \equiv \left| \frac{\partial F}{\partial z}(z) \right| + \left| \frac{\partial F}{\partial \bar{z}}(z) \right| \leq M|z|^{p-1} \text{ for } |z| \geq 1$$

Growth condition

Range of possible  $p$ ,  $1 < p < \frac{d+2}{d-2} \textcircled{*}$

If  $F(z) = |z|^{2\sigma}z$ , then  $\textcircled{*} \Leftrightarrow \sigma < \frac{2}{d-2}$

$$\begin{aligned} \text{(DH)} \quad \psi(t) &= e^{i\frac{t}{2}\Delta}\psi_0 + i \int_0^t e^{i\frac{t-s}{2}\Delta} F(\psi(s)) ds \\ &= (G_0\psi_0)(t) + iGF(\psi) \end{aligned}$$

**Lemma.**

$$(i) \quad G_0 : L^2 \rightarrow L_t^r(L_x^q) \quad \frac{2}{r} + \frac{d}{q} = \frac{d}{2} \quad 2 < r \leq \infty$$

which includes  $L^2 \rightarrow L_t^\infty(L_x^2)$

$$(ii) \quad G : L_t^1(L_x^2) \rightarrow L_t^r(L_x^q) \quad \leftarrow \text{includes } L_t^1(L_x^2) \rightarrow L_t^\infty(L_x^2)$$

$$L_t^{r'}(L_x^q) \rightarrow L_t^\infty(L_x^2)$$

$$(iii) \quad G : L_t^{r'}(L_x^{q'}) \rightarrow L_t^r(L_x^q)$$

**REMARK:** These are homogeneous estimates (scaling invariant) thus you can't improve the constant by localizing  $0 \leq t \leq T$ .

**Lemma.** With  $F$  as above, one has  $F = F_1 + F_2$

$$\begin{aligned} \text{where } |F_1'| &\leq M & |F_2(z) - F_2(w)| &\leq M(|z|^{p-1} + |w|^{p-1})|z - w| \\ |F_1(z) - F_1(w)| &\leq M|z - w| \end{aligned}$$

*Proof.*  $F_1 = \chi F$ ,  $F_2 = (1 - \chi)F$ , where  $\chi$  is smooth cut-off of radius  $\sim 1$ . □

Idea of proof of  $H^1$  well-posedness.

Formally  $\partial\psi(t) = G_0\partial\psi + iG(F'(\psi) \cdot \partial\psi)$

$$\begin{aligned} F(\psi) &= |\psi|^{2\sigma}\psi = |\psi\bar{\psi}|^\sigma\psi \\ \partial_j F^2(\psi) &= \sigma|\psi|^{2(\sigma-1)}(\partial_j\psi\bar{\psi} + \psi\partial_j\bar{\psi}) + |\psi|^{2\sigma}\partial_j\psi \\ &= (\sigma+1)|\psi|^{2\sigma}\partial_j\psi + \sigma|\psi|^{2(\sigma-1)}\psi\partial_j\bar{\psi} \end{aligned}$$

$$\begin{aligned} \frac{d}{d\varepsilon}\Big|_0 F(z + \varepsilon w) &= \frac{\partial F}{\partial z}(z)w + \frac{\partial F}{\partial \bar{z}}(z)w \stackrel{\text{def}}{=} F'(z) \cdot w \\ \partial\psi(t) &= G\partial\psi_0 + G(F'(\psi) \cdot \partial\psi) \end{aligned}$$

$$\begin{aligned} \text{show} \quad \sup_{0 \leq t \leq T} \|\partial\psi(t)\|_2 &\leq \|\partial\psi_0\|_2 + \sup_{0 \leq t \leq T} \|G(F'(t) \cdot \partial\psi)(t)\|_2 \\ &\leq \|\partial\psi_0\|_2 + C \|F'(\psi) \cdot \partial\psi\|_{L_t^1(L_x^2)} \end{aligned}$$

If  $F'$  were bounded, then

$$\|\partial\psi\|_{L^\infty(I, L_x^2)} \leq \|\partial\psi_0\|_2 + C \|F'\|_\infty \cdot T \cdot \|\partial\psi\|_{L^\infty(I, L_x^2)}$$

Since  $F'$  isn't bounded, do the following

$$\begin{aligned} \|F_1'(\psi) \cdot \partial\psi\|_{L^1(I, L_x^2)} &\lesssim T \|\partial\psi\|_{L^\infty(I, L_x^2)} \\ \|F_2'(\psi) \cdot \partial\psi\|_{L^{r'}(I, L_x^{q'})} &\lesssim \|\psi(t)\|_{L_x^q}^{p-1} \cdot \|\partial\psi(t)\|_{L_x^q} \|_{L^{r'}(I)} \quad ** \end{aligned}$$

Guess proper value of  $q'$ :

$$\begin{aligned} \| |f|^{p-1}g \|_{q'} &\leq \left( \int |f|^{(p-1)q'} |g|^{q'} dx \right)^{\frac{1}{q'}} \\ &\leq \left( \int |f|^{(p-1)q's} dx \right)^{\frac{1}{sq'}} \left( \int |g|^{q's'} dx \right)^{\frac{1}{s'q'}} \end{aligned}$$

Choose 
$$\left. \begin{aligned} q's' &= q \\ (p-1)q's &= q \end{aligned} \right\} \rightarrow \frac{q'}{q} + \frac{(p-1)q'}{q} = \frac{1}{s} + \frac{1}{s'} = 1$$

$$\frac{pq'}{q} = \frac{p}{q-1} = 1 \Rightarrow \boxed{q = p+1} > 2$$

By Sobolev,  $(**) \lesssim \| \psi \|_{L^\infty(I, L_x^q)}^{p-1} \cdot \| \partial\psi \|_{L^{r'}(I, L_x^q)} \quad H^1 \hookrightarrow L^q \quad \frac{1}{2} - \frac{1}{q} = \frac{1}{d}$

$$\lesssim \| \psi \|_{L^\infty(I, H_x^1)}^{p-1} \cdot \| \partial\psi \|_{L^{r'}(I, L_x^q)} \quad p+1 = q < \frac{2d}{d-2} \quad p < \frac{d+2}{d-2}$$

$$\lesssim \| \psi \|_{L^\infty(I, H_x^1)}^{p-1} \cdot T^{\frac{1}{r'} - \frac{1}{r}} \| \partial\psi \|_{L^r(I, L_x^q)} \quad \text{Note: } r' < 2 < r$$

Similarly, one obtains

$$\begin{aligned} &\| \partial\psi \|_{L^\infty(I, L_x^2)} + \| \partial\psi \|_{L^r(I, L_x^q)} \\ &\leq \| \partial\psi_0 \|_{L^2} + T \| \partial\psi \|_{L^\infty(I, L_x^2)} + CT^{\frac{1}{r'} - \frac{1}{r}} \| \psi \|_{L^\infty(I, H^1)}^{p-1} \| \partial\psi \|_{L^r(I, L^q)} \end{aligned}$$

Define  $LHS = \mu$

Then

$$\begin{aligned} \mu &\lesssim \| \partial\psi_0 \|_2 + T_\mu + CT^\alpha \mu^p, \\ \Rightarrow \mu(T) &\leq C \| \partial\psi_0 \|_2 \end{aligned}$$

assuming  $L^2$  conservation and  $\mu$  continuous provided  $T < T^* = T_p^*(\| \psi_0 \|_{H^1})$ .

*Careful Proof* We'll show the following

**Theorem.** *The equation  $\psi = G_0\phi_0 + iGF(\phi)$  for every  $\phi_0 \in H^1$  has a unique solution*

$$\phi \in C([0, T], H^1) \cap C^1([0, T], H^{-1})$$

$$\text{with } T = T(\| \phi_0 \|_{H^1}) > 0.$$

Moreover, the solution depends continuously on  $\phi_0$  in the  $H^1$  norm.

$$\text{Set } q = p+1, < \frac{2d}{d-2}. \quad \frac{2}{r} + \frac{d}{q} = \frac{d}{2}, \text{ so } r > 2, \text{ (since } q < \frac{2d}{d-2}\text{)}$$

$$\text{Set } \begin{cases} X_0 &= L^\infty(I, L^2 \cap L^q) \\ X &= L^\infty(I, L_x^2) \cap L^r(I, L_x^q) \\ X' &= L^1(I, L^2) + L^{r'}(I, L^{q'}) \\ \bar{X} &= C(I, L^2) \cap L^r(I, L^2) \end{cases}$$

**Lemma.**

$$\begin{aligned} G_0 &: L^2 \rightarrow X \\ G &: X' \rightarrow X \end{aligned}$$

Moreover:

$$G_0 : L^2 \rightarrow \bar{X} \text{ and } G : X' \rightarrow \bar{X}$$

*Proof.* First two statements are just Lemma 1.

$$\| e^{it\frac{\Delta}{2}} f - e^{is\frac{\Delta}{2}} f \|_2 = \| (e^{-i(t-s)\frac{|\xi|^2}{2}} - 1) \hat{f}(\xi) \|_{L^2(\xi)} \rightarrow 0$$

by Dominated Convergence theorem as  $t - s \rightarrow 0$  □

**Lemma.**  $F : X_o \rightarrow X'$  continuously and boundedly.

Moreover

$$\| F_1(v) - F_1(w) \|_{1,2} \lesssim T \| v - w \|_{\infty,2}$$

$$\| F_2(v) - F_2(w) \|_{r',q'} \lesssim T^\alpha (\| v \|_{X_0}^{p-1} + \| w \|_{X_0}^p) \| v - w \|_{r,q}$$

where  $\alpha = \frac{1}{r'} - \frac{1}{r} > 0$ ,

On a ball  $B_R(X_0)$ ,  $F$  is Lipschitz:  $X_0 \rightarrow X'_1$  (and constant may be made small for fixed  $R_1$  and small  $T$ ).

*Proof.*  $F = F_1 + F_2$

$$\begin{aligned} |F_1(v) - F_1(w)|(x, t) &\lesssim |v(x, t) - w(x, t)| \\ \int_0^T \| F_1(v(t, x)) - F_1(w(t, x)) \|_{L_x^2} dt &\lesssim \int_0^T \| v(x, t) - w(x, t) \|_{L_x^2} dt \\ &\lesssim T \| v - w \|_{L^\infty(I, L_x^2)} \end{aligned}$$

$$|F_2(v(x, t) - F_2(w(x, t)))| \lesssim (|v(x, t)|^{p-1} + |w(x, t)|^{p-1}) |v(x, t) - w(x, t)|$$

$$\| F_2(v) - F_2(w) \|_{L^r(I, L^q)} \lesssim \left( \int_0^T (\| v(t) \|_{L_x^q}^{p-1} + \| w(t) \|_{L_x^q}^{p-1})^{r'} \| v(t) - w(t) \|_{L_x^q}^{r'} dt \right)^{\frac{1}{r'}}$$

Hölder  $q = p + 1$

$$\lesssim (\| v \|_{X_0}^{p-1} + \| w \|_{X_0}^{p-1}) T^\alpha \| v - w \|_{L^r(I, L_x^q)}$$

□

**Corollary.**  $GF : X_o \rightarrow \bar{X}$ , and on each ball  $B_R(X_0)$ ,  $GF$  is a contraction in  $X$  norm for small  $T$ .

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$$\begin{aligned} I &= [0, T], q = p + 1 \\ X_0 &= L^\infty(I, L_x^2 \cap L_x^q) \\ X &= L^\infty(I, L^2) \cap L^r(I, L^q) \\ X &= C(I, L^2) \cap L^r(I, L^q) \\ X' &= L^1(I, L^2) + L^{r'}(I, L^{q'}) \end{aligned}$$

$$\begin{aligned}
(\text{SE}) \quad & i\partial_t \psi + \frac{1}{2}\Delta\psi + F(\psi) = 0 \\
& \psi|_{t=0} = \psi_0 \in H^1 \\
& F(0) = 0, F \in C^1(\mathbb{R}^2, \mathbb{R}^2), \\
& \left| \frac{\partial F}{\partial z}(z) \right| + \left| \frac{\partial F}{\partial \bar{z}}(z) \right| \leq M|z|^{p-1}, \quad |z| \geq 1 \\
& 1 < p < \frac{d+2}{d-2}
\end{aligned}$$

$$(\text{DH}) \quad \psi = \underbrace{G_0}_{e^{i\frac{t}{2}\Delta}} \psi_0 + \underbrace{i GF(\psi)}_{\int_0^t e^{i(t-s)\frac{\Delta}{2}} F(\psi(s)) ds}$$

$\rightarrow$  strong solution of SE?  
 $\psi \in C(I, H^1) \cap C^1(I, H^{-1})$   
 remark :  $F : X_0 \rightarrow X'$   
 $C(I, H^1) \hookrightarrow X_0$  by Sobolev embedding  
 $\left( H^1 \hookrightarrow L^q = L^{p+1}, \frac{1}{2} - \frac{1}{q} \leq \frac{1}{d} \right)$   
 $X' \hookrightarrow L^1(I, H^{-1})$   
 $(H^1 \hookrightarrow L^2 \cap L^q, L^2 + L^q \hookrightarrow H^{-1})$

**Lemma.**  $GF : X_0 \rightarrow \bar{X}$  continuously and boundedly. On each ball  $B_R(X_0)$ ,  $GF$  is a contraction, with  $T$  suff. small (dep. on  $R$ ). In fact,  $\forall u, v \in B_R(X_0) : \|GF(u) - GF(v)\|_{\bar{X}} \leq C(T + T^\alpha R^{p-1})\|v - u\|_X$ .

**Corollary.** Suppose  $\psi \in X_0$  is a weak solution of (DH) (check meaningful). Then  $\psi$  is unique.

*Proof.*

$$\begin{aligned}
\text{Given } \psi_0 \in L^2, \text{ suppose } \psi &= G_0\psi_0 + iGF(\psi) \\
\tilde{\psi} &= G_0\psi_0 + iGF(\tilde{\psi})
\end{aligned}$$

$$\begin{aligned}
\psi - \tilde{\psi} &= iGF(\psi) - iGF(\tilde{\psi}) \\
\|\psi - \tilde{\psi}\|_X &\leq \delta \|\psi - \tilde{\psi}\|_X \quad \text{by contraction} \\
\Rightarrow \psi &= \tilde{\psi} \text{ on } [0, T_0] \quad (T_0 \text{ chosen so small that } \delta < \frac{1}{2})
\end{aligned}$$

Repeat  $[T_0, 2T_0]$  but:

**Remark:** not clear we can talk about  $\psi(T_0)$  and  $\tilde{\psi}(T_0)$ . However,  $\psi$  and  $\tilde{\psi} \in C(I, H^{-1})$  because of (DH) :  $G_0\psi_0 \in C(I, L^2)$  and

$$\int_0^t e^{i(t-s)\frac{\Delta}{2}} \underbrace{F(\psi(s))}_{\in L^1(I, H^{-1})} ds \in C(I, H^{-1})$$

So talk about  $\psi(T_0), \tilde{\psi}(T_0) \in H^{-1}$ . Since

$$\begin{aligned} \psi, \tilde{\psi} &\in L^\infty(I, L^2) \\ \Rightarrow \psi(T_0) \text{ and } \tilde{\psi}(T_0) &\in L^2 \end{aligned}$$

$$\begin{aligned} |\langle \psi(T_0), \phi \rangle| &= \lim_{t_j \rightarrow T_0} |\langle \psi(t_j), \phi \rangle| \quad \text{for some seq } t_j \\ &\leq \|\psi\|_{L^\infty(I, L^2)} \cdot \|\phi\|_2 \end{aligned}$$

**Q:** need to check that

$$\psi(T_0 + h) = e^{it\frac{\Delta}{2}} \psi(T_0) + i \int_{T_0}^{T_0+h} e^{i(T_0+h-s)\frac{\Delta}{2}} F(\psi(s)) ds .$$

OK because

$$\begin{aligned} \psi(T_0 + h) &= e^{i(T_0+h)\frac{\Delta}{2}} \psi(0) + i \int_0^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta} F(\psi(s)) ds \\ &= e^{\frac{i}{2}h\Delta} e^{\frac{i}{2}T_0\Delta} \psi(0) + i e^{i\frac{h}{2}\Delta} \int_0^{T_0} e^{\frac{i}{2}(T_0-s)\Delta} F(\psi(s)) ds \\ &\quad + i \int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta} F(\psi(s)) ds \\ &= e^{\frac{i}{2}h\Delta} \psi(T_0) + i \int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta} F(\psi(s)) ds \end{aligned}$$

**NB:**  $T_0$  depends only on  $R(\psi \in X_0, \tilde{\psi} \in X_0)$

**Or:** extend from a point between  $\frac{T_0}{2}$  and  $T_0$  where  $\psi$  and  $\tilde{\psi}$  agree.

**NB:** Because of  $R$ , cannot prove existence immediately.

$$Y = \{v \in X, \partial_x v \in X\} \subseteq L^\infty(I, H^1)$$

$$\bar{Y} = \{v \in \bar{X}, \partial_x v \in \bar{X}\} \subseteq C(I, H^1)$$

$$Y' = \{v \in X', \partial_x v \in X'\}$$

Sobolev embedding:

$$\begin{aligned} Y &\subset X_0, \\ \bar{Y} &\subset X, Y \subset X, Y' \subset X'. \end{aligned}$$

□

**Lemma.**

$$G_0 : H^1 \rightarrow \bar{Y}$$

$$G : Y' \rightarrow \bar{Y}$$

*Proof.*

$$\begin{aligned} v \in L^2, \quad \|G_0 v\|_{L^\infty(I, L^2)} &= \|e^{\frac{i}{2}t\Delta} v\|_{L^\infty(I, L^2)} \\ &= \|v\|_{L^2} \end{aligned}$$

$$\left(\frac{2}{r} + \frac{d}{q} = \frac{d}{2}\right) \|G_0 v\|_{L^r(I, L_x^q)} \leq C \|v\|_{L^2}$$

So  $G_0 : L^2 \rightarrow \bar{X}$ .

Since  $\partial G_0 = G_0 \partial$ , we see that  $G_0 = H^1 \rightarrow \bar{Y}$  and  $X' \rightarrow \bar{X}$  and  $\partial G = G \partial$ , we have  $G : Y' \rightarrow \bar{Y}$ .  $\square$

**Lemma.**  $F : Y \rightarrow Y'$  continuously and boundedly and  $\|F(v)\|_{Y'} \leq C (T + T^\alpha \|v\|_Y^{p-1}) \|v\|_Y$ .

*Proof.*

$$\begin{aligned} \|F(v)\|_{X'} &\leq C(T + T^\alpha \|v\|_{X_0}^{p-1}) \|v\|_X \\ &\leq C(T + T^\alpha \|v\|_Y^{p-1}) \|v\|_Y. \end{aligned}$$

$$(\tau_h)(x) = f(x+h) \quad (\text{dunno } F \in C^1)$$

$$\left\| \frac{1}{|h|} (\tau_h - 1) F_1(v) \right\|_{L^1(I, L_x^2)} \leq C \frac{1}{|h|} \|(\tau_h - 1)v\|_{L^1(I, L^2)}$$

$$\leq CT \|\partial v\|_{L^\infty(I, L^2)} \quad (\text{fundamental})$$

$$|\langle F_1(v), \partial \phi \rangle| \leq CT \|\partial v\|_{L^\infty(I, L^2)}$$

$$\text{means } \|\partial F_1(v)\|_{L^1(I, L^2)} \leq CT \|\partial v\|_X$$

$$\|\partial F_2(v)\|_{L^{r'}(I, L_x^{q'})} \leq C \| |v|^{p-1} \partial v \|_{L^{r'}(I, L_x^{q'})}$$

( $\| |v|^p \|_{L_x^{q'}}$  ok because  $pq' = q \Rightarrow p+1 = q$ )

$$\leq C \left( \int (\|v(t)\|_{L_x^q}^{p-1} \|\partial v(t)\|_{L_x^q}^{p'} dt) \right)^{\frac{1}{r'}}$$

$$\leq C \|v\|_{L^\infty(I, L^q)}^{p'} T^\alpha \|\partial v\|_{L^r(I, L_x^q)}$$

$$\leq CT^\alpha \|v\|_Y^{p-1} \|\partial v\|_X$$

$\square$

*Proof.* Let  $\psi_0 \in H^1$ . Then there exists  $T = T(\|\psi_0\|_{H^1}) > 0$  so that there exists a unique  $u \in \bar{Y}([0, T])$ , with  $u = G_0 \psi_0 + iGF(u)$ .  $\square$

*Proof.* Let  $\Phi(u) = G_0 \psi_0 + iGF(u)$

$$\Phi = B_R(Y) \circlearrowleft \text{ provided } R = C_1 \|\psi_0\|_{H^1}$$

where  $C_1 = \text{big abs. const.}$

$$\|\Phi(u)\|_Y \leq C \|\psi_0\|_{H^1} + C (T + T^\alpha \|u\|_Y^{p-1}) \|u\|_Y$$

$$\begin{aligned} &\leq \underbrace{C\|\psi_0\|_{H^1}}_{<R/2} + \underbrace{C(T + T^\alpha R^{p-1})}_{<1/2} R \\ &\leq R \end{aligned}$$

Let  $E = \{B_R(Y), \|\cdot\|_X\}$ . Control in stronger norm, contract in weaker norm.

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_X &\leq C(T + T^\alpha R^{p-1})\|u - v\|_X \\ &< \frac{1}{2}\|u - v\|_X. \end{aligned}$$

Fixed point is in  $\bar{Y}$ .

Is  $Y$  complete with respect to  $\|\cdot\|_X$ ?

Suppose  $\|u_n - u_m\|_X \rightarrow 0$  as  $n, m \rightarrow \infty$ .

$$\begin{aligned} \{u_n\} &\subset B_R(Y) \\ u_n &\xrightarrow{\|\cdot\|_X} u \\ \langle \partial u_n(t), \phi(t) \rangle &= -\langle u_n(t), \partial \phi(t) \rangle \\ &\quad \downarrow \\ &= -\langle u(t), \partial \phi(t) \rangle \end{aligned}$$

has estimate uniform on  $n$ , not involving  $\partial \phi$ . □

**Lemma.** Let  $u = u_{\psi_0}$  be the solution from the previous proposition. Then  $\psi_0 \rightarrow u_{\psi_0}$  is a continuous map from  $H^1 \rightarrow \bar{Y}$  in the following sense. Given  $\psi_0 \in H^1$ , there exists  $T = T(\psi_0) > 0$  so that for every  $\varepsilon > 0, \exists \delta > 0$ , if  $\psi_1 \in H^1, \|\psi_0 - \psi_1\|_{H^1} < \delta \Rightarrow u = u_{\psi_0}$  and  $v = v_{\psi_1}$  exists in  $\bar{Y}(0, T)$  and  $\|u - v\|_Y < \varepsilon$ .

*Proof.* It is clear that there exists  $T = T(\psi_0)$ ,

$$\Phi(w) = \Phi_{\psi_0}(w) = G_0 \psi_0 + iGF(w)$$

$$u = \lim_{n \rightarrow \infty} \Phi_{\psi_0}^n \quad (\text{iterate})$$

$$\|u - \Phi_{\psi_0}^n(\psi_0)\|_X < \delta^n \quad (\text{not the same } \delta \text{ as above})$$

$$\|v - \Phi_{\psi_1}^n(\psi_1)\|_X < \delta^n$$

$$\|u - v\|_X \leq 2\delta^n + \|\Phi_{\psi_0}^n(\psi) - \Phi_{\psi_1}^n(\psi_1)\|_X$$

Inductively:

$$\begin{aligned} \|\Phi_{\psi_0}(\psi_0) - \Phi_{\psi_1}(\psi_1)\|_X &\leq \|G(F\psi_0 - F\psi_1)\|_X + \|G_0(\psi_0 - \psi_1)\|_X \\ &\leq C\|\psi_0 - \psi_1\|_2 + C(T + T^\alpha R^{p-1})\|\psi_0 - \psi_1\|_X \\ &\leq C\|\psi_0 - \psi_1\|_2 + C\|\psi_0 - \psi_1\|_{H^1} \end{aligned}$$

$$\begin{aligned} \|\Phi_{\psi_0}^2(\psi_0) - \Phi_{\psi_1}^2(\psi_1)\|_X &\leq C\|\psi_0 - \psi_1\|_2 + C(T + T^\alpha R^{p-1})\|\Phi_{\psi_0}(\psi_0) - \Phi_{\psi_1}(\psi_1)\|_X \\ &\leq \underbrace{C}_{\substack{\text{does not} \\ \text{grow for } n \\ \text{(but does not} \\ \text{matter anyway)}}} \|\psi_0 - \psi_1\|_2 + \underbrace{C_1}_{C_1 = C_0 + \delta C_1} \|\psi_0 - \psi_1\|_2. \end{aligned}$$

□

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$$\begin{aligned}
\text{Recall } I &= [0, T] \\
X_0 &= L^\infty(I; L_x^q \cap L_x^2) \\
X &= L^\infty(I; L^2) \cap L^r(I; L^q) \\
X &= C(I; L^2) \cap L^r(I; L^q) \\
X' &= L^1(I; L^2) + L^{r'}(I; L^{q'})
\end{aligned}$$

$$(\text{SE}) \quad \left[ \begin{array}{l} i\partial_t \psi + \frac{1}{2} \Delta \psi + F(\psi) = 0 \\ \psi|_{t=0} = \psi_0 \in H^1 \\ F(0) = 0, F \in C^1(\mathbb{R}^2; \mathbb{R}^2), \\ \left| \frac{\partial F}{\partial z}(z) \right| + \left| \frac{\partial \bar{F}}{\partial \bar{z}}(z) \right| \leq M|z|^{p-1}, \quad |z| \geq 1 \\ 1 < p < \frac{d+2}{d-2} \end{array} \right.$$

$$(\text{DH}) \quad \left[ \begin{array}{l} \psi = \underbrace{G_0 \psi_0} + \underbrace{iGF(\psi)} \\ e^{i\frac{t}{2}\Delta} \psi_0 \quad \int_0^t e^{(t-s)\frac{\Delta}{2}} F(\psi(s)) ds \end{array} \right.$$

We're trying to show that there exists a unique strong solution of (SE)

$$\psi \in C(I, H^1) \cap C^1(I; H^{-1})$$

Remark:  $F : X_0 \rightarrow X'$

$C(I; H^1) \hookrightarrow X_0$  by Sobolev embedding

$$\left( H^1 \hookrightarrow L^q = L^{p+1}; \frac{1}{2} - \frac{1}{q} \leq \frac{1}{d} \right)$$

$$X' \hookrightarrow L^1(I; H^{-1})$$

$$\left( H^1 \hookrightarrow L^2 \cap L^q, \xrightarrow{\text{duality}} L^2 + L^{q'} \hookrightarrow H^{-1} \right)$$

Last time we did:

**Lemma.**  $GF : X_0 \rightarrow \bar{X}$  continuous and bounded and on each ball  $B_R(X_0)$ ,  $GF$  is a contraction with  $T$  sufficiently small (depending on  $R$ ):

$$\begin{aligned}
\|GF(u) - GF(v)\|_{\bar{X}} &\leq C(T + T^\alpha R^{p-1}) \|u - v\|_X \\
&\forall u, v \in B_R(X_0)
\end{aligned}$$

**Corollary.** Suppose  $\psi \in X_0$  is a weak solution of (DH). Then  $\psi$  is unique.

*Proof.* Given  $\psi_0 \in L^2$  suppose

$$\begin{aligned}\psi &= G_0\psi_0 + iGF\psi \\ \tilde{\psi} &= G_0\psi_0 + iGF\tilde{\psi} \\ \Rightarrow \psi - \tilde{\psi} &= iG(F(\psi) - F(\tilde{\psi})) \\ \Rightarrow \|\psi - \tilde{\psi}\|_X &\leq \delta\|\psi - \tilde{\psi}\|_X ; \text{ provided } T \text{ is small}\end{aligned}$$

$$\Rightarrow \psi = \tilde{\psi} \text{ on } [0, T_0] \text{ (with } T_0 \text{ so that } \delta < \frac{1}{2}\text{)}$$

To extend to any existence interval, repeat. Careful: It is not clear we can talk about  $\psi(T_0)$  or  $\tilde{\psi}(T_0)$ . However  $\psi, \tilde{\psi} \in C(I; H^{-1})$  since clearly  $G_0\psi_0 \in C(I; L^2)$  and

$$\int_0^t e^{i(t-s)\frac{\Delta}{2}} \underbrace{F(\psi(s))}_{\in L^1(I; H^{-1})} ds \in C(I; H^{-1})$$

So we can talk about  $\psi(T_0), \tilde{\psi}(T_0) \in H^{-1}$ . But since  $\psi, \tilde{\psi} \in L^\infty(I; L^2) \xrightarrow{?} \psi(T_0), \tilde{\psi}(T_0) \in L^2!$

$$|\langle \psi(T_0), \phi \rangle| = \lim_{t_j \rightarrow T_0} |\langle \psi(t_j), \phi \rangle| \leq \|\psi\|_{L^\infty(I; L^2)} \|\phi\|_2$$

Question:

$$\begin{aligned}\psi(T_0 + h) &= e^{i\frac{h}{2}\Delta}\psi(T_0) + i \int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta} F(\psi(s)) ds \\ \psi(T_0 + h) &= e^{i\frac{T_0+h}{2}\Delta}\psi(0) + i \int_0^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta} F(\psi(s)) ds \\ &= e^{\frac{i}{2}h\Delta} e^{\frac{i}{2}T_0\Delta}\psi(0) + e^{i\frac{h}{2}\Delta} i \int_0^{T_0} e^{\frac{i}{2}(T_0-s)\Delta} F(\psi(s)) ds \\ &\quad + i \int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta} F(\psi(s)) ds \\ &= e^{\frac{i}{2}h\Delta}\psi(T_0) + i \int_{T_0}^{T_0+h} e^{\frac{i}{2}(T_0+h-s)\Delta} F(\psi(s)) ds.\end{aligned}$$

□

New spaces:

$$\begin{aligned}Y &= \{v \in X, \partial_x v \in X\} \\ \bar{Y} &= \{v \in \bar{X}, \partial_x v \in \bar{X}\} \\ Y' &= \{v \in X', \partial_x v \in X'\}\end{aligned}$$

Comments

$$\begin{aligned} Y &\subset L^\infty(I; H^1) \\ \bar{Y} &\subset C(I; H^1) \end{aligned}$$

Note

$$\begin{aligned} Y &\subset X_0 && \text{(by Sobolev embedding)} \\ \bar{Y} &\subset \bar{X} \\ Y &\subset X \\ Y' &\subset X' \end{aligned}$$

**Lemma.**

$$\begin{cases} G_0 : H^1 &\rightarrow \bar{Y} \\ G : Y^1 &\rightarrow \bar{Y} \end{cases}$$

*Proof.*

$$v \in L^2 \implies \|G_0 v\|_{L^\infty(I; L^2)} = \|e^{i\frac{t}{2}\Delta} v\|_{L^\infty(I; L^2)} = \|v\|_{L^2}$$

$$\|G_0 v\|_{L^r(I; L_x^q)} \leq C \|v\|_{L^2}$$

$$\left( \frac{2}{r} + \frac{d}{q} = \frac{d}{2} \right)$$

So

$$\left. \begin{aligned} G_0 : L^2 &\longrightarrow \bar{X} \\ [G_0, \partial_x] &= 0 \end{aligned} \right\} \implies G_0 : H' \longrightarrow \bar{Y}$$

Since

$$\left. \begin{aligned} G : X' &\longrightarrow \bar{X} \\ [G, \partial_x] &= 0 \end{aligned} \right\} \implies G : Y' \longrightarrow \bar{Y}$$

□

**Lemma.**  $F : Y \longrightarrow Y'$  continuously and boundedly and

$$\|F(v)\|_{Y'} \leq C \left( T + T^\alpha \|v\|_Y^{p-1} \right) \|v\|_Y$$

*Proof.*

$$\begin{aligned} \|F(v)\|_{X'} &\leq C \left( T + T^\alpha \|v\|_{X_0}^{p-1} \right) \|v\|_X \\ &\leq C \left( T + T^\alpha \|v\|_Y^{p-1} \right) \|v\|_Y \\ &\quad \uparrow \\ &\quad \text{Sobolev embedding} \end{aligned}$$

Let  $(\tau_h f)(x) = f(x+h)$ 

$$\begin{aligned} \left\| \frac{1}{h} (\tau_h - 1) F_1(v) \right\|_{L^1(I; L_x^2)} &\leq C \frac{1}{|h|} \left\| (\tau_h - 1)v \right\|_{L^1(I; L^2)} \\ &\leq CT \|\partial_x v\|_{L^\infty(I; L^2)} \\ \implies |\langle F_1(v), \partial \phi \rangle| &\leq CT \|\partial v\|_{L^\infty(I; L^2)} \end{aligned}$$

$$\text{Thus } \|\partial F_1(v)\|_{L^1(I; L^2)} \leq CT \|\partial v\|_X$$

$$\begin{aligned}
\|\partial F_2(v)\|_{L^{r'}(I;L_x^{q'})} &\leq C\| |v|^{p-1}\partial v\|_{L^{r'}(I;L_x^{q'})} \\
&\leq C\left(\int\left(\|v(t)\|_{L_x^q}^{p-1}\|\partial_x v(t)\|_{L_x^q}\right)^{r'}dt\right)^{\frac{1}{r'}} \\
&\leq C\|v\|_{L^\infty(I;L^q)}^{p-1}T^\alpha\|\partial_x v\|_{L^r(I;L_x^q)} \\
&\leq CT^\alpha\|v\|_Y^{p-1}\|\partial v\|_X.
\end{aligned}$$

□

**Proposition** Let  $\psi_0 \in H^1$ . Then there exists  $T = T(\|\psi_0\|_{H^1}) > 0$  so that there is a unique  $u \in \bar{Y}([0, T])$  with  $u = G_0\psi_0 + iGFu$ .

*Proof.* Let  $\Phi(u) = G_0\psi_0 + iGF(u)$ .

Then  $\Phi : B_R(Y) \circlearrowleft$  provided  $R = C_1\|\psi_0\|_{H^1}$ ,  $C_1 \gg 1$ , absolute constant.

$$\begin{aligned}
\|\Phi(u)\|_Y &\leq C\|\psi_0\|_{H^1} + C\left(T + T^\alpha\|u\|_Y^{p-1}\right)\|u\|_Y \\
&\leq C\|\psi_0\|_{H^1} + \underbrace{C(T + T^\alpha R^{p-1})}_{< \frac{1}{2}}R \leq R
\end{aligned}$$

for  $R = 2C\|\psi_0\|_{H^1}$  and  $T = T(R) \ll 1$ .

Let  $E = \{B_R(Y), \|\cdot\|_X\}$ .

$$\|\Phi(u) - \Phi(v)\|_X \leq C(T + T^\alpha R^{p-1})\|u - v\|_X < \frac{1}{2}\|u - v\|_X.$$

Suppose  $\|u_n - u_m\|_X \xrightarrow{n, m \rightarrow \infty} 0$  and  $(u_n) \subset B_R(Y)$

$$\begin{aligned}
&\Rightarrow u_n \rightarrow u \text{ in } \|\cdot\|_X \\
\langle \partial u_n(t), \phi(t) \rangle &= -\langle u_n(t), \partial \phi(t) \rangle \xrightarrow{n \rightarrow \infty} -\langle u(t), \partial \phi(t) \rangle \\
&\Rightarrow u \in B_R(Y)
\end{aligned}$$

Thus  $E$  is complete.

So by contraction argument we are done. □

**Lemma.** Let  $u = u_{\psi_0}$  be the solution from the previous Proposition. Then  $H^1 \ni \psi_0 \mapsto u_{\psi_0} \in \bar{Y}$  is continuous in the following sense:

Given  $\psi_0 \in H^1$ , there exists  $T = T(\psi_0) > 0$  so that  $\forall \varepsilon > 0, \exists \delta > 0$  such that if  $\psi_1 \in H^1$ ,  $\|\psi_0 - \psi_1\|_{H^1} \leq \delta \Rightarrow u = u_{\psi_0}$  and  $v = v_{\psi_1}$  exists in  $\bar{Y}([0, T])$  and  $\|u - v\|_Y \leq \varepsilon$ .

*Proof.* It is clear that we have  $T = T(\|\psi_0\|)$ .

$$\begin{aligned}
\Phi(w) &= \Phi_{\psi_0}(w) = G_0\psi_0 + iGFw \\
&\Rightarrow u = \lim_{n \rightarrow \infty} \Phi_{\psi_0}^n(\psi_0)
\end{aligned}$$

$$\begin{aligned}
\|u - \Phi_{\psi_0}^n(\psi_0)\|_X &< \delta^n \\
\|v - \Phi_{\psi_1}^n(\psi_1)\|_X &< \delta^n \\
\Rightarrow \|u - v\|_X &\leq 2\delta^n + \|\Phi_{\psi_0}^n(\psi_0) - \Phi_{\psi_1}^n(\psi_1)\|_X
\end{aligned}$$

Induction:

$$\begin{aligned}
\|\Phi_{\psi_0}(\psi_0) - \Phi_{\psi_1}(\psi_1)\|_X &\leq \|GF(\psi_0) - GF(\psi_1)\|_X + \|G_0(\psi_0 - \psi_1)\|_X \\
&\leq C\|\psi_0 - \psi_1\|_{L^2} + C(T + T^\alpha R^{p-1})\|\psi_0 - \psi_1\|_X \\
&\leq C\|\psi_0 - \psi_1\|_{L^2} + C\|\psi_0 - \psi_1\|_{H^1}
\end{aligned}$$

because  $\Phi_\psi(\psi) \in B_R(Y)$

$$\begin{aligned}
\|\Phi_{\psi_0}^2(\psi_0) - \Phi_{\psi_1}^2(\psi_1)\|_X &\leq C\|\psi_0 - \psi_1\|_2 + C(T + T^\alpha R^{p-1})\|\Phi_{\psi_0}(\psi_0) - \Phi_{\psi_1}(\psi_1)\|_X \\
&\leq \dots
\end{aligned}$$

$\Rightarrow$  (by induction)

$$\|u - v\|_X < \varepsilon.$$

(To be continued)

□

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Last time: we showed that for every  $\psi_0 \in H^1$

$$(DH) \quad u = G_0\psi_0 + i GF(u)$$

has a unique solution

$$u \in \bar{Y} = \{v \in \bar{Y} : \partial v \in \bar{X}\} \subset C(I, H^1) \\ I = [0, T], \quad T = T(\|\psi_0\|_{H^1})$$

• continuity in the following sense:

if  $\psi_n \xrightarrow{H^1} \psi_0$  then for some  $T = T(\|\psi_0\|_{H^1}) > 0$  one has  $\|u_n - u\|_X \rightarrow 0$  where  $u_n, u$  are solutions of (DH) with  $\psi_0, \psi$  respectively.

Further observation: can take  $X_0$  instead of  $X$

$$\|u_n(t) - u(t)\|_{L^q(\mathbb{R}^d)} \leq C \|u_n(t) - u(t)\|_{L^2}^\theta \|\partial u_n(t) - \partial u(t)\|_{L^2}^{1-\theta}$$

for some  $0 < \theta < 1$  provided  $q < \frac{2d}{d-2}$

$$\text{Also : } \begin{cases} \|u_n(t) - u(t)\|_{L^2}^\theta \rightarrow 0 \\ \|\partial u_n(t) - \partial u(t)\|_{L^2}^{1-\theta} \leq \|\partial u\|_{L^\infty(I, L^2)}^{1-\theta} + \|\partial u_n\|_{L^\infty(I, L^2)}^{1-\theta} \end{cases}$$

Thus  $\|u_n - u\|_{L^\infty(I; L^q)} \mapsto 0$

For future reference note that  $\|u_n - u\|_{L^\infty(I, L^{q+\varepsilon})} \rightarrow 0$  for small  $\varepsilon > 0$ .

**Claim:**  $\|u_n - u\|_Y \rightarrow 0$  as  $n \rightarrow \infty$

$$\begin{aligned} \partial u - \partial u_n &= G_0(\partial\psi_0 - \partial\psi_n) + iG(F'(u)\partial u - F'(u_n)\partial u_n) \\ &= G_0(\partial\psi_0 - \partial\psi_n) + iG\left[F'(u_n)(\partial u - \partial u_n)\right] + iG\left[(F'(u) - F'(u_n))\partial u\right] \end{aligned}$$

$$\begin{aligned} \|G_0(\partial\psi_0 - \partial\psi_n)\|_X &\leq C\|\partial\psi_0 - \partial\psi_n\|_2 \rightarrow 0 \\ \|F'(u_n)(\partial u - \partial u_n)\|_{X^1} &\leq \|F'_1(u_n)(\partial u - \partial u_n)\|_{L^1(I; L^2)} \\ &\quad + \|F'_2(u_n)(\partial u - \partial u_n)\|_{L^{r'}(I; L_x^{q'})} = A + B \end{aligned}$$

Repeat the contraction argument:

$$\left. \begin{aligned} A &\leq CT\|\partial u - \partial u_n\|_{L^\infty(I, L^2)} \\ \text{Also, for } B : B &\leq CT^\infty\|u_n\|_{X_0}^{p-1}\|\partial u - \partial u_n\|_{L^r(I, L_x^q)} \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow A + B \leq \delta \|\partial u - \partial u_n\|_X$$

$$\textcircled{*} \quad \|(F'_1(u) - F'_1(u_n))\partial u\|_{L^1(I, L^q)} \stackrel{\text{claim}}{\mapsto} 0$$

$$\textcircled{*}\textcircled{*} \quad \|(F'_2(u) - F'_2(u_n))\partial u\|_{L^{r'}(I; L^{q'})} \mapsto 0$$

For  $\circledast$ ,

$$|(F'_1(u) - F'_1(u_n))(t, x) \cdot \partial u(t, x)| \leq C |\partial u(t, x)| \in L^\infty(I; L^2)$$

It suffices to show that:

$$\|(F'_1(u) - F'_1(u_n))\partial u(t)\|_{L^2_x} \rightarrow 0$$

for a.e.  $t \in I$ .

Assume that

$$u_n(t, \cdot) \rightarrow u(t, \cdot) \text{ a.e.} \Rightarrow \underbrace{(F'_1(u_n)) - F'_1(u)(t) \cdot \partial u(t)}_{\substack{\downarrow \\ \text{a.e.} \\ 0}}$$

And it has a positive  $L^2$ -majorant.

Fix "a.e."  $t$ . Then  $u_{n_j}(t, \cdot) \rightarrow u(t, \cdot)$  a.e. Therefore:

$$\|(F(u_{n_j}) - F(u))(t)(\partial u)(t)\|_2 \rightarrow 0.$$

$$u_n(t) \xrightarrow{L^{q+\varepsilon}, L^q} u(t)$$

Given  $\varepsilon$ , there exists  $\rho(t) > 0$  s.t.

$$\sup_n \left[ \|u_n(t)\|_{L^q(c_{B_{\rho(t)}})} + \|u(t)\|_{L^q(c_{B_{\rho(t)}})} \right] < \varepsilon$$

$$\begin{aligned} \|(F'_2(u_n) - F'_2(u))(t)\partial u(t)\|_{L^{q'}(c_{B_{\rho(t)}})} &\leq C (\|u_n(t)\|_{L^q(c_{B_{\rho(t)}})}^{p-1} \\ &+ \|u(t)\|_{L^q(c_{B_{\rho(t)}})}^{p-1}) \|\partial u(t)\|_{L^q} < c\varepsilon^{p-1} \|\partial u(t)\|_{L^q} \end{aligned} \quad \circledast \circledast \circledast$$

It suffices to show that

$$\|(F'_2(u_n) - F'_2(u))(t)\partial u(t)\|_{L^{q'}_x} \xrightarrow{\text{a.e. } t} 0$$

$$\begin{aligned} \text{Since } lhs &\leq C \left( \|u_n(t)\|_q^{p-1} + \|u(t)\|_q^{p-1} \right) \|\partial u(t)\|_{L^q} \\ &\leq C R^{p-1} \|\partial u(t)\|_{L^q} \in L^r(I) \hookrightarrow L^{r'}(I). \end{aligned}$$

By  $\circledast \circledast \circledast$  it suffices to show that

$$\int_{|x| < \rho(t)} \left| (F'_2(u_n(t, x)) - F'_2(u(t, x))) \partial u(t, x) \right|^{q'} dx \mapsto 0$$

$$S_{\delta_1, \delta_2} = \left\{ |x| < \rho(t) : |u_n(t, x) - u(t, x)| < \delta_1, |u_n(t, x)| < \frac{1}{\delta_2}, |u(t, x)| < \frac{1}{\delta_2} \right\}$$

$$|B_{\rho(t)} \setminus S_\delta| \leq \frac{1}{\delta_1^q} \|u_n(t) - u(t)\|_q^q + \delta_2^q \left( \|u_n(t)\|_q^q + \|u(t)\|_q^q \right)$$

So

$$\begin{aligned} & \int_{|X| < \rho(t)} \left| [F_2'(u_n(t, x)) - F_2'(u(t, x))] \partial u(t, x) \right|^{q'} dx \\ & \leq \int_{S_{\delta_1, \delta_2}} \varepsilon^{q'} |\partial u(t, x)|^{q'} dx + \underbrace{\left( \int_{B_{\rho(t)} \setminus S_{\delta_1, \delta_2}} |\partial u(t, x)|^q dx \right)}_{< \varepsilon} \underbrace{\left( \|u_n(t)\|_q^{q(p-1)} + \|u(t)\|_q^{q(p-1)} \right)}_{< C \text{ unif. in } n}. \end{aligned}$$

Therefore

$$\|\partial(u_n - u)\|_X \leq C \|\psi_n - \psi_0\|_{H^1} + \delta \|\partial(u_n - u)\|_X + 0(1) \text{ as } n \rightarrow \infty.$$

Conclusion:  $\|u_n - u\|_y \mapsto 0$ .

**Lemma.**  $\psi \in C(I, H^1) \Rightarrow F(\psi) \in C(I, H^1)$

*Proof.*

$$\begin{aligned} \|F_1(\psi(t)) - F_1(\psi(s))\|_2 & \leq C \|\psi(t) - \psi(s)\|_2 \longrightarrow 0 \text{ as } s \longrightarrow t \\ \|F_2(\psi(t)) - F_2(\psi(s))\|_{L^{q'}} & \leq C \left( \|\psi(t)\|_q^{p-1} + \|\psi(s)\|_q^{p-1} \right) \underbrace{\|\psi(t) - \psi(s)\|}_{\substack{\rightarrow 0 \text{ as } s \rightarrow t \\ \text{by Sobolev embedding}}} \end{aligned}$$

We have shown  $F(\psi) \in C(I; L^2 + L^{q'}) \hookrightarrow L(I, H^{-1})$  by duality (since  $H^1 \hookrightarrow L^2 + L^q$ ,  $L^2 + L^{q'} \hookrightarrow H^{-1}$ )  $\square$

**Corollary.** Let  $\psi$  be the unique local solution of  $\psi = G_0\psi_0 + iGF(\psi)\phi_0 \in H^1$  which is given by the previous construction.

Then  $\partial_t \psi \in C(I; H^{-1})$  and  $i\partial_t \psi + \frac{1}{2}\Delta \psi + F(\psi) = 0$  pointwise in  $t \in I$  as an identity in  $H^{-1}$ .

*Proof.*

$$\begin{aligned} \psi(t) & = e^{i\frac{t}{2}\Delta} \psi_0 + i \int_0^t e^{i(t-s)\frac{\Delta}{2}} F(\psi(s)) ds. \\ \frac{1}{h}(\psi(t+h) - \psi(t)) & = \frac{1}{h}(e^{i\frac{h}{2}\Delta} - I)e^{it\frac{\Delta}{2}} \psi_0 + i\frac{1}{h}(e^{ih\frac{\Delta}{2}} - I) \\ & \cdot \underbrace{\int_0^t e^{i(t-s)\frac{\Delta}{2}} F(\psi(s)) ds}_{GF(\psi) \in H^1} + i\frac{1}{h} \int_t^{t+h} e^{i(t+h-s)\frac{\Delta}{2}} F(\psi(s)) ds + i\frac{1}{h} \\ & \cdot \int_t^{t+h} e^{i(t+h-s)\frac{\Delta}{2}} [F(\psi(s)) - F(\psi(t))] ds \end{aligned}$$

We prove

$$\left\| \frac{1}{h}(e^{i\frac{h}{2}\Delta} - I)f - \frac{i}{2}\Delta f \right\|_{H^1} \longrightarrow 0 \text{ as } h \rightarrow 0 \text{ for every } f \in H^1$$

because

$$\int \underbrace{\left| \left[ \frac{1}{h} (e^{-i\frac{h}{2}\xi^2} - 1) + \frac{i}{2}\xi^2 \right] \right|^2}_{\leq C|\xi|^4 \text{ unif. in } h} |\hat{f}(\xi)|^2 (1 + |\xi|^2) d\xi \xrightarrow{\text{LDCT}} 0$$

$$i \frac{1}{h} \int_t^{t+h} e^{i(t+h-s)\frac{\Delta}{2}} [F(\psi(s)) - F(\psi(t))] ds \longrightarrow 0 \text{ in } H^{-1},$$

because

$$\int \left| \frac{1}{h} \int_0^h (e^{-ih\frac{\xi^2}{2}} du - 1) \right|^2 (1 + |\xi|^2)^{-1} |\hat{f}(\xi)|^2 d\xi \xrightarrow{\text{LDCT}} 0$$

□

**Corollary.** Suppose  $\psi_0 \in H^1$ ,  $\phi \in C(I; H^1) \cap C^1(I; H^1)$

$$\begin{cases} i\partial_t \psi + \frac{1}{2} \Delta \psi + F(\psi) = 0 \\ \psi|_{t=0} = \psi_0 \end{cases}$$

Then  $\psi_0 = G_0 \psi_0 + i GF(\psi)$ .

*Proof.*

$$i\partial_t (e^{-it\frac{\Delta}{2}} \psi(t)) = ie^{-it\frac{\Delta}{2}} F(\psi(t))$$

$$i\partial_t (e^{-it\frac{\Delta}{2}} \psi(t)) = i \left( -i \frac{\Delta}{2} e^{-it\frac{\Delta}{2}} \psi(t) + e^{-it\frac{\Delta}{2}} \partial_t \psi(t) \right)$$

$$= -e^{-it\frac{\Delta}{2}} F(\psi(t)).$$

$$ie^{-it\frac{\Delta}{2}} \psi(t) - i\psi(0) = - \int_0^t e^{-is\frac{\Delta}{2}} F(\psi(s)) ds$$

$$\psi(t) = e^{it\frac{\Delta}{2}} \psi(0) + i \int_0^t e^{i(t-s)\frac{\Delta}{2}} F(\psi(s)) ds$$

□

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**Theorem.**

- 1) Let  $0 < \sigma < \frac{2}{d-2}$ . For every  $\psi_0 \in H^1$   
 $\exists T_* > 0$  (max. time of existence),  $T_* = T_*(\|\psi_0\|_{H^1})$  so that  $\exists! \psi \in C([0, T_*], H^1) \cap C^1([0, T_*], H^{-1})$ ,

$$i\partial_t \psi + \frac{1}{2} \Delta + \lambda |\psi|^{2\sigma} \psi = 0 \quad (\text{pointwise in } H^{-1}).$$

Moreover,  $\psi$  depends continuously on  $\psi_0$  (w.r.t.  $H^{-1}$ ).

- 2) If  $\psi_0 \in H^2$ , then  $\psi \in C([0, T_*], H^2) \cap C^1([0, T_*], L^2)$ .

*Remark*

- 1) If  $T_* < \infty$ , then  $\lim_{t \rightarrow T_*} \|\psi(t)\|_{H^1} = \infty$ .

(because if  $\lim_{t \rightarrow T_*} \|\psi(t)\|_{H^1} < \infty$ , then take  $t_0 < T_*$  so close to  $T_*$  that you can continue beyond  $T_*$ ).

- 2) You can include a potential term  $V(t, x)\psi$  in part 1) of the theorem, provided

$$\sup_t \left[ \|V(t, x)\|_{L_x^\infty + L_x^{d/2}} + \|\nabla_x V(t, x)\|_{L_x^\infty + L_x^{d/2}} \right] < \infty.$$

Back to contraction argument for Kato's theorem.

$$\tilde{F}(t, x, u) = V(t, x)u + \underbrace{F(u)}_{\lambda|u|^{2\sigma}u}$$

we needed

$$\begin{aligned} F &: X_0 \rightarrow X' \\ \tilde{F} &: Y \rightarrow Y' \end{aligned}$$

$$X_0 = L^\infty(I; L^2 \cap L^q) \quad q = 2\sigma + 2 < \frac{2d}{d-2}$$

$$X' = L^1(I; L^2) + L^{r'}(I, L^{q'})$$

$$\text{Put } F = 0 \text{ and } V(t, x) = \underbrace{V_1(t, x)}_{L^\infty} + \underbrace{V_2(t, x)}_{L^{d/2}}$$

$$\|V_1(t, x)\psi\|_{L^1(I, L^2)} \leq CT \|\psi\|_{L^\infty(I, L^2)}$$

$$\begin{aligned} \|V_2(t, x)\psi\|_{L^{r'}(I, L^2)} &= \left( \int_0^T \|V_2(t, \cdot)\psi\|_{L^{q'}}^{r'} dt \right)^{1/r'} \\ &\leq \left( \int_0^T \|V_2(t, \cdot)\|_{q'(\frac{q}{q'})}^{r'} \|\psi(t)\|_q^{r'} dt \right)^{1/r'} \\ &\leq \|V_2\|_{L^\infty(\mathbb{R}, L^{q'(\frac{q}{q'})})} \underbrace{T^\alpha}_{\substack{\text{gain} \\ \text{from} \\ r \text{ to } r'}} \|\psi\|_{L^r(I, L^q)} \end{aligned}$$

$$m = \frac{q}{q-1} \frac{q-1}{q-2} = \frac{q}{q-2} \Rightarrow \frac{1}{m} = 1 - \frac{2}{q} < 1 - \frac{d-2}{d} = \frac{2}{d}$$

$$\infty > m > \frac{d}{2} \quad \text{so ok.}$$

Conserved quantities. Let  $\psi_0, \psi$  be as in the theorem.  
Then

$$(i) \quad \|\psi(t)\|_2 = \|\psi_0\|_2$$

$$(ii) \quad \mathcal{H}[\psi(t)] = \int \left\{ \frac{1}{2} |\nabla \psi(t, x)|^2 - \frac{\lambda}{\sigma+1} (\psi)^{2\sigma+2}(t, x) \right\} dx = \mathcal{H}[\psi_0].$$

*Proof.*

$$(i) \quad \langle i\partial_t \psi, \psi \rangle + \frac{1}{2} \underbrace{\langle \Delta \psi, \psi \rangle + \lambda \langle |\psi|^{2\sigma} \psi, \psi \rangle}_{\text{real}} = 0$$

$$\underbrace{i\langle \partial_t \psi, \psi \rangle + i\langle \psi, \partial_t \psi \rangle}_{= i \frac{d}{dt} \langle \psi, \psi \rangle(t)} = 0$$

$$(ii) \quad \langle i\partial_t \psi + \frac{1}{2} \Delta \psi + \lambda |\psi|^{2\sigma} \psi = 0, \partial_t \psi \rangle = 0$$

need  $\psi \in H^2$  and then pass to the limit.

$$i \|\partial_t \psi\|_2^2 + \frac{1}{2} \underbrace{\langle \Delta \psi, \partial_t \psi \rangle}_{-\langle \nabla \psi, \partial_t \nabla \psi \rangle} + \lambda \langle |\psi|^{2\sigma} \psi, \partial_t \psi \rangle = 0$$

(add conjugate)

$$-\frac{1}{2} \langle \nabla \psi, \partial_t \nabla \psi \rangle - \frac{1}{2} \langle \partial_t \nabla \psi, \nabla \psi \rangle + \lambda (\langle |\psi|^{2\sigma} \psi, \partial_t \psi \rangle + \langle \partial_t \psi, |\psi|^{2\sigma} \psi \rangle) = 0$$

$$0 = -\frac{1}{2} \frac{d}{dt} \|\nabla \psi\|_2^2 + \frac{\lambda}{\sigma+1} \frac{d}{dt} \int |\psi|^{2\sigma+2} dx$$

$$\text{If } \psi_0^{(n)} \xrightarrow{H^1} \psi_0 \Rightarrow \psi^{(n)} \longrightarrow \psi \text{ in } C([0, T_*], H^1) \cap C^1([0, T_n], H^{-1})$$

then

$$\mathcal{H}[\psi^{(n)}(t)] = \mathcal{H}[\psi_0^{(n)}]$$

$$\downarrow \text{(Sobolev embedding)}$$

$$\mathcal{H}[\psi(t)] = \mathcal{H}[\psi_0]$$

□

**Theorem.** (Global existence) Let  $\lambda \leq 0$  (good) and  $\sigma < \frac{2}{d-2}$ , or  $\lambda > 0$  and  $\sigma < \frac{2}{d}$ .  
Then  $T_* = \infty, \forall \psi_0 \in H^1$

*Proof.*

- 1)  $\sup_{t < T^*} \frac{1}{2} \|\nabla \psi(t)\|_2^2 \leq \mathcal{H}[\psi_0] \Rightarrow T^* = \infty$ .  
 2) For  $\lambda > 0$ , we need Gagliardo-Nirenberg-Sobolev:

$$\|u\|_q \leq C \|\nabla u\|_2^\theta \|u\|_2^{1-\theta}$$

$$\text{need } 2 \leq q \leq 2^* = \frac{2d}{d-2}$$

$$\left| \begin{array}{l} u(x) \longrightarrow u(\rho x) \\ \rho^{-d/q} = \rho^\alpha \rho^{-d\theta/2} \rho^{-d(1-\theta)/2} \\ \theta = \frac{d}{2} - \frac{d}{q} = d \left( \frac{1}{2} - \frac{1}{q} \right) \end{array} \right.$$

□

*Proof of GNS.* Show  $\|u\|_{2^*} \leq C \|\nabla u\|_2$ ,  $\forall u \in \mathcal{S}$   
 (endpoint is sharp Sobolev emb.)  
 (other endpoint is  $\|u\|_2 \leq \|u\|_2$ ), then interpolation

$$\hat{u}(\xi) = \frac{\xi}{|\xi|^2} \widehat{\nabla u}(\xi)$$

$$\|u\|_{2^*} = \left\| \left( \frac{\xi}{|\xi|^2} \widehat{\nabla u}(\xi) \right)^\vee \right\|_{2^*} \stackrel{?}{\leq} C \|\nabla u\|_2$$

→ need the multiplier  $\frac{\xi}{|\xi|^2}$  to take  $L^2 \rightarrow L^{2^*}$ .

compare to  $m_0(\xi) = \frac{1}{|\xi|}$ ,  $\check{m}_0(x) = \frac{C}{|x|^{d-1}}$ ,  
 fractional integration, takes  $L^p \rightarrow L^r$ ,  $1 < r < \infty$

$$\text{with } 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{\frac{d}{d-1}} \Rightarrow \frac{1}{d} = \frac{1}{p} - \frac{1}{r}$$

$$\text{and } \frac{1}{d} = \frac{1}{2} - \frac{1}{2^*} \quad \text{works}$$

To go from  $m_0$  to

$$\vec{m} = \left( \frac{\xi_j}{|\xi|^2} \right)_{j=1}^d = \left( \hat{R}_j \frac{1}{|\xi|} \right)_{j=1}^d$$

and  $\hat{R}_j$  is bounded from  $L^2$  to  $L^2$  and  $L^{2^*}$  to  $L^{2^*}$

$$\begin{aligned} \text{then } \int \frac{1}{2} |\nabla \psi(t, x)|^2 dx &\leq \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1} \int |\psi|^{2\sigma+2} dx \\ &\leq \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1} \|\nabla \psi(t)\|_2^{\theta(2\sigma+2)} \underbrace{\|\psi(t)\|_2^{(1-\theta)(2\sigma+2)}}_{\|\psi_0\|_2^{(1-\theta)(2\sigma+2)}} \end{aligned}$$

$$\text{need } \begin{array}{c} \theta \\ \downarrow \\ d \left( \frac{1}{2} - \frac{1}{q} \right) \end{array} (2\sigma+2) < 2$$

$$d \left( \frac{1}{2} - \frac{1}{2\sigma + 2} \right) (2\sigma + 2) < 2, \quad \sigma < \frac{2}{d} \quad \text{OK.}$$

□

When  $\sigma = \frac{2}{d}$ , Ok when small  $L^2$  data.

**Remark** (small data global existence)

$$1) \quad \sigma = \frac{2}{d} \text{ and } \|\psi_0\|_2 < \varepsilon = \varepsilon(\lambda, \sigma, d) \Rightarrow T_* = \infty.$$

$$2) \quad \frac{2}{d} < \sigma < \frac{2}{d-2} \text{ and } \|\psi_0\|_{H^1} < \varepsilon \Rightarrow T_* = \infty.$$

*Proof.*

$$1) \quad \frac{1}{2} \|\nabla \psi(t)\|_2^2 \leq \mathcal{H}[\psi_0] + \underbrace{\frac{\lambda}{\sigma+1} \varepsilon^{2\sigma}}_{< \frac{1}{4} \text{ ok.}} \|\nabla \psi(t)\|_2^2$$

$$2) \quad \frac{1}{2} \|\nabla \psi(t)\|_2^2 \leq \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1} \|\nabla \psi(t)\|_2^{d\sigma} \underbrace{\|\psi_0\|_2^{2\sigma+2-d\sigma}}_{= \frac{\lambda}{\sigma+1} \|\nabla \psi(t)\|_2^{d\sigma-2} \|\psi_0\|_2^{2\sigma+2-d\sigma}} \\ \leq \frac{1}{4} \|\nabla \psi\|_2^2$$

$$\|\nabla \psi(t)\|_2^{d\sigma-2} \|\psi_0\|_2^{2\sigma+2-d\sigma}$$

$$\leq (4\mathcal{H}[\psi_0])^{\frac{d\sigma-2}{2}} \|\psi_0\|_2^{2\sigma+2-d\sigma}$$

need  $< \frac{1}{4} \cdot \frac{\sigma+1}{\lambda}$  then proceed “inductively.”

□

NB:  $\lambda \leq 0$  = defocussing

$\lambda > 0$  = focussing

**Theorem.** (Glassey) Let  $\lambda > 0$  and  $\frac{2}{d} \leq \sigma < \frac{2}{d-2}$   
 If  $\psi_0 \in H^1$ ,  $x\psi_0 \in L^2$  and  $\mathcal{H}[\psi_0] < 0$ .  
 $\Rightarrow T_* < \infty$ .

*Proof.* We need to show that

$$\frac{d^2}{dt^2} \int |x|^2 |\psi(t, x)|^2 dx = 4\mathcal{H}[\psi_0] - \frac{2\lambda}{\sigma+1} (d\sigma - 2) \int_{\mathbb{R}^d} |\psi|^{2\sigma+2}(t, x) dx$$

(nonlinear Viriel identity)

then

$$\frac{d^2}{dt^2} \int |x|^2 |\psi(t, x)|^2 dx < 0$$

$\int$  must become  $< 0$  at some point, blowup (focusing at  $x = 0$ ,  $L^\infty$  also blows up)  $\square$

**Prop** Let  $\psi \in C([0, T_*), H^1) \cap C^1([0, T_*), H^1)$  be a solution of

$$\begin{cases} i\partial_t \psi + \frac{1}{2}\Delta + \lambda|\psi|^{2\sigma} = 0 \\ \psi \Big|_{t=0} = \psi_0 \in H^1. \end{cases}$$

Suppose  $x\psi_0 \in L^2 \Rightarrow x\psi(t, x) \in L^2$  for all  $0 < t < T_*$  and  $t \rightarrow \int |x|^2 |\psi(t, x)|^2 dx$  is  $C^2$  and Viriel holds. Moreover, one has the pseudoconformal identity

$$\begin{aligned} \|(x - t_p)\psi(t)\|_2^2 - 2t^2 \frac{\lambda}{\sigma+1} \int_{\mathbb{R}} |\psi(t, x)|^{2\sigma+2} dx \\ = \|x\psi_0\|_2^2 + 2 \frac{xd\sigma - 2}{\sigma+1} \int_0^t \tau \int_{\mathbb{R}^d} |\psi(\tau, x)|^{2\sigma+2} dx d\tau. \end{aligned}$$

$\square$

*Proof.*  $\varepsilon > 0$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|e^{-\varepsilon|x|^2} |x| \psi(t, x)\|_2^2 \\ &= \operatorname{Re} \langle e^{-2\varepsilon|x|^2} |x|^2 \psi_t, \psi \rangle \\ &= \operatorname{Re} i \langle e^{-2\varepsilon|x|^2} |x|^2 \psi, i \psi_t \rangle \\ &= -\operatorname{Re} i \langle e^{-2\varepsilon|x|^2} |x|^2 \psi, \frac{1}{2} \Delta \psi + \lambda |\psi|^{2\sigma} \psi \rangle \\ &= \frac{1}{2} \operatorname{Re} i \langle \nabla (e^{-2\varepsilon|x|^2} |x|^2 \psi), \nabla \psi \rangle \\ &= \operatorname{Re} i \langle e^{-2\varepsilon|x|^2} |x| \cdot \psi, \nabla \psi \rangle \\ &\quad - 2 \operatorname{Re} i \langle e^{2\varepsilon|x|^2} \varepsilon x |x|^2 \psi, \nabla \psi \rangle \end{aligned}$$

Gronwall uniformly in  $\varepsilon$ .

$$\begin{aligned}
\frac{1}{2} \left\| e^{-\varepsilon|x|^2} |x| \psi(t, x) \right\|_2^2 &\leq \left\| |x| \psi_0 \right\|_2^2 \\
&+ \int_0^t \left\| e^{-\varepsilon|x|^2} |x| \psi(\tau) \right\|_2 \left\| \nabla \psi(\tau) \right\|_2 d\tau \\
&+ 2 \int_0^t \underbrace{\left\| e^{-\varepsilon|x|^2} \varepsilon |x|^2 \right\|_\infty}_{\substack{\text{bdd OK} \\ \text{unif. in } \varepsilon}} \left\| e^{-\varepsilon|x|^2} x \psi(\tau) \right\|_2 \underbrace{\left\| \nabla \psi(\tau) \right\|_2}_{\substack{\text{OK because} \\ H^1 \text{ solutions} \\ \text{away from } T_*}} d\tau \\
&\leq \left\| |x| \psi_0 \right\|_2^2 + \sup_{0 \leq \tau \leq T < T_*} \left\| \nabla \psi(\tau) \right\|_2 (1 + 2C_0) \\
&\quad \int_0^t \left\| e^{-\varepsilon|x|^2} x \psi(\tau) \right\|_2 d\tau \\
&\leq \left\| |x| \psi_0 \right\|_2^2 + K(T)^2 + T \int_0^t \left\| e^{-\varepsilon|x|^2} x \psi(\tau) \right\|_2^2 d\tau \\
&\quad \longrightarrow \text{expo growth, } \sup_{\varepsilon > 0} \left\| e^{-\varepsilon|x|^2} x \psi(\tau) \right\|_2 < \infty . \\
&\quad \quad \quad \forall 0 < \tau < T_*
\end{aligned}$$

□

4/1/2004

Last term we proved most of the following:

**Theorem.**

- 1)  $0 < \sigma < \frac{2}{d-2}$ . Then for every  $\psi_0 \in H^1$   $\exists 0 < T_* \leq \infty$  (max time of existence).  
 $T_* = T_*(\|\psi_0\|_{H^1})$ , so that  $\exists! \psi \in C([0, T_*]; H^1) \cap C^1([0, T_*], H^1)$

$$\text{with } \begin{cases} i\partial_t \psi + \frac{1}{2}\Delta \psi + \lambda|\psi|^{2\sigma}\psi = 0 \\ \psi|_{t=0} = \psi_0 \end{cases}$$

Moreover,  $\psi$  depends continuously on  $\psi_0 \in H^1$ .

- 2) If  $\psi_0 \in H^2$ , then  $\psi \in C([0, T]; H^2) \cap C^1([0, T]; L^2)$ .

**Remarks:**

- 1) If  $T_* < \infty$ , then  $\lim_{t \rightarrow T_*} \|\psi(t)\|_{H^1} = \infty$  (obvious from)  $T_* = T_*(\|\psi_0\|_{H^1})$ .  
 2) We can include a potential  $V(t, x)$  in part (1) of theorem to treat  $i\partial_t \psi + \frac{1}{2}\Delta \psi + V(t, x)\psi + \lambda|\psi|^{2\sigma}\psi = 0$

Assumption:  $\sup_t [\|V(t, x)\|_{L^\infty + L^{d/2}} + \|\nabla_x V(t, x)\|_{L^\infty + L^{d/2}}] < \infty$ .

We need to go back to contraction argument of Kato's Theorem.

$$\tilde{F}(t, x; u) = V(t, x)u + F(u) = V(t, x)u + \lambda|u|^{2\sigma}\psi$$

$$\tilde{F} : X_0 \rightarrow X^1, \tilde{F} : Y \rightarrow Y^1$$

$$X_0 = L^\infty(I; L^2 \cap L^q), X^1 = L^1(I; L^q) + L^{r^1}(I; L^{q'}) \quad , \quad q = 2\sigma + 2 < \frac{2d}{d-2}$$

Set  $F = 0$  and  $V(t, x) = V_1(t, x) + V_2(t, x)$

$$\|V_1(t, x)\psi\|_{L^1(I; L^2)} \leq CT\|\psi\|_{L^\infty(I; L^2)}$$

$$\begin{aligned} \|V_2(t, x)\psi\|_{L^{r'}(I; L^{q'})} &= \left( \int_0^T \|V_2(t, \cdot)\psi\|_{L^{q'}}^{r'} dt \right)^{\frac{1}{r'}} \leq \left( \int_0^T \|V_2(t, \cdot)\|_{q'(\frac{q}{q'})}^{r'} \|\psi\|_q^{r'} dt \right)^{\frac{1}{r'}} \\ &\leq \|V_2\|_{L^\infty(\mathbb{R}, L^{q'}(\frac{q}{q'}))} T^\alpha \|\psi\|_{L^r(I; L^q)} \end{aligned}$$

$$\text{with } m = q' \left( \frac{q}{q'} \right)' = \frac{q}{q-2} > \frac{d}{2} \text{ and } \alpha = \frac{1}{r'} - \frac{1}{r}.$$

**Conserved quantities:** Let  $\psi_0, \psi$  be as in Theorem. Then:

$$(i) \quad \|\psi(t)\|_2 = \|\psi_0\|_2$$

$$(ii) \quad \mathcal{H}[\psi(t)] = \int_{\mathbb{R}^d} \left\{ \frac{1}{2} |\nabla \psi(t, x)|^2 - \frac{\lambda}{\sigma+1} |\psi(t, x)|^{2\sigma+2} \right\} dx = \mathcal{H}[\psi_0].$$

*Proof.* (i) We have

$$i\langle \partial_t \psi, \psi \rangle + \frac{1}{2} \underbrace{\langle \Delta \psi, \psi \rangle + \lambda \langle |\psi|^{2\sigma} \psi, \psi \rangle}_{\text{real}} = 0$$

$$i\langle \partial_t \psi, \psi \rangle + i\langle \psi, \partial_t \psi \rangle = 0$$

$$i\frac{d}{dt} \langle \psi, \psi \rangle = 0$$

(ii)  $\langle Eq, \partial_t \psi \rangle = 0$  so for  $\psi_0 \in H^2$  :

$$i \|\partial_t \psi\|_2^2 + \frac{1}{2} \underbrace{\langle \Delta \psi, \partial_t \psi \rangle}_{-\langle \nabla \psi, \partial_t \nabla \psi \rangle} + \lambda \langle |\psi|^{2\sigma} \psi, \partial_t \psi \rangle = 0$$

$$0 = -\frac{1}{2} \langle \nabla \psi, \partial_t \nabla \psi \rangle - \frac{1}{2} \langle \partial_t \nabla \psi, \nabla \psi \rangle + \lambda (\langle |\psi|^{2\sigma} \psi, \partial_t \psi \rangle + \langle \partial_t \psi, |\psi|^{2\sigma} \psi \rangle)$$

$$0 = -\frac{1}{2} \frac{d}{dt} \|\nabla \psi\|_2^2 + \frac{\lambda}{\sigma+1} \frac{d}{dt} \int |\psi|^{2\sigma+2} dx$$

Now if  $\psi_0 \in H^1$ , take  $\psi^{(n)} \rightarrow \psi_0$  in  $H^1 \Rightarrow \psi^{(n)} \rightarrow \psi$  in  $C([0, T_*]; H^1) \cap C^1([0, T_*]; H^{-1})$

Hence

$$\begin{array}{ccc} \mathcal{H}[\psi^{(n)}(t)] & = & \mathcal{H}[\psi^{(n)}] \\ \downarrow & & \downarrow \\ \mathcal{H}[\psi(t)] & & \mathcal{H}[\psi_0] \end{array}$$

□

**Theorem.** (Global existence) Let  $\lambda \leq 0$  and  $\sigma < \frac{2}{d-2}$ , or  $\lambda > 0$  and  $\sigma < \frac{2}{d}$ .

Then  $T_* = \infty \quad \forall \psi_0 \in H^1$

*Proof.*

$$\lambda \leq 0 \quad \Rightarrow \quad \sup_{t \leq T_*} \frac{1}{2} \|\nabla \psi(t)\|_2^2 \leq \mathcal{H}[\psi_0] \Rightarrow T_* = \infty$$

$\lambda > 0$ — we need Gagliardo-Nirenberg-Sobolev:

$$\begin{aligned} \|u\|_q &\leq C \|\nabla u\|_2^\theta \|u\|_2^{1-\theta} \\ 2 \leq q \leq 2^* &= \frac{2d}{d-2}, \quad \frac{1}{2} - \frac{1}{2^*} = \frac{1}{d} \\ \theta &= d \left( \frac{1}{2} - \frac{1}{q} \right) \text{ by scaling} \end{aligned}$$

*Proof of G-N-S:*  $\|u\|_{2^*} \leq C \|\nabla u\|_2 \forall u \in \mathcal{S}$  (then extend to  $H^1$ ) (plus Hölder for  $2 < q < 2^*$ )

$$\hat{u}(\xi) = \frac{\xi}{|\xi|^2} \widehat{\nabla u}(\xi), \text{ so want } \left\| \left( \frac{\xi}{|\xi|^2} \widehat{\nabla u}(\xi) \right)^\vee \right\|_{2^*} \leq C \|\nabla u\|_2$$

So want to bound the multiplier  $\vec{m}(\xi) = \frac{\xi}{|\xi|^2}$  on  $L^2 \rightarrow L^{2^*}$

Let  $m_0(\xi) = \frac{1}{|\xi|}$ ;  $\check{m}_0(x) = \frac{C}{|x|^{d-1}}$  is kernel  $L^p \rightarrow L^r$  with  $1 < r < \infty$  and  $1 + \frac{1}{r} = \frac{1}{p} + \frac{d-1}{d} = 1 + \frac{1}{p} - \frac{1}{d}$ , so  $\frac{1}{d} = \frac{1}{p} - \frac{1}{r}$  ( $= \frac{1}{2} - \frac{1}{2^*}$ )

$$\text{Now } \vec{m} = \left( \frac{\xi_j}{|\xi|^2} \right)_{j=1}^d = \left( \hat{R}_j \frac{1}{|\xi|} \right)_{j=1}^d$$

with  $R_j$  - Riesz transf. (in Fourier space  $\hat{R}_j = \frac{\xi_j}{|\xi|}$ )

since  $\frac{1}{|\xi|} : L^2 \rightarrow L^{2^*}$  and  $R_j : L^p \rightarrow L^p$  ( $1 < p < \infty$ ), done

$$\begin{aligned} \int \frac{1}{2} |\nabla \psi(t, x)|^2 dx &\leq \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1} \int |\psi_0(t, x)|^{2\sigma+2} dx \\ &\leq \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1} \|\nabla \psi(t)\|_2^{\theta(2\sigma+2)} \underbrace{\|\psi(t)\|_2^{(1-\theta)}}_{\|\psi_0\|_2^{(1-\theta)(2\sigma+2)}} (2\sigma+2) \end{aligned}$$

$$\text{Need } \theta(2\sigma+2) < 2, \text{ so } d \left( \frac{1}{2} - \frac{1}{2\sigma+1} \right) (2\sigma+2) < 2, \text{ i.e., } \sigma < \frac{2}{d}$$

□

**Remark:** (small data global existence for  $\lambda > 0$ )

- 1)  $\sigma = \frac{2}{d}$  and  $\|\psi_0\|_2 < \varepsilon = \varepsilon(\lambda, \sigma, d) \Rightarrow T_* = \infty$
- 2)  $\frac{2}{d} < \sigma < \frac{2}{d-2}$  and  $\|\psi_0\|_{H^1} < \varepsilon \Rightarrow T_* = \infty$

*Proof.* 1)  $\frac{1}{2} \|\nabla \psi(t)\|_2^2 \leq \mathcal{H}[\psi_0] + \underbrace{\frac{\lambda}{\sigma+1} \varepsilon^{2\sigma}}_{\leq \frac{1}{4} \Rightarrow \text{done}} \|\nabla \psi(t)\|_2^2$  (see previous proof)

2)  $\frac{1}{2} \|\nabla \psi(t)\|_2^2 \leq \mathcal{H}[\psi_0] + \frac{\lambda}{\sigma+1} \|\nabla \psi(t)\|_2^{d\sigma} \|\psi_0\|_2^{2\sigma+2-d\sigma}$  ( $d\sigma = \theta(2\sigma+2)$ )  
 Write  $\|\nabla \psi\|_2^{d\sigma} = \|\nabla \psi\|_2^{d\sigma-2} \|\nabla \psi\|_2^2$  and need  $\frac{\lambda}{\sigma+1} \|\nabla \psi\|_2^{d\sigma-2} \|\psi_0\|_2^{2\sigma+2-d\sigma} \leq \frac{1}{4}$   
 So assume  $\|\nabla \psi\|_2^2 \leq 8\mathcal{H}[\psi_0]$  and take  $\varepsilon$  s.t.  $\frac{\lambda}{\sigma+1} (8\mathcal{H}[\psi_0])^{\frac{d\sigma-2}{2}} \|\psi_0\|_2^{2\sigma+2-d\sigma} \leq \frac{1}{4}$   
 Then get  $\|\nabla \psi\|_2^2 \leq 4\mathcal{H}[\psi_0]$  up to that  $T$  using  $\nearrow \Rightarrow$  can extend past  $T$

□

**Glassey's Theorem:** Let  $\lambda > 0$ ,  $\frac{2}{d} \leq \sigma < \frac{2}{d-2}$ . If  $\psi_0 \in H^1$ ,  $x\psi_0 \in L^2$ , and  $\mathcal{H}[\psi_0] < 0$ , then  $T_* < \infty$ .

*Proof.*

$$\frac{d^2}{dt^2} \int |X|^2 |\psi(x, t)|^2 dx \stackrel{\text{Virial identity}}{=} 4\mathcal{H}[\psi_0] - \frac{2\lambda}{\sigma+1} (d\sigma - 2) \int_{\mathbb{R}^d} |\psi(t, x)|^{2\sigma+2} dx \leq 4\mathcal{H}[\psi_0] < 0$$

□

**Proposition:** Let  $\psi \in C([0, T_*]; H^1) \cap C^1([0, T_*]; H^{-1})$  be a solution if

$$\begin{cases} i\partial_t \psi + \frac{1}{2} \Delta \psi + \lambda |\psi|^{2\sigma} \psi = 0 \\ \psi|_{t=\infty} = \psi_0 \in H^1 \end{cases}$$

Suppose  $x\psi_0 \in L^2 \Rightarrow x\psi(t, x) \in L^2$  for all  $0 < t < T_*$  and  $t \mapsto \int |x|^2 |\psi(t, x)|^2 dx$  is  $C^2$  and *Virial identity* holds. Moreover, one has the *pseudo-conformal identity*.

$$\|(x-tp)\psi(t)\|_2^2 - 2t^2 \frac{\lambda}{\sigma+1} \int_{\mathbb{R}^d} |\psi(t, x)|^{2\sigma+2} dx = \|x\psi_0\|_2^2 + \frac{2\lambda(d\sigma-2)}{\sigma+1} \int_0^t \tau \int_{\mathbb{R}^d} |\psi(t, x)|^{2\sigma+2} dx d\tau$$

*Proof.*

$$\varepsilon > 0; \frac{1}{2} \frac{d}{dt} \|e^{-\varepsilon|x|^2} |x| \psi(t, x)\|_2^2 = \text{Re } i \langle e^{-2\varepsilon|x|^2} |x|^2 \psi, i\psi_t \rangle$$

$$\begin{aligned}
&= -\operatorname{Re} i \langle e^{-2\varepsilon|x|^2} |x|^2 \psi, \frac{1}{2} \Delta \psi + \lambda |\psi|^{2\sigma} \psi \rangle \\
&= \frac{1}{2} \operatorname{Re} i \langle \nabla (e^{-2\varepsilon|x|^2} |x|^2) \psi, \nabla \psi \rangle \\
&= \operatorname{Re} i \langle e^{-2\varepsilon|x|^2} x \psi, \nabla \psi \rangle - 2 \operatorname{Re} i \langle e^{-2\varepsilon|x|^2} \varepsilon x |x|^2 \psi, \nabla \psi \rangle
\end{aligned}$$

So undo derivative:

$$\begin{aligned}
\frac{1}{2} \|e^{-2\varepsilon|x|^2} x \psi\|_2^2 &\leq \| |x| \psi_0 \|_2^2 + \int_0^t \|e^{-\varepsilon|x|^2} x \psi(\tau, x)\|_2 \|\nabla \psi(\tau, x)\|_2 d\tau \\
&\quad + 2 \int_0^t \underbrace{\|e^{-\varepsilon|x|^2}\|_\infty}_{\leq C \text{ unif. in } \varepsilon} \|e^{-\varepsilon|x|^2} x \psi(\tau)\|_2 \|\nabla \psi(\tau)\|_2 d\tau \\
&\leq \| |x| \psi_0 \|_2^2 + \underbrace{\sup_{0 \leq \tau \leq T < T_*} \|\nabla \psi(\tau)\|_2 (1 + 2C_0)}_{\leq K(T)} \int_0^t \|e^{-\varepsilon|x|^2} x \psi(\tau)\|_2 d\tau \\
&\leq \| |x| \psi_0 \|_2^2 + K(T) \int_0^t \|e^{-\varepsilon|x|^2} x \psi(\tau)\|_2^2 d\tau
\end{aligned}$$

So  $\sup_{\varepsilon > 0} \|e^{-\varepsilon|x|^2} x \psi(\tau)\|_2 < \infty \quad \forall 0 < t < T_*$  (by Gronwall)  $\Rightarrow x \psi(\tau, x) \in L^2$ .  $\square$

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**Version 2**

$H^1$  critical equation  $i\partial_t \psi + \frac{1}{2} \Delta \psi - |\psi|^4 \psi = 0$  in  $\mathbb{R}^3$  ( $\sigma = \frac{2}{d-2s}$ ,  $s = 1$ ,  $d = 3$ )

Major open problem to prove global existence —easy to show local existence for  $\sigma = \frac{2}{d-2}$ ,  $d \geq 3$ .

Main difference:  $T_* = T_*(\psi_0)$  but do not know if  $T_* = T_*(\|\psi_0\|_{H^1})$

Ideas: Strichartz and Sobolev.

**Theorem.** (Cazenave-Weissler)

Let  $d \geq 3$ ,  $\sigma = \frac{2}{d-2}$ . For every  $\psi_0 \in H^1$ , there exists

$$\psi \in C([0, T_*], H^1) \cap C^1([0, T_*], H^{-1}),$$

$$T_* = T_*(\psi_0) = 0,$$

of

$$\begin{cases} i\partial_t \psi + \frac{1}{2} \Delta \psi + \lambda |\psi|^{2\sigma} \psi = 0. \\ \psi|_{t=0} = \psi_0. \end{cases}$$

This solution belongs to  $L_{loc}^\gamma([0, T_*], W^{1,\rho})$ , where  $\gamma = \gamma(d)$ ,  $\rho = \rho(d)$  are a particular pair of Strichartz exponents

$$\left( \frac{2}{\gamma} + \frac{d}{\rho} = \frac{d}{2} \right) \quad \text{and}$$

$\psi$  is unique in  $L_{loc}^\infty([0, T_*], H^1) \cap L_{loc}^\gamma([0, T_*], W^{1,\rho})$ . Finally, there is  $L^2$  and energy cons.

**Idea:**

$$\text{set } f_n(u) = \lambda \begin{cases} u^\sigma, & 0 \leq u \leq n \\ \text{smooth} & n < u < 4n \\ (2n)^\sigma & u \geq 4n. \end{cases}$$

$$\text{s.t. } |f'_n(u)| \leq C \cdot |u|^{\sigma-1}$$

Solve 
$$\begin{cases} i\partial_t \psi^{(n)} + \frac{1}{2} \Delta \psi^{(n)} + f_n(|\psi^{(n)}|^2) \psi^{(n)} = 0 \\ \psi^{(n)}|_{t=0} = \psi_0 \end{cases}$$

→ globally in  $H^1$  because nonlinearity bounded. Thus find *a priori* bounds on  $L^\gamma([0, T_*], W^{1,\rho})$  and (determine  $T_*$ ).

Call  $(r, q)$  admissible if  $\frac{2}{r} + \frac{d}{q} = \frac{d}{2}$ ,  $2 < r \leq \infty$

**Lemma.** Let  $(r, q)$  be admissible and  $\gamma = \frac{2d}{d-2}$  and  $p = \frac{2d^2}{d^2-2d+4}$ . Let  $r \geq \frac{d}{d-2}$ . Then

$$\begin{aligned} \sup_n \|\nabla [f_n(|\psi|^2)\psi]\|_{L^{r'} I, L^{q'}} &\leq C \cdot \|\psi\|_{L^\gamma(I, L^{\rho^*})}^{2\sigma} \cdot \|\nabla \psi\|_{L^a(I, L^b)} \\ &\leq C \cdot \|\nabla \psi\|_{L^\gamma(I, L^\rho)}^{2\sigma} \cdot \|\nabla \psi\|_{L^a(I, L^b)} \end{aligned}$$

where

$$\begin{aligned} \frac{2\sigma}{p^*} + \frac{1}{b} &= \frac{1}{q'} \\ \frac{2\sigma}{\gamma} + \frac{1}{a} &= \frac{1}{r'} \text{ and } (a, b) \text{ satisfies} \\ \frac{2}{a} + \frac{d}{b} &= \frac{d}{2} \end{aligned}$$

Note:  $(\gamma, \rho)$  det as follows: take  $b = q$ ,  $a = r$ , then actually  $b = q = \rho$ ,  $a = r = \gamma$ .

*Proof.*

$$\begin{aligned} \left| \nabla [f_n(|\psi|^2)\psi] \right| &\leq |f'_n(|\psi|^2)| |\psi|^2 |\nabla \psi| \\ &+ |f_n(|\psi|^2)| |\nabla \psi| \leq C |\psi|^{2\sigma} |\nabla \psi| \\ \|\psi|^{2\sigma} |\nabla \psi|\|_{L_x^{q'}} &\leq \|\psi\|_{L^\gamma}^{2\sigma} \|\nabla \psi\|_b \\ 1 \leq m \leq \infty \text{ with } \left. \begin{aligned} mq' &= b \\ 2\sigma q' m' &= \rho^* \end{aligned} \right\} &\Rightarrow \frac{q'}{b} + \frac{2\sigma q'}{\rho^*} = 1 \end{aligned}$$

$$\begin{aligned} \|\psi|^{2\sigma} |\nabla \psi|\|_{L^{r'}(I, L^{q'})} &\leq \|\psi\|_{L^\gamma(I, L^{\rho^*})}^{2\sigma} \|\nabla \psi\|_{L^a(I, L^b)} \\ 1 \leq \nu \leq \infty \text{ s.t. } \left. \begin{aligned} 2\sigma r' \nu' &= \gamma \\ r' \nu &= a \end{aligned} \right\} &\Rightarrow \frac{2\sigma}{\gamma} + \frac{1}{q} = \frac{1}{r'} \end{aligned}$$

Check  $(a, b)$  is admissible: (still check  $a \geq 2$  afterwards).

$$\begin{aligned} \frac{2}{a} + \frac{d}{b} &= 2\left(\frac{1}{r'} - \frac{2\sigma}{\gamma}\right) + d\left(\frac{1}{q'} - \frac{2\sigma}{\rho^*}\right) \\ &= 2 + d - \left(\frac{2}{r} + \frac{d}{q}\right) - 2\sigma\left(\frac{2}{\gamma} + \frac{d}{\rho^*}\right) \end{aligned}$$

$$\begin{aligned}
&= 2 + d - \frac{d}{2} - \frac{4}{d-2} \underbrace{\left( \frac{2}{\gamma} + \frac{d}{\rho} \right)}_{\frac{d}{2}} + 2\gamma \\
&= 2 + d - \frac{d}{2} - \frac{4}{d-2} \frac{d}{2} + \frac{4}{d-2} \\
&= 2 + \frac{d}{2} - \frac{4}{d-2} \left( \frac{d}{2} - 1 \right) = \frac{d}{2}.
\end{aligned}$$

$$\left( \begin{array}{l} r \geq \frac{d}{d-2} \Rightarrow \text{bounds on } m, \nu \\ \Rightarrow a \geq 2 \end{array} \right)$$

□

Recall

$$(Gf)(t) = \int_0^t e^{i(t-s)\Delta/2} f(s) ds$$

**Lemma.**

$$\| Gf \|_{L^r(I, L^q)} \leq C \cdot \| f \|_{L^{r'}(I, L^{q'})}$$

for any admissible  $(r, q), (\gamma, q)$

*Proof.*

$$\left\{ \begin{array}{ll} G : L^1(I, L^2) \rightarrow L^r(I, L^q) & (a) \\ G : L^{r'}(I, L^{q'}) \rightarrow L^r(I, L^q) & (b) \\ G : L^{r'}(I, L^{q'}) \rightarrow L^\infty(I, L^2) & (c) \end{array} \right.$$

NB:

$$\frac{2}{r} + \frac{d}{q} = \frac{d}{2} \begin{cases} 2 < r \leq \infty \\ 2 \leq q \leq \frac{2d}{d-2} \end{cases}$$

Interp.

$$\begin{array}{l} (a), (b) \quad q' \leq \rho' \leq 2, \quad 2 \leq \rho \leq q. \\ (b), (c) \quad 2 \leq q \leq \rho. \end{array}$$

□

**Corollary.**  $\forall$  admissible  $(r, q)$ , take  $(\gamma, \rho)$  as before

$$(1) \quad \| \nabla G(f_n(|\psi|^2\psi)) \|_{L^r(I, L^q)} \leq C \| \nabla \psi \|_{L^\gamma(I, L^\rho)}^{2\sigma+1}$$

$$\begin{aligned}
(2) \quad & \| G(f_n(|\psi|^2\psi)) - G(f_n(|\phi|^2\phi)) \|_{L^r(I, L^q)} \\
& \leq C \left( \| \nabla \psi \|_{L^\gamma(I, L^\rho)}^{2\sigma} + \| \nabla \phi \|_{L^\gamma(I, L^\rho)}^{2\sigma} \right) \\
& \quad \| \psi - \phi \|_{L^\gamma(I, L^\rho)}
\end{aligned}$$

(will be applied with  $(r, p) = (\gamma, \rho)$ ).

*Proof.* Combine lemmas 1 and 2.

□

*Proof of Theorem.* Switch to Duhamel.

$$\begin{aligned}\psi^{(n)} &= G_0\psi_0 + iGF_n(\psi^{(n)}) \\ F_n(\psi) &= f_n(|\psi|^2)\psi\end{aligned}$$

(equivalent to strong version for  $H^1$  solution).

$$\|\nabla\psi^{(n)}\|_{L^\gamma(I,L^\rho)} \leq \|G_0\nabla\psi_0\|_{L^\gamma(I,L^\rho)} + C_0 \cdot \|\nabla\psi^{(n)}\|_{L^\gamma(I,L^\rho)}^{2\sigma+1}$$

We need  $\sup_n \|\nabla\psi^{(n)}\|_{L^\gamma(I,L^\rho)} < \infty$

$$\rightarrow \text{take } \|G_0\nabla\psi_0\|_{L^\gamma(I,L^\rho)} < \delta \leq \|\nabla\psi_0\|_2$$

then

$$\sup_n \|\nabla\psi^{(n)}\|_{L^\gamma(I,L^\rho)} \leq 2\delta \text{ if } 2C_0(\delta)^{2\sigma} < 1$$

Det  $\cdot T_*$  so that  $< \delta \leq \|\nabla\psi_0\|_2$  (or small initial data)

Use  $H^1$  existence  $\Rightarrow \nabla\psi_0 \in L^2$  but what guarantees  $\nabla\psi_0 \in L^\rho$ ? need to show that  $\|\nabla\psi^{(n)}\|_{L^\gamma(I,L^\rho)} < \infty$ .

$$\text{Kato} \Rightarrow \nabla\psi \in L^{\tilde{r}}(I, L^{\tilde{q}}), \quad \tilde{q} = 2\tilde{\sigma} + 2.$$

where  $\tilde{\sigma}$  is arbitrary subcritical

$$\begin{aligned}\tilde{\sigma} &< \frac{2}{d-2} \\ \rho &= 2\tilde{\sigma} + 2,\end{aligned}$$

$$d = 3. \quad \frac{18}{7} = 2\tilde{\sigma} + 2 < 6.$$

Conclusion: Uniform boundedness in  $n$ .

**NB:** we also have  $\|\nabla\psi^{(n)}\|_{L^\infty(I,L^2)}$

$$\leq \|G_0\nabla\psi_0\|_2 + C \|\nabla\psi^{(n)}\|_{L^\gamma(I,L^\rho)}^{2\sigma+1}$$

(by the corollary before).

$$\begin{aligned}&\leq C \cdot \|\nabla\psi_0\|_2 + C \cdot (2\delta)^{2\sigma} \|\nabla\psi_0\|_2 \\ &\leq C \cdot \|\nabla\psi_0\|_2\end{aligned}$$

and also  $\boxed{\sup_n \|\nabla\psi^{(n)}\|_{L^\gamma(I,L^q)} < \infty.}$

Estimates on

$$\|\psi^{(n)}\|_{L^\gamma(I,L^\rho)} \leq \|G_0\psi_0\|_{L^\gamma(I,L^\rho)} + \overbrace{C \cdot (2\delta)^{2\sigma}}^{< \frac{1}{2}} \cdot \|\psi^{(n)}\|_{L^\gamma(I,L^\rho)}$$

$$\boxed{\sup_n \|\psi^{(n)}\|_{L^\gamma(I,L^\rho)} < \infty.}$$

Pass to a limit

$$\|\psi^{(n)} - \psi^{(m)}\|_{L^\gamma(I,L^\rho)} = \|GF_n(\psi^{(n)}) - GF_m(\psi^{(m)})\|_{L^\gamma(I,L^\rho)}$$

$$\leq \| G[F_n(\psi^{(n)}) - F_n(\psi^{(m)})] \|_{L^\gamma(I, L^\rho)}$$

(assume  $n > m$ )

$$\begin{aligned} & + \| G[F_n(\psi^{(m)}) - F(\psi^{(m)})] \|_{L^\gamma(I, L^\rho)} \\ & + \| G[F_m(\psi^{(m)}) - F(\psi^{(m)})] \| \\ & \lesssim \underbrace{(\| \nabla \psi^{(n)} \|_{L^\gamma(I, L^\rho)}^{2\sigma} + \| \nabla \psi^{(m)} \|_{L^\gamma(I, L^\rho)}^{2\sigma})}_{\leq (2\delta)^{2\sigma}} \| \psi^{(n)} - \psi^{(m)} \|_{L^\gamma(I, L^\rho)} \\ & + 2 \underbrace{\| \nabla \psi^{(m)} \|^{2\sigma}}_{\leq (2\delta)^{2\sigma}} \| \underbrace{\chi_{[|\psi^{(m)}| > m]} \psi^{(m)}}_{\leq m^{-\frac{(2-\rho)}{\rho}} (\int_I \|\nabla \psi^{(m)}(t)\|_{L^q}^{\frac{2^*}{\rho} \cdot \gamma} dt)^{\frac{1}{\gamma}}} \|_{L^\gamma(I, L^\rho)} \end{aligned}$$

because  $\leq \| F_n(\psi^{(m)}) - F(\psi^{(m)}) \|_{L^{\gamma'}(I, L^{\rho'})}$ , Hölder and apply Sobolev emb. to get  $\nabla$ .

$$\begin{aligned} \text{(because } \| \chi_{[|\psi^{(m)}| > m]} \psi^{(m)} \|_{L^\rho}^\rho & \leq m^{-(2^*-\rho)} \int |\psi^{(m)}|^{2^*} dx \\ & \leq m^{-(2^*-\rho)} \| \nabla \psi^{(m)} \|_{L^2}^{2^*} \end{aligned}$$

Cauchy seq.  $\rightarrow$  strong limit that still satisfies the bounds.

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**Version 2**

Conclusion: Provided  $\| G_0 \nabla \psi_0 \|_{L^\gamma(I, L^\rho)} < \delta$  small then

- (a)  $\sup_n \| \nabla \psi^{(n)} \|_{L^\gamma(I, L^\rho)} \leq 2\delta$
  - (b)  $\sup_n \| \nabla \psi^{(n)} \|_{L^r(I, L^q)} < \infty$  ( $r, q$ ) admissible
  - (c)  $\sup_n \| \psi^{(n)} \|_{L^r(I, L^q)} < \infty$
- and then  $\| \psi^{(n)} - \psi^{(m)} \|_{L^\gamma(I, L^\rho)} \rightarrow 0$  as  $n, m \rightarrow \infty$ .

$$\psi^{(n)} \rightarrow \psi \text{ in } L^\gamma(I, L^\rho)$$

and  $\psi$  still satisfies (a), (b), (c)  
(because e.g.,  $\nabla \psi \in L^\infty(I, L^2)$ ,

$$\begin{aligned} |\langle \psi(t), \nabla \chi(t) \rangle| & = \lim |\langle \psi^{(n)}(t), \nabla \chi(t) \rangle| \\ & = \lim |\langle \nabla \psi^{(n)}(t), \chi(t) \rangle| \\ & \leq \sup_n \| \nabla \psi^{(n)}(t) \|_2 \cdot \| \chi(t) \|_2 \end{aligned}$$

$$\int |\langle \psi(t), \nabla \chi(t) \rangle| dt \leq \sup_n \| \nabla \psi^{(n)} \|_{L^\infty(I, L^2)} \cdot \| \chi \|_{L^1(I, L^2)}$$

Then check eq. satisfied

$$\psi^{(n)} = G_0\psi_0 + \underbrace{iGF_n(\psi^{(n)})}_{\rightarrow 0} \text{ in } L^\gamma(I, L^\rho)$$

↓

$$\psi \text{ in } L^\gamma(I, L^\rho) \underbrace{G[F_n(\psi^{(n)}) - F_n(\psi)]}_{\rightarrow 0} + G[F_n(\psi) - F(\psi)] + GF(\psi)$$

$$\text{because } \leq C(\|\nabla\psi^{(n)}\|^{2\sigma} + \|\nabla\psi\|^{2\sigma}) \|\psi^{(n)} - \psi\|$$

and

$$G[F_n(\psi) - F(\psi)] \rightarrow 0 \quad \text{because}$$

$$\begin{aligned} & \|GF_n(\psi) - GF(\psi)\|_{L^\gamma(I, L^\rho)} \\ & \leq \|F_n(\psi) - F(\psi)\|_{L^{\gamma'}(I, L^{\rho'})} \\ & \leq \|f_n(|\psi|^2)\psi - f(|\psi|^2)\psi\|_{L^{\gamma'}(I, L^{\rho'})} \\ & \lesssim \|\psi\|^{2\sigma} \psi \chi_{[|\psi|^2 > n]} \|_{L^{\gamma'}(I, L^{\rho'})} \\ & \lesssim \|\nabla\psi\|_{L^\gamma(I, L^\rho)}^{2\sigma} \underbrace{\|\psi \chi_{[|\psi|^2 > n]}\|_{L^{\gamma'}(I, L^{\rho'})}}_{\rightarrow 0} \end{aligned}$$

Also,  $\psi \in C(I, H^1)$  is easy.

Uniqueness in  $L^\infty(I, H^1) \cap L^\gamma(I, W^{1, \rho})$ .

$$\psi = G_0\psi_0 + iGF(\psi)$$

$$\begin{aligned} \|\nabla\psi\|_{L^\gamma(I, L^\rho)} & \leq \overbrace{\|G_0\nabla\psi_0\|_{L^\gamma(I, L^\rho)}}^{< \delta} \\ & \quad + C \|\nabla\psi\|_{L^\gamma(I, L^\rho)}^{2\sigma+1} \end{aligned}$$

take  $2C_0(2\delta)^{2\sigma} < 1$

(these estimates derived directly from eq.)

$$\begin{aligned} \Rightarrow \|\nabla\psi\|_{L^\gamma(I, L^\rho)} & < 2\delta. \\ \|\psi - \phi\|_{L^\gamma(I, L^\rho)} & \leq \|GF(\psi) - GF(\phi)\|_{L^\gamma(I, L^\rho)} \\ & \leq C(\|\nabla\psi\|_{\gamma, \rho}^{2\sigma} + \|\nabla\phi\|_{\gamma, \rho}^{2\sigma}) \|\psi - \phi\|_{\gamma, \rho} \\ \Rightarrow \psi & = \phi \end{aligned}$$

Conservation laws

$$\begin{aligned} (i) \quad \|\psi^n(t)\|_2 & = \|\psi_0\|_2 \\ \|\psi^{(n)} - \psi^{(m)}\|_{L^\infty(I, L^2)} & \leq \|GF_n(\psi^{(n)}) - GF_m(\psi^{(m)})\|_{\infty, 2} \\ & \leq \dots \leq \|\psi^{(n)} - \psi^{(m)}\|_{\gamma, \rho} \rightarrow 0. \end{aligned}$$

for almost every  $t$ ,  $\exists$  subseq s.t.

$$\psi^{(n)} \rightarrow \psi \text{ in } L_x^2$$

for a.e.  $t$ ,  $\|\psi(t)\|_2 = \|\psi_0\|_2$  since  $\psi(t)$  is co. in time, OK.

$$(ii) \quad \begin{aligned} \mathcal{H}_n[\psi^n(t)] &= \mathcal{H}_n[\psi_0] \\ &= \int \left[ \frac{1}{2} |\nabla \psi_0|^2 - G_n(\psi_0) \right] dx \end{aligned}$$

where  $G_n(u) = \int_0^u F_n(v) dv$

$$\mathcal{H}_n[\psi^{(n)}(t)] = \int \left[ \frac{1}{2} |\nabla \psi^{(n)}(t, x)|^2 - G_n(\psi^n(t, x)) \right] dx$$

weak convergence only  $\rightarrow$  one-sided control by semi-continuity.

$$\begin{aligned} \sup_n |G_n(u)| &\leq C |u|^{2\sigma+2}, \quad 2\sigma + 2 = \frac{4}{d-2} + 2 = \frac{2d}{d-2} = 2^* \\ \|\psi^{(n)} - \psi^{(m)}\|_{2^*} &\stackrel{?}{\rightarrow} 0 \end{aligned}$$

(not from  $\|\psi^{(n)} - \psi^{(m)}\|_{L^r(I, L^q)} \leq \dots \rightarrow 0$  because  $r = 2$ ,  $q = 2^*$  is a forbidden endpoint.)

$$\begin{aligned} \text{Yes, } \quad &\|\psi^{(n)}(t) - \psi^{(m)}(t)\|_{2^*} \\ &\leq C \|\nabla \psi^{(n)}(t) - \nabla \psi^{(m)}(t)\|_q^\theta \\ &\quad \|\psi^{(n)}(t) - \psi^{(m)}(t)\|_2^{1-\theta} \\ (q > 2, \frac{1}{q} - \frac{1}{q^*} = \frac{1}{d} \text{ with } \frac{1}{q^*} = \frac{1}{q} - \frac{1}{d} < \frac{1}{2} - \frac{1}{d} = \frac{1}{s^*} \Rightarrow q^* > 2^*) \end{aligned}$$

$\exists$  subseq, for a.e.  $t$ ,  $\|\nabla - \nabla\|$  bdd,

$$\|\psi^{(n)}(t) - \psi^{(m)}(t)\| \rightarrow 0.$$

$$\Rightarrow \int G_n(\psi^{(n)}(t, x)) dx \rightarrow \int G(\psi(t, x)) dx$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{H}_n[\psi_0] &= \mathcal{H}[\psi_0] \\ &= \lim_{n \rightarrow \infty} \mathcal{H}_n[\psi^{(n)}] \\ &= \lim_{n \rightarrow \infty} \mathcal{H}_n[\psi^{(n)}] \\ &\geq \mathcal{H}[\psi(t)] \end{aligned}$$

Concl.

$$\mathcal{H}[\psi_0] \geq \mathcal{H}[\psi(t)]$$

Time-reversal, previous result do not depend on it (though equation changes)

$$\begin{aligned} t_0 = 0 &\leq t \leq t_1 \\ \phi(t) &= \psi(t_1 - t) \end{aligned}$$

$$\psi(t) = e^{it\frac{\Delta}{2}}\psi_0 + i \int_0^t e^{i(t-s)\frac{\Delta}{2}} F(\psi(s)) ds .$$

**Claim:**  $\phi$  satisfies

$$\begin{aligned} \phi(t) &\stackrel{?}{=} e^{-it\frac{\Delta}{2}}\psi(t_1) - i \int_0^t e^{-i(t-s)\frac{\Delta}{2}} F(\phi(s)) ds \\ &(-i\partial_t\psi + \frac{1}{2}\Delta\psi + F(\phi) = 0) \end{aligned}$$

$$\begin{aligned} \phi(t) &= e^{-it\frac{\Delta}{2}} \left\{ e^{it_1\frac{\Delta}{2}}\psi_0 + i \int_0^{t_1} e^{i(t_1-\tau)\frac{\Delta}{2}} F(\psi(\tau)) d\tau \right\} \\ &- i \int_{t_1-t}^{t_1} e^{-i(t-t_1+s)\frac{\Delta}{2}} F(\psi(s)) ds = e^{i(t_1-t)\frac{\Delta}{2}}\psi_0 \\ &+ ie^{-it\frac{\Delta}{2}} \int_0^{t_1-t} e^{i(t_1-t)\frac{\Delta}{2}} F(\psi(\tau)) d\tau \end{aligned}$$

(so Backwards Duhamel is satisfied)

Check  $C^1(I, H^{-1})$

$$i\partial_t\psi + \frac{1}{2}\Delta\psi + \lambda \underbrace{|\psi|^{2\sigma}\psi}_{\text{co. in } H^{-1}} = 0 .$$

d=3

$$\sigma = 2, \quad |\psi|^4\psi .$$

$$t \rightarrow \psi \in C(I, H^1) .$$

$$H^1 \hookrightarrow L^6(\mathbb{R}^3), \quad L^{\frac{6}{5}} \hookrightarrow H^{-1}$$

$$t \rightarrow |\psi|^4\psi \in C(I, L^{\frac{6}{5}})$$

$$\text{and } \in C(I, H^{-1})$$

the rest is okay.

Look at  $i\partial_t\psi + \frac{1}{2}\Delta\psi - |\psi|^2\psi = 0$ .

$$\begin{aligned} s = 1 &\text{ subcritical for } s > \frac{1}{2} \\ s = \frac{1}{2} &\text{ critical for } s = \frac{1}{2}. \end{aligned}$$

$$\sigma_{\text{critical}} = 1 = \frac{2}{d-2s} = \frac{2}{3-2s}, \quad s = \frac{1}{2} .$$

**Claim:** We have all the methods to prove local existence

$$\exists ce, \quad \text{both } s > \frac{1}{2}, \quad s = \frac{1}{2}$$

$$\begin{aligned}
\psi &= G_0\psi_0 + \lambda iG(|\psi|^2\psi) \\
\| \langle \nabla \rangle^s \psi \|_{L^\gamma(I, L^q)} &= \| ((1 + |\xi|^2)^{\frac{s}{2}} \hat{\psi})^\vee \|_{\gamma, q} \\
&\downarrow &\leq \| G_0 \langle \nabla \rangle^s \psi_0 \|_{\gamma, q} \\
(1 - \Delta)^{\frac{s}{2}} &+ C \cdot \| \langle \nabla \rangle^s |\psi|^2 \psi \|_{a', b'} \\
&\leq \| \langle \nabla \rangle^s \psi_0 \|_2 \\
&+ CT^\alpha \| \psi \|_{L^\gamma(I, L^p)}^2 \cdot \| \langle \nabla \rangle^s \psi \|_{\infty, 2}
\end{aligned}$$

$\alpha > 0$  because subcritical

**Ma 142**

**4/15/2003**

Last time:

$$\begin{aligned}
i\partial_t \psi + \frac{1}{2} \Delta \psi - |\psi|^2 \psi &= 0 \\
\psi|_{t=0} &= \psi_0 \in H^1
\end{aligned}$$

We proved global well-posedness, energy conservation,  $L^2$  conservation. Moreover, we have regularity in  $H^2$  (Kato's paper.)

We proved pseudo-conformal identity:

$$\| (x - tp)\psi(t) \|_2^2 + t^2 \| \psi(t) \|_4^4 = \| x\psi_0 \|_2^2 - \int_0^t \tau \| \psi(\tau) \|_4^4 d\tau$$

which implies

$$\| \psi(t) \|_4^4 \leq \frac{\| x\psi_0 \|_2^2}{t^2},$$

which implies that there are no solitons.

Where can we expect well-posedness (local)?

$$\sigma_{\text{crit}} = \frac{2}{d - 2s}, \quad d = 3, \quad s = \frac{1}{2}.$$

If  $s > \frac{1}{2}$ , we can prove local well-posedness:

We have real formulation  $\psi = G_0\psi_0 + iG|\psi|^2\psi$  and

$$\| \langle \nabla \rangle^s \psi \|_{L^r(I; L^q)} \leq C \| \langle \nabla \rangle^s \psi_0 \|_2 + C \| \langle \nabla \rangle^s |\psi|^2 \psi \|_{L^{a'}(I; L^{b'})}$$

If  $\langle \nabla \rangle$  would be an ordinary  $\nabla$ , we would have, by Leibnitz,

$$\| \langle \nabla \rangle^s (|\psi|^2 \psi) \|_{b'} \lesssim \| \langle \nabla \rangle^s \psi \|_b \| |\psi|^2 \|_g.$$

We need to control the nonlinear term by means of  $\| \langle \nabla \rangle^s \psi \|_b$ .

$b < 6 = \frac{2d}{d-2}$  (the endpoint for Strichartz) and by Sobolev embedding  $1 - \frac{1}{\frac{2b}{b-2}} \leq \frac{s}{3}$ , i.e.,  $s \geq \frac{3}{b}$ . So we have to have  $s > \frac{1}{2}$ .

$$\begin{aligned}
\| \langle \nabla \rangle^s \psi \|_{r, q} &\lesssim \| \psi_0 \|_{H^s} + \left( \int_0^T \| \langle \nabla \rangle^s \psi(t) \|_{2^{a'}}^{2a'} \| \langle \nabla \rangle^s \psi \|_{L_x^{b'}}^{a'} dt \right)^{\frac{1}{a'}} \\
&\lesssim \| \psi_0 \|_{H^s} + \| \langle \nabla \rangle^s \psi \|_{L^\infty(I; L^2)}^2 T^{1-\frac{2}{a}} \| \langle \nabla \rangle^s \psi \|_{L^a(I; L^b)} \quad (I = [0, T]).
\end{aligned}$$

By admissibility,  $\frac{2}{a} + \frac{3}{b} = \frac{3}{2}$ , so  $b < 6 \Leftrightarrow a > 2$ , so  $1 - \frac{2}{a} > 0$ .

Set  $r = \infty$ ,  $g = 2$ , and let  $\mu := \|\langle \nabla \rangle^s \psi\|_{\infty, 2} + \|\langle \nabla \rangle^s \psi\|_{a, b}$

So,  $\mu \leq C_0 \|\psi_0\|_{H^s} + C_1 T^{1-\frac{2}{a}} \mu^3$  and, for  $T$  small enough,  $\mu \leq 2C_0 \|\psi_0\|_{H^s}$

For  $s = \frac{1}{2}$ , we can still prove the well-posedness.

**EXERCISE:** Prove it.

As for global well-posedness, we don't know about it for every  $s > \frac{1}{2}$  (at the moment we know for  $s > \frac{4}{5}$ ).

Prove local and global well-posedness of  $i\partial_t \psi + \frac{1}{2} \Delta \psi + \lambda |\psi|^{2\sigma} \psi = 0$  when  $d\sigma < 2$ .

Now we want to prove Leibnitz for  $\langle \nabla \rangle^s$ ,  $s > 0$ , i.e.,

$$\|\langle \nabla \rangle^s (fg)\|_p \lesssim \|\langle \nabla \rangle^s f\|_{p_1} \|g\|_{p_2} + \|\langle \nabla \rangle^s g\|_{p_3} \|f\|_{p_4},$$

where  $1 < p < \infty$ ,  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$ ,  $p_1, p_2, p_3, p_4 \in \langle 1, \infty \rangle$ .

(Note that the estimate  $\|\langle \nabla \rangle^s (|\psi|^2 \psi)\|_{b'} \lesssim \|\langle \nabla \rangle^s \psi\|_b \|\psi\|_q^2$  follows from Leibnitz for  $\langle \nabla \rangle^s$  applied to the product  $\psi \psi \bar{\psi}$ .)

The question is: is  $\frac{\nabla}{\langle \nabla \rangle} = \frac{\xi}{(1+|\xi|^2)^{\frac{1}{2}}}$  a bounded multiplier?

The idea of the proof of Leibnitz for  $\langle \nabla \rangle^s$  is as follows:

$$f = \sum_{j \geq 0} \Delta_j f, \quad g = \sum_{k \geq 0} \Delta_k g.$$

$$\begin{aligned} \|\langle \nabla \rangle^s (fg)\|_p &= \left\| \sum_{j, k \geq 0} \langle \nabla \rangle^s \Delta_j f \Delta_k g \right\|_p \\ &= \sum_j \sum_{k \leq j-10} \|\langle \nabla \rangle^s \Delta_j f \Delta_k g\|_p \quad \leftarrow A \\ &\quad + \sum_k \sum_{j \leq k-10} \|\langle \nabla \rangle^s \Delta_j f \Delta_k g\|_p \quad \leftarrow B \\ &\quad + \left\| \sum_{|j-k| < 10} \langle \nabla \rangle^s \Delta_j f \Delta_k g \right\|_p \quad \leftarrow C \end{aligned}$$

A should behave like this:

$$\begin{aligned} A &\lesssim \left\| \left( \sum_{j \geq 0} 2^{2js} |\Delta_j f|^2 \sum_{k \leq j-10} |\Delta_k g|^2 \right)^{\frac{1}{2}} \right\|_p \quad \leftarrow \text{this is called the paraproduct} \\ &\lesssim \left\| \left( \sum_{j \leq 0} 2^{2js} |\Delta_j f|^2 \right)^{\frac{1}{2}} M g \right\|_p \\ &\lesssim \left\| \left( \sum_{j > 0} 2^{2js} |\Delta_j f|^2 \right)^{\frac{1}{2}} \right\|_{p_1} \|M g\|_{p_2} \\ &\lesssim \|f\|_{W^{s, p_1}} \|g\|_{p_2} \end{aligned}$$

the same for B

We expect

$$\begin{aligned}
C &\lesssim \left\| \sum_{|j-k|<10} 2^{js} |\Delta_j f| |\Delta_k g| \right\|_p \\
&\lesssim \left( \sum_j 2^{2js} |\Delta_j f|^2 \right)^{\frac{1}{2}} \left( \sum_k |\Delta_k g|^2 \right)^{\frac{1}{2}} \Big\|_p \\
&\lesssim \left\| \left( \sum_j 2^{2js} |\Delta_j f|^2 \right)^{\frac{1}{2}} \right\|_{p_1} \left\| \left( \sum_k |\Delta_k g|^2 \right)^{\frac{1}{2}} \right\|_{p_2} \lesssim \|f\|_{W^{s,p_1}} \|g\|_{p_2}
\end{aligned}$$

**Lemma.**

(i) Let  $\{f_k\}_{k \geq 0}$  be some functions with  $\text{supp } \hat{f}_k \subseteq \{c_0 2^k \leq |\xi| < C_0 2^k\}$ , for  $k \geq 1$ , and  $\text{supp } \hat{f}_0 \subseteq \{|\xi| \leq C_0\}$ .

$$\text{Then } \|\langle \nabla \rangle^s \sum_{k=0}^{\infty} f_k\|_p \lesssim \left\| \left( \sum_{k=0}^{\infty} 2^{ks} |f_k|^2 \right)^{\frac{1}{2}} \right\|_p.$$

The reverse holds if  $f_k = \Delta_k f$ .

(ii) Let  $\{f_k\}_{k \geq 0}$  be such that  $\text{supp } \hat{f}_k \subset B(0, C_0 2^k)$ .

$$\text{Then } \|\langle \nabla \rangle^s \sum_{k=0}^{\infty} f_k\|_p \lesssim \left\| \left( \sum_{k=0}^{\infty} 2^{2ks} |f_k|^2 \right)^{\frac{1}{2}} \right\|_p.$$

**Exercise:**

Clean up A,B,C by the lemma.

(In part C, don't do C – S.

Comment:

$$\begin{aligned}
&\left\| \sum_{|j-k| \leq 10} \langle \nabla \rangle^s \Delta_j f \Delta_k g \right\|_p \lesssim \left\| \left( \sum 2^{2sj} |\Delta_j f|^2 |\Delta_k g|^2 \right)^{\frac{1}{2}} \right\|_p \\
&\lesssim \left\| \left( \sum_{j \geq 0} 2^{2sj} |\Delta_j f|^2 \right)^{\frac{1}{2}} M g \right\|_p \lesssim \|f\|_{W^{s,p_1}} \|g\|_{p_2} .
\end{aligned}$$

*Proof of the lemma.* Pick  $\nu$  is large that  $\text{dist}(\text{supp } \hat{f}_k, \text{supp } \hat{f}_{k+\nu}) \sim 2^k, \forall k$  (So,  $c_0 2^\nu > C_0$ )

$$\begin{aligned}
\left\| \langle \nabla \rangle^s \sum_{k \geq 0} f_k \right\|_p &\leq \sum_{j=0}^{\nu-1} \left\| \langle \nabla \rangle^s \sum_{k \equiv j \pmod{\nu}} f_k \right\|_p \\
&= \sum_{j=0}^{\nu-1} \left\| \sum_{k \equiv j} \langle \nabla \rangle^s 2^{-ks} \psi_k * 2^{2k} f_k \right\|_p \\
&= \sum_{j=0}^{\nu-1} \left\| (m_{s,j} \hat{F}_j)^\vee \right\|_p,
\end{aligned}$$

where  $F_j = \sum_{k \equiv j} 2^{ks} f_k$  and

$$m_{s,j}(\xi) = \sum (1 + |\xi|^2)^{\frac{s}{2}} 2^{-ks} \hat{\psi}_k(\xi)$$

If  $\psi_k$  is obtained by scaling,  $m_{s,j}$  is a Mikhlin multiplier and we have

$$\begin{aligned} \sum_{j=0}^{\nu-1} \| (m_{s,j} \hat{F}_j)^\vee \|_p &\lesssim \sum_{j=0}^{\nu-1} \| F_j \|_p \lesssim \sum_{j=0}^{\nu-1} \left\| \left( \sum_{k \equiv j} 2^{2ks} |f_k|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\lesssim \left\| \left( \sum_{k \geq 0} 2^{2ks} |f_k|^2 \right)^{\frac{1}{2}} \right\|_p . \end{aligned}$$

□

**Ma 142**  
**4/17/2003**

Lemma from last time

- (i)  $\{f_k\}_{k \geq 0}$  so that  $\text{supp } \hat{f}_k \subset \{C_0 2^k \leq |\xi| \leq C_0 2^k\}$  if  $k \geq 1$  and  $\text{supp } \hat{f}_0 \subset \{|\xi| \leq C_0\}$ .  
Then

$$\| \langle \nabla \rangle^s \sum_{k \geq 0} f_k \|_{L^p} \lesssim \left\| \left( \sum_{k \geq 0} 2^{2sk} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p}$$

if  $1 < p < \infty$  and  $s \in \mathbb{R}$ . Conversely, if  $f_k = \Delta_k(f)$  (= Littlewood-Paley projection), then the converse holds.

- (ii) If  $\{f_k\}_{k \geq 0}$  are such that  $\text{supp } \hat{f}_k \subset \{|\xi| \leq C_0 2^k\}$  for  $k \geq 0$ , then

$$\| \langle \nabla \rangle^s \sum_{k \geq 0} f_k \|_p \lesssim \left\| \left( \sum_{k \geq 0} 2^{2sk} |f_k|^2 \right)^{\frac{1}{2}} \right\|_p$$

if  $s > 0$  and  $1 < p < \infty$ .

*Proof* (continuation): The converse

$$\left\| \left( \sum_{k \geq 0} 2^{2sk} |\Delta_k(f)|^2 \right)^{\frac{1}{2}} \right\|_p = \left\| \left( \sum_{k \geq 0} 2^{2sk} |\Delta_k(\tilde{\psi} \star f)|^2 \right)^{\frac{1}{2}} \right\|_p$$

where  $\widehat{\Delta_k(f)}(\xi) = \hat{\psi}(2^{-k}\xi) \hat{f}(\xi)$ .

( $\hat{\psi} = 1$  on  $\text{supp } \hat{\psi}$  and  $\hat{\psi}$  does not pile up:  $\sum_k |\hat{\psi}(\xi)|^2 \leq 3$ .)

$$\hat{\psi}_k(\xi) = \hat{\psi}(2^{-k}\xi)$$

$$\lesssim \left\| \sum_{k \geq 0} 2^{sk} \tilde{\psi}_k \star f \right\|_p$$

↑

by LP

$$= \left\| \underbrace{\left( \sum_{k \geq 0} 2^{sk} (1 + |\xi|^2)^{-\frac{s}{2}} \hat{\psi}(2^{-k}\xi) (1 + |\xi|^2)^{+\frac{s}{2}} \hat{f}(\xi) \right)^\vee}_{m_s(\xi)} \right\|_p$$

with  $\|m_s\|_\infty$  and  $|\partial^\alpha m_s(\xi)| \leq C_\alpha (1 + |\xi|)^{-|\alpha|}$  because, for example,

$$|\nabla m_s(\xi)| \lesssim \sum (1 + |\xi|^2)^{-\frac{s}{2}} \{ (1 + |\xi|)^{-1} 2^{sk} |\hat{\psi}(2^{-k}\xi)| \}$$

$$+ 2^{sk} \underbrace{2^{-k} |(\nabla \hat{\psi})(2^{-k}\xi)|}_{\sim (1+|\xi|)^{-1} |(\nabla \hat{\psi})(2^{-k}\xi)|} \}$$

So  $m_s$  is a Mihklin multiplier.

$$\lesssim \| ((1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi))^{\vee} \|_p = \| f \|_{W^{s,p}}$$

Remark for (ii), what about  $s = 0$ ?

$$p = 2, s = 0, \quad \left\| \sum_k f_k \right\|_2^2 \stackrel{?}{\lesssim} \sum_k \| f_k \|_2^2$$

would be true only if there was some orthogonality, not in general. (e.g., take  $\hat{f}_k = \chi_{[0,1]}$ ,  $N^2 \stackrel{?}{\lesssim} N$ , no.)

*Proof of (ii).* Use fact that weight  $(1 + |\xi|^2)^{\frac{s}{2}}$  is bigger at the endpoints of the interval  $[-C_0 2^k, C_0 2^k]$  in 1-D.

From (i),

$$\begin{aligned} \left\| \sum_{k \geq 0} f_k \right\|_{W^{s,p}} &\lesssim \left\| \left( \sum_{l \geq 0} 2^{2sl} \left| \Delta_l \sum_{k \geq l} f_k \right|^2 \right)^{\frac{1}{2}} \right\|_p \\ &= \left\| \sum_{l \geq 0} \left| \sum_{k \geq l} 2^{s(l-k)} \psi_l * \underbrace{2^{sk} f_k}_{w_k} \right|^2 \right)^{\frac{1}{2}} \right\|_p \\ &\stackrel{?}{\lesssim} \left\| \left( \sum_{k \geq 0} \underbrace{2^{2sk} |f_k|^2}_{|w_k|^2} \right)^{\frac{1}{2}} \right\|_p \end{aligned}$$

is the statement

$$\begin{aligned} \left\| T(\{w_k\}_{k \geq 0}) \right\|_{L^p(l^2)} &\stackrel{?}{\lesssim} \left\| \{w_k\}_{k \geq 0} \right\|_{L^p(l^2)} \\ T(\{w_k\}_{k \geq 0}) &= \left\{ \sum_{k \geq l \geq 0} 2^{s(l-k)} \psi_l * w_k \right\}_{l \geq 0} \end{aligned}$$

→ vector-valued singular integral.

By Hilbert space-valued Calderon-Zygmund theory,

- α)  $T : L^2(l^2) \rightarrow L^2(l^2)$  bounded
- β) kernel matrix element  $K_{kl} = 2^{s(l-k)} \psi_l$ ,  
 $\| K(x) \|_{l^2 \rightarrow l^2} \leq B \cdot |x|^{-d}$
- γ)  $\| \nabla K(x) \|_{l^2 \rightarrow l^2} \leq B \cdot |x|^{-d-1}$

why γ) is needed?  $1/|x| * f$  does not make sense.

CZ-theorem: α), β), γ)  $\Rightarrow \| T \|_{L^p(l^2) \rightarrow L^p(l^2)} \leq C_{d,p} B$ ,

$$1 < p < \infty$$

Show  $\alpha$ )

$$\begin{aligned}
\| T\{w_k\}_{k \geq 0} \|_{L^2(\mathbb{R}^2)}^2 &= \int \left( \sum_{l \geq 0} \left| \sum_{k \geq l \geq 0} K_{kl} * w_k \right|^2 \right) dx \\
&= \sum_{l \geq 0} \int \left| \sum_{k \geq l} 2^{s(l-k)} \widehat{\psi}(2^{-l}\xi) \widehat{w}_k(\xi) \right|^2 d\xi \\
&\lesssim \int \underbrace{\sum_{l \geq 0} |\widehat{\psi}(2^{-l}\xi)|^2}_{\leq C} \underbrace{\sum_{k \geq l} 2^{2s(l-k)} \sum_{k \geq l} |\widehat{w}_k(\xi)|^2}_{\leq C(s)} d\xi \\
&\leq C(s) \| \{w_k\}_{k \geq 0} \|_{L^2(\mathbb{R}^2)}^2 .
\end{aligned}$$

Show  $\beta$ ): Schur's test

$$\begin{aligned}
(1) \quad \sup_l \sum_{k \geq l} |K_{kl}(x)| &\stackrel{?}{\leq} B|x|^{-d} \\
(2) \quad \sup_k \sum_{k \geq l \geq 0} |K_{kl}(x)| &\stackrel{?}{\leq} B|x|^{-d}
\end{aligned}$$

$$\begin{aligned}
(1) \quad \sum_{k \geq l} 2^{s(l-k)} |\psi_l(x)| &\leq C(s) |\psi_l(x)| = C(s) 2^{ld} |\psi(2^l x)| \\
&\leq C_N(s) 2^{ld} (1 + 2^l |x|)^{-N} \quad N \text{ big.} \\
&\lesssim \begin{cases} |x|^{-d} & \text{for } 2^l |x| < 1 \\ 2^{l(d-N)} |x|^{-N} < |x|^{-d} & \text{for } 2^l |x| > 1 . \end{cases} \\
(2) \quad \sum_{k \geq l \geq 0} 2^{s(l-k)} |\psi_l(x)| &\leq C_N \sum_{k \geq l \geq 0} 2^{s(l-k)} (1 + 2^l |x|)^{-N} \\
&\lesssim \underbrace{\sum_{\substack{k \geq l \geq 0 \\ 2^l |x| < 1}} \underbrace{2^{s(l-k)} 2^{ld}}_{\leq 1}}_{\lesssim |x|^{-d}} + \underbrace{\sum_{\substack{k \geq l \geq 0 \\ 2^l |x| > 1}} \underbrace{2^{s(l-k)} 2^{ld}}_{\leq 1} (2^l |x|)^{-N}}_{\text{take } N = d + 1,} \\
&\lesssim |x|^{-d-1} (|x|^{-1})^{-1} = |x|^{-d}
\end{aligned}$$

Show  $\gamma$ ) idem,  $2^{ld}$  becomes  $2^{l(d+1)}$ , etc.

Note: Vector-valued CZ theory trivially follows from scalar C2 theory.

## Standing Waves and Calculus of Variations

$$i\partial_t\psi + \frac{1}{2}\Delta\psi + f(|\psi|^2)\psi = 0$$

Try solutions of the form  $\psi(t, x) = e^{i\frac{x^2}{2}t}\phi(x)$

$$0 = -\frac{\alpha^2}{2}\phi e^{i\frac{\alpha^2}{2}t} + e^{i\frac{\alpha^2}{2}t}\frac{1}{2}\Delta\phi + f(|\phi|^2)\phi e^{i\frac{\alpha^2}{2}t}$$

$$0 = -\frac{\alpha^2}{2}\phi + \frac{1}{2}\Delta\phi + f(|\phi|^2)\phi \quad (\text{time indep.})$$

**Comments:**  $f(u) = \lambda u^\sigma$ ,  $\sigma > 0$ .

If  $\lambda = -1$  (defocussing), it follows from the pseudo-conformal identity that

$$\|\psi(t)\|_{2\sigma+2}^{2\sigma+2} \lesssim t^{-2}.$$

$\Rightarrow$  You can't have standing waves for  $\lambda < 0$ . Furthermore, you want  $\phi$  to decay nicely at  $\infty$ . The nonlinear term is higher-order  $\Rightarrow$  negligible at  $\infty$ . Look at the linear part, solutions look like  $\phi(x) \sim e^{-\alpha|x|}$ , okay. ( $\alpha^2 > 0$  ensures we are above/below the spectrum of  $-\Delta/\Delta$ ).

Moreover

$$0 = \int -\frac{\alpha^2}{2}|\phi|^2 - \frac{1}{2}|\nabla\phi|^2 + \lambda|\phi|^{2\sigma+2} dx \Rightarrow \lambda > 0$$

Side note:  $i\partial_t\psi + \frac{1}{2}\Delta\psi + V\psi + f(|\psi|^2)\psi = 0$

$$\begin{aligned} \mathcal{H}(\psi(t)) &= \int \left\{ \frac{1}{2}|\nabla\psi|^2 - V|\psi|^2 - F(\psi) \right\} dx \\ &= \mathcal{H}(\psi_0) \end{aligned}$$

Next time: non-uniqueness of solutions to

$$\begin{cases} -\Delta u = mu - |u|^{2\sigma}u \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

$$\mathcal{L}[u] = \int_{\Omega} \left\{ \frac{1}{2}|\nabla u|^2 - F(u) \right\} dx$$

i.e., non-uniqueness of the minimizers (local minima).

$$\inf_{\gamma} \max_{0 \leq t \leq 1} \mathcal{L}[\gamma(t)].$$

is here a saddle point.

**Ma 142 4/22/2003**

Please look through Exercises 4)–11) within this coming week (most important 4)–9)).

Last time we started with *standing wave*. Solutions to

$$\begin{aligned} i\partial_t\psi + \frac{1}{2}\Delta\psi + f(|\psi|^2)\psi &= 0 \\ \psi(t, x) &= e^{i\frac{\alpha^2}{2}t}\phi(x) \\ \Rightarrow -\frac{1}{2}\alpha^2\phi + \frac{1}{2}\Delta\phi + f(\phi^2)\phi &= 0 \end{aligned}$$

Continuing formally

classically:

$$\begin{aligned} \mathcal{G}_{\vec{v}, y}(t) &:= e^{-it\frac{|v|^2}{2}} e^{-i\vec{v}\cdot x} e^{i(y+t\vec{v})\cdot\vec{p}} \\ x &\mapsto x - y - t\vec{v} \\ \xi &\mapsto \xi - \vec{v} \\ \mathcal{G}_{\vec{v}, y}(t)e^{i\frac{t}{2}\Delta} &= e^{i\frac{t}{2}\Delta}\mathcal{G}_{\vec{v}, y}(0) \end{aligned}$$

Define  $\phi(t) = \mathcal{G}_{\vec{v}, y}(t)\psi(t)$  for  $\psi$  a solution.

Then

$$\begin{aligned} \phi(t) &= \mathcal{G}_{\vec{v}, y}(t)\left\{e^{i\frac{t}{2}\Delta}\psi(0) + i\int_0^t e^{i(t-s)\frac{\Delta}{2}} f(|\psi(s)|^2)\psi(s) ds\right. \\ &= e^{i\frac{t}{2}\Delta}\phi(0) + i\int_0^t e^{i(t-s)\frac{\Delta}{2}} \underbrace{\mathcal{G}_{\vec{v}, y}(s)f(|\psi(s)|^2)\psi(s)}_{f(|\phi(s)|^2)\phi(s)} ds \\ &\Rightarrow \begin{cases} i\partial_t\phi + \frac{1}{2}\Delta\phi + f(|\phi|^2)\phi = 0 \\ \phi|_{t=0} = \mathcal{G}_{\vec{v}, y}(0)\psi_0 \end{cases} \end{aligned}$$

So, again for standing waves

$$e^{i\gamma}\mathcal{G}_{\vec{v}, y}(t)[e^{i\frac{\alpha^2}{2}t}\phi] = e^{\frac{i}{2}(\alpha^2 - |v|^2)t} e^{i\gamma} e^{i\vec{v}\cdot x} \phi(x - y - t\vec{v})$$

depends on  $(\alpha, \gamma, y, \vec{v}) \in \mathbb{R}^{2d+2}$ .

**Solve:** Let  $m \geq 0$

$$\begin{cases} -\Delta u + mu = f(u) & \text{in } \Omega \subset \mathbb{R}^d \text{ bdd.} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\mathcal{L}[u] = \int_{\Omega} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{m}{2}u^2 - F(u) \right\}(x) dx$$

$$u \in C_0^\infty(\Omega)$$

$$u_\varepsilon(x) = u(x) + \varepsilon\phi(x)$$

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \mathcal{L}[u_\varepsilon] = \int_{\Omega} (\nabla u \cdot \nabla \phi + mu\phi - f(u)\phi)(x) \, dx \\
&\quad \uparrow \\
&\quad \forall \phi \\
&= \int_{\Omega} (-\Delta u \cdot \phi + mu\phi - f(u)\phi)(x) \, dx \\
&\Rightarrow \Delta u - mu + f(u) = 0 .
\end{aligned}$$

Abstract machinery for finding critical points

$\mathcal{H}$  = Hilbert space

$I \in C^1(\mathcal{H}; \mathbb{R})$

$I[u + w] = I[u] + \langle I'[u], w \rangle + o(\|w\|^2)$

$I' : \mathcal{H} \rightarrow \mathcal{H}^*$  locally-Lipschitz

$\mathcal{A}_c \{u \in \mathcal{H} | I[u] \leq c\}$

$\mathcal{K}_c \{u \in \mathcal{H} | I[u] = c, I'[u] = 0\}$

Palais-Smale condition (PS):

if  $\{u_k\}_{k=1}^\infty$  in  $\mathcal{H}$  s.t.  $\{I[u_k]\}_k$  is bounded and  $I'[u_k] \rightarrow 0$ , then  $\{u_k\}_k$  precompact in  $\mathcal{H}$ .

Remark:

For  $F(u) = |u|^4$ ,  $\mathcal{H} = H_0^1$

**MP Theorem** Let  $I$  be as above and (PS). Assume  $I[0] = 0$ ,  $\exists r, a > 0$  with  $I[u] > a$  if  $\|u\| = r$  and  $\exists v \in \mathcal{H}$ ,  $\|v\| > r$ ,  $I(v) \leq 0$ . Let

$$C = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I[\gamma(t)], \quad \Gamma = \{\gamma : (0, 1) \rightarrow \mathcal{H} : \gamma(0) = 0; \gamma(1) = v\}$$

Then  $C$  is a critical value.

**Deformation Theorem**  $I$  satisfies (PS) and  $\mathcal{K}_c = \emptyset$ . Then  $\exists \varepsilon > \delta > 0$  and

$$\eta : [0, 1] \times \mathcal{H} \rightarrow \mathcal{H} \text{ with}$$

- (i)  $\eta_0(u) = u$ ,
- (ii)  $\eta(t, u) = u$ ,  $\forall t \in [0, 1]$ ,  $\forall u$  with  $I[u]$  and  $(c - \varepsilon, c + \varepsilon)$
- (iii)  $\eta_1(\mathcal{A}_{c+\delta}) \subset \mathcal{A}_{c-\delta}$ ;
- (iv)  $I[\eta_t(u)] \leq I[u]$ ,  $\forall u \in \mathcal{H}$ ,  $\forall t \in [0, 1]$

*Proof.* Idea is to solve:

$$\begin{cases} \dot{\eta}(t) = -I'[\eta] \\ \eta(0) = u \end{cases}$$

$$\frac{d}{dt} I[\eta] = \langle I'[\eta], \dot{\eta} \rangle = - \|I'[\eta]\|^2$$

modify this flow to move *only* those  $u$  with  $I[u]$  close to  $c$ !

**Claim:**  $\exists \sigma, \varepsilon > 0$  so that

$$\inf_{u: |I[u]-c| \leq \varepsilon} \|I'[u]\| > \sigma > 0 .$$

If not:

$$\left. \begin{array}{l} I'[u_k] \rightarrow 0 \\ I[u_k] \rightarrow C \end{array} \right\} \xrightarrow{PS} \begin{array}{l} u_k \xrightarrow{\mathcal{H}} u_\infty, \\ I'[u_\infty] = 0; \\ I[u_\infty] = c \end{array} \\ \Rightarrow u_\infty \in \mathcal{K}_c = \emptyset : \text{contradiction}$$

So solve:

$$\left\{ \begin{array}{l} \dot{\eta}(t) = -I'[\eta]g(\eta) \\ \eta(0) = u \end{array} \right. \left[ \Rightarrow \frac{d}{dt} I[\eta] = - \| I'[\eta] \|^2 g(\eta) \right]$$

Define

$$\begin{aligned} A &= \{u \in \mathcal{H} : I[u] \in (c - \delta, c + \delta)\} \\ B &= \{u \in \mathcal{H} : I[u] \notin (c - \varepsilon, c + \varepsilon)\} \\ h(u) &= \frac{\text{dist}(u, B)}{\text{dist}(u, B) + \text{dist}(u, A)} \Rightarrow \begin{array}{l} h = 0 \text{ on } B \\ h = 1 \text{ on } A \end{array} \\ g(\eta) &= h(\eta) \end{aligned}$$

So solve on  $t \in [0, 1]$ ;

If  $I[\eta_t(u)] \leq c - \delta$  for some  $t$  and  $u \Rightarrow$  okay.

WLOG:

$$\begin{aligned} c + \delta &> I[\eta_t(u)] > c - \delta \text{ for every } 0 \leq t \leq 1 \\ \Rightarrow I[\eta_1(u)] &< c + \delta - \sigma^2 < c - \delta \text{ (contradiction for } \delta \ll 1). \end{aligned}$$

□

*Proof of MP.* Suppose  $\mathcal{K}_c = \emptyset$ . Pick  $\varepsilon, \delta > 0$  like in Deformation Theorem and

$$\max_{0 \leq t \leq 1} I[\gamma(t)] < c + \delta$$

Define  $\tilde{\gamma}(t) = \eta_1[\gamma(t)]$

$$\begin{aligned} \tilde{\gamma} &\in \Gamma, \eta_1(v) = 0, n_1(v) = v \\ \text{as } 0 &= I(0) \notin (c - \varepsilon, c + \varepsilon) \\ 0 &\geq I(v) \notin (c - \varepsilon, c + \varepsilon) \\ \max_{0 \leq t \leq 1} I[\tilde{\gamma}(t)] &< c - \delta : \text{contradiction} \end{aligned}$$

□

**Remark** Existence of solution to (\*) on  $[0, 1]$ :

Define  $T = \sup\{t : \exists \eta\}$

$$\begin{aligned} T < \infty &\Rightarrow \int_0^T \| I'[\eta(t)] \|^2 g(\eta(t)) dt = \infty \\ &\Rightarrow \exists T_1 < T \text{ s.t. } I[\eta(T_1)] < c - 10^6 \\ &\text{as } \int_0^T \| I'[\eta] \|^2 g(\eta) dt = \infty \end{aligned}$$

$\Rightarrow \forall t < T_1, \| I'(\eta) \| g(\eta) = 0$  : contradiction

$$I[u] = \int_{\Omega} \left\{ \frac{1}{2} |\nabla u|^2 + \frac{m}{2} u^2 - F(u) \right\} dx, \quad m \geq 0.$$

**Theorem.** Suppose  $f$  is such that

$$\begin{aligned} |f'(z)| &\leq C(1 + |z|^{p-1}) \\ |f(z)| &\leq C(1 + |z|^p), \quad p < \frac{d+2}{d-2}. \end{aligned}$$

$$F(z) = \int_0^z f(u) du \quad \text{with } 0 \leq F(z) \leq \gamma z f(z), \quad \gamma < \frac{1}{2}$$

$\forall z \in \mathbb{R}.$

and  $F(z) > \beta > 0$  if  $|z| > R$

Then  $\exists$  a non-zero weak solution  $u \in H_0^1(\Omega)$  of  $-\Delta u + mu = f(u)$ , i.e.,

$$\begin{aligned} \int_{\Omega} \nabla u \cdot \nabla v + muv &= \int_{\Omega} f(u)v \\ \forall v &\in H_0^1(\Omega) \\ (H_0^1(\Omega) &= \overline{C_0^\infty}^{\|\cdot\|_{H^1}}) \end{aligned}$$

Let  $\mathcal{H} = H_0^1$

**Lemma.**  $I \in C^1(\mathcal{H}; \mathbb{R})$ ,  $I' \in Lip_{loc}$  and

$$\langle I'[u], w \rangle = \int_{\Omega} \{ \nabla u \cdot \nabla w + muw - f(u)w \} dx$$

*Proof.*

$$\begin{aligned} I[u+w] &= I[u] + \overbrace{\int_{\Omega} \{ \nabla u \cdot \nabla w + muw - f(u)w \} dx}^{\stackrel{def}{\langle I'[u], w \rangle}} \\ &\quad + \underbrace{\int_{\Omega} \left( \frac{1}{2} |\nabla w|^2 + \frac{m}{2} w^2 \right) dx}_{=O(\|w\|_{\mathcal{H}}^2)} + \underbrace{\int_{\Omega} (-F(u+w) + f(u)w + F(u)) dx}_R \end{aligned}$$

Now  $\| u \|_{\mathcal{H}} = \int_{\Omega} |\nabla u|^2$

$$| -R | = \left| \int_{\Omega} (F(u+w) - F(u) - f(u)w) dx \right|$$

$$\begin{aligned}
&\leq \int_{\Omega} \left( \int_0^1 (1-s) |f'(u+sw)| |w|^2 ds \right) dx \\
&\lesssim \int_{\Omega} (1 + |u|^{p-1} + |w|^{p-1}) |w|^2 dx \\
&\lesssim \left( \int_{\Omega} (1 + |u| + |w|)^{p+1} \right)^{\frac{p-1}{p+1}} \underbrace{\left( \int_{\Omega} |w|^{p+1} \right)^{\frac{2}{p+1}}}_{0(\|w\|_{\mathcal{H}}^2)} \quad (\text{by Sobolev Embedding})
\end{aligned}$$

$$\begin{aligned}
\| I'[u_1] - I'[u_2] \|_{\mathcal{H}^*} &= \sup_{\|v\|_{\mathcal{H}} \leq 1} |\langle I'[u_1] - I'[u_2], v \rangle| \\
&\lesssim \sup_{\|v\|_{\mathcal{H}} \leq 1} \left\{ \int_{\Omega} (|\nabla(u_1 - u_2)| |\nabla v| + m|u_1 - u_2| |v|) dx + \int_{\Omega} |f(u_1) - f(u_2)| |v| dx \right\} \\
&\lesssim \|u_1 - u_2\|_{\mathcal{H}} + \sup_{\|v\|_{\mathcal{H}} \leq 1} \int_{\Omega} (1 + |u_1|^{p-1} + |u_2|^{p-1}) |u_1 - u_2| |v| \\
&\lesssim \|u_1 - u_2\|_{\mathcal{H}} + \sup_{\|v\|_{\mathcal{H}} \leq 1} \underbrace{\left( \int_{\Omega} 1 + |u_1|^{p+1} + |u_2|^{p+1} \right)^{\frac{p-1}{p+1}}}_{C_R} \underbrace{\|u_1 - u_2\|_{p+1}}_{\|u_1 - u_2\|_{\mathcal{H}}} \|v\|_{p-1}
\end{aligned}$$

□

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We are solving

$$\begin{cases} -\Delta u + mu = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

conditions:

$$\begin{aligned} m \geq 0, \quad & |f(z)| \leq C(1 + |z|^p) \\ & |f'(z)| \leq C(1 + |z|^{p-1}) \end{aligned}$$

$$1 < p < \frac{d+2}{d-2}$$

$$\left( \text{point: } p+1 < \frac{2d}{d-2} = 2^* \right)$$

$$F(z) := \int_0^z f(w) dw, \quad 0 \leq F(z) \leq \gamma z f(z)$$

$$0 < \gamma < \frac{1}{2}$$

$$\mathcal{H} := H_0^1(\Omega), \quad \|u\|_{\mathcal{H}}^2 := \int_{\Omega} |\nabla u|^2$$

$\Omega \subset \mathbb{R}^d$  bounded, nice boundary.

**Lemma.** *Let*

$$I[u] = \int_{\Omega} dx \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2} m u^2 - F(u) \right)$$

Then  $I \in C^1(\mathcal{H}, \mathbb{R})$ ,  $I' \in Lip_{loc}$

$$\langle I'[u], v \rangle = \int_{\Omega} (\nabla u \nabla v + m u v - f(u) v) dx$$

**Lemma.** *I satisfies Palais-Smale condition*

*Proof.*  $\{u_k\} \in \mathcal{H}$ ,  $I[u_k]$  bounded and

$$I'[u_k] \xrightarrow{\|\cdot\|_{\mathcal{H}^*}} 0$$

$$|\langle I'[u_k], u_k \rangle| < \varepsilon \|u_k\|_{\mathcal{H}}$$

$$-C < \int_{\Omega} \frac{1}{2} |\nabla u_k|^2 + \frac{m}{2} u_k^2 - F(u_k) < C$$

$$-\varepsilon \|u_k\|_{\mathcal{H}} < \int_{\Omega} \frac{1}{2} |\nabla u_k|^2 + \frac{m}{2} u_k^2 - \frac{1}{2} f(u_k) u_k < \varepsilon \|u_k\|_{\mathcal{H}}$$

$$\left(\frac{1}{2} - \gamma\right) \int_{\Omega} u_k f(u_k) \leq \int_{\Omega} \frac{1}{2} f(u_k) u_k - F(u_k) < C + \varepsilon \|u_k\|_{\mathcal{H}}$$

$$\frac{\frac{1}{2} - \gamma}{\gamma} \int_{\Omega} F(u_k)$$

$$\Rightarrow \frac{1}{2} \|u_k\|_{\mathcal{H}}^2 = \int_{\Omega} \frac{1}{2} |\nabla u_k|^2 < C + \frac{\gamma}{\frac{1}{2} - \gamma} \left( C + \varepsilon \|u_k\|_{\mathcal{H}} \right)$$

$$\sup_k \|u_k\|_{\mathcal{H}} < \infty$$

$$\Rightarrow u_{kj} \rightarrow u_{\infty} \text{ in } H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega), H_0^1(\Omega) \hookrightarrow W^{p+1, \varepsilon}$$

$$\|-\Delta u_k + m u_k - f(u_k)\|_{\mathcal{H}^*} \rightarrow 0$$

Exp of non-compact embedding  $H_0^1(\Omega) \hookrightarrow L^{2^*}$

$$\left\| \phi\left(\frac{x}{\delta}\right) \delta^{-\frac{d}{2}} \right\|_2$$

$$u_{k_n} \rightarrow u_{\infty} \text{ in } L^{p+1}(\Omega)$$

$$\|u_{kj} - u_{kl}\|_2 \rightarrow 0 \Rightarrow \|u_{kj} - u_{kl}\|_{\mathcal{H}^*}$$

$$= \sup |\langle u_{kj} - u_{kl}, v \rangle_{L^2}|$$

$$0 \stackrel{?}{\leftarrow} \|f(u_{kj}) - f(u_{kl})\|_{\mathcal{H}^*} \geq \|v\|_{H_0^1(\Omega)}$$

$$= \sup_{\|v\|_{H_0^1(\Omega)}} |\langle f(u_{kj}) - f(u_{kl}), v \rangle| \leq \|u_{kj} - u_{kl}\|_2 \|v\|_2$$

$\leq 1$

Thus

$$\|u_{kj} - u_{kl}\|_{\mathcal{H}^*} \rightarrow 0$$

$$\begin{aligned} &\lesssim \|f(u_{kj}) - f(u_{kl})\|_{L^{\frac{p+1}{p}}} \\ &\lesssim \|(1 + |u_{kj}|^{p-1} + |u_{kl}|^{p-1})|u_{kj} - u_{kl}|\|_{L^{\frac{p+1}{p}}} \\ &\lesssim \|1 + |u_{kj}| + |u_{kl}|\|_{p+1}^{p-1} \|u_{kj} - u_{kl}\|_{p+1} \rightarrow 0 \\ &0 \leftarrow \|-\Delta u_{kj} + \Delta u_{kl}\|_{\mathcal{H}^*} \end{aligned}$$

$$\begin{aligned} \int |\nabla(u_{kj} - u_{kl})|^2 &\leq \varepsilon \|u_{kj} - u_{kl}\|_{\mathcal{H}} \\ &= \varepsilon \left( \int |\nabla(u_{kj} - u_{kl})|^2 \right)^{\frac{1}{2}} \end{aligned}$$

□

Comment on last lecture:

Use/Solve:

$$\dot{\eta}_t = -\frac{I'[\eta_t]}{1 + I'[\eta_t]} g(\eta_t)$$

**Theorem.** Assume in addition that

$$F(z) = \beta > 0 \text{ if } |z| > R.$$

Then  $\exists$  a weak solution  $u \neq 0$  of

$$\begin{cases} -\Delta u + mu = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \in H_0^1(\Omega) \end{cases}$$

*Proof.*

$$\text{check these 3 } \begin{cases} I[0] = 0 \\ I[u] > a > 0 & \text{if } \|u\|_{\mathcal{H}} = \tau > 0 \\ \exists v \in \mathcal{H}, \|v\|_{\mathcal{H}} > \tau, I[v] \leq 0 \end{cases}$$

$$(1) \quad F(0) = 0$$

$$\begin{aligned} (2) \quad I[u] &\geq \frac{1}{2} \|u\|_{\mathcal{H}}^2 - \int_{\{|u|>\delta\}} |F(u)| dx - \int_{\{|u|<\delta\}} |F(u)| dx \\ &> \frac{1}{2} \|u\|_{\mathcal{H}}^2 - C_\delta \int_{\Omega} |u|^{p+1} - \varepsilon \int_{\Omega} |u|^2 dx \\ &> \left( \frac{1}{2} - C_{\text{Poincare}}^\varepsilon \right) \|u\|_{\mathcal{H}}^2 - C \|u\|_{\mathcal{H}}^{p+1} \quad \text{Sobolev embed.} \\ &> \frac{1}{4} r^2 - C r^{p+1} \end{aligned}$$

$$\int |u|^2 \leq C_{\text{Poincare}} \int |\nabla u|^2$$

$$\begin{aligned}
(3) \quad (z^{-\mu} F(z))' &= z^{-\mu-1} (zF' - \mu F(z)) \\
&\geq z^{-\mu-1} \left( \frac{1}{\gamma} - \mu \right) F(z) > \beta \left( \frac{1}{\gamma} - \mu \right) z^{-\mu-1} \\
z^{-\mu} F(z) - R^{-\mu} F(R) &> \beta \left( \frac{1}{2} - \mu \right) \left( \frac{R^{-\mu}}{\mu} - \frac{z^{-\mu}}{\mu} \right) \quad \text{if } z \geq R \\
&\Rightarrow z > 200R \text{ implies} \\
F(z) &> z^\mu \frac{\beta \left( \frac{1}{\gamma} - \mu \right)}{2\mu} R^{-\mu} \\
F(z) &> a_1 z^\mu - A_1, \forall z
\end{aligned}$$

(Mountain pass in Rabinowitz)

$$\begin{aligned}
I[tv] &\leq C_v t^2 + A_1 |\Omega| - a_1 \int_{\Omega} t^\mu |v|^\mu dx \\
&\longrightarrow -\infty \text{ for } t \rightarrow \infty
\end{aligned}$$

By MP  $\exists u \neq 0$  with  $\langle I'[u], v \rangle = 0 \forall v \in \mathcal{H}$

$$\Rightarrow \int \nabla u \nabla v + muv - f(u)v = 0 \quad \forall v \in H_0^1(\Omega)$$

this is a weak solution □

— Regularity;

— Is the weak sol. a classical solution?

— Positivity of solutions?

— Is it true that the solution  $\in C^\infty(\bar{\Omega})$ ?

Regularity:

$$\begin{aligned}
-\Delta u = f \in L_{loc}^2(\Omega) &\Rightarrow u \in W_{loc}^{2,2}(\Omega) \\
\chi \text{ comp. supp } -\Delta(\chi u) &= -\chi f - 2\nabla\chi\nabla u - u\nabla\chi
\end{aligned}$$

$$-\Delta u = f(u) \stackrel{?}{\in} L^2$$

$$\int |u|^{2p}, \infty ?$$

$$\frac{1}{2} - \frac{1}{2p} < \frac{1}{d}, \quad \frac{1}{p'} < \frac{2}{d}, \quad p' > \frac{d}{2}$$

$$p < \frac{d}{d-2}, \quad p < \frac{d+2}{d-2}$$

**Theorem.** *Let  $f$  be as at the start at this lecture*

$$\begin{aligned}
-\Delta u = f(u) \quad &\text{weak solution in } \Omega \\
&\Rightarrow u \in W_{loc}^{2,2}(\Omega)
\end{aligned}$$

$$\begin{aligned}
\Omega' + h\vec{e}_i &\in \Omega \\
v_h(x) &= \frac{1}{h}(u(x + h\vec{e}_i) - u(x)) \\
-\Delta v_h &= \frac{1}{h}(f(u(x + h\vec{e}_i)) - f(u(x))) \\
\sup_{|h|<\varepsilon} \|v_h\|_{H^1(\Omega')} &< \infty
\end{aligned}$$

$$\begin{aligned}
\int \nabla v_h \nabla(\phi^2 v_h) &= \int \frac{1}{h}(f(u(x + h\vec{e}_i)) - f(u(x))) \phi^2 v_h \, dx \\
\int_{\Omega} \phi |\nabla v_h|^2 &\leq 2 \int_{\Omega} \phi |\nabla \phi| |v_h| |\nabla v_h| \\
&+ C \int_{\Omega} (1 + |u(x + h\vec{e}_i)|^{p-1} + |u(x)|^{p-1}) |v_h(x)|^2 \phi^2 \, dx \\
&\leq \frac{1}{2} \int \phi^2 |\nabla v_h|^2 + 2 \int |\nabla \phi|^2 |v_h|^2 + \\
&+ \left( \int (1 + |u(x + h\vec{e}_i)| + |u(x)|)^{(p-1)m'} \right)^{\frac{1}{m'}} \times \left( \int |\phi v_h|^{2m} \right)^{\frac{1}{m}} = X \\
&\left[ \begin{array}{l} 2m < 2^* \\ (p-1)m' < 2^* \end{array} \right.
\end{aligned}$$

$$\begin{aligned}
2m = 2^* &\Rightarrow m = \frac{2^*}{2} = \frac{d}{d-2} \\
m' &= \frac{d}{2} \\
(p-1)\frac{d}{2} &\stackrel{?}{<} 2^* = \frac{2d}{d-2} \\
p-1 &< \frac{4}{d-2} \\
p &< \frac{d+2}{d-2} \quad \checkmark
\end{aligned}$$

$$\textcircled{*} \quad 0 < \theta < 1$$

$$\begin{aligned}
\left( \int |\phi v_h|^{2m} \right)^{\frac{1}{m}} &\leq \left( \int |\nabla(\phi v_h)|^2 \right)^{\theta} \times \left( \int |\phi v_h|^2 \right)^{(1-\theta)} \\
\int_{\Omega'} |v_h|^2 \, dx &< C \int_{\Omega} |\nabla u|^2 \, dx \\
\sup_{|h|<\varepsilon} \int_{\Omega'} |\nabla v_h|^2 \, dx &< \infty
\end{aligned}$$

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**Sogge's Estimate:**

$$\| H_k f \|_{L^{p'}(S^{d-1})} \leq C(1+k)^{1-\frac{2}{d}} \| f \|_{L^p(S^{d-1})} \quad p = \frac{2d}{d+2}, \quad p' = \frac{2d}{d-2}$$

( $H_k$  = orthogonal projection onto  $k^{\text{th}}$  spherical harmonics,  $k \geq 0$ )

$$(-\Delta_{S^{d-1}} Y_k = k(k+d-2)Y_k)$$

**Carleman:**

$$\begin{aligned} \| |x|^{-\tau} u \|_{L^{q'}(\mathbb{R}^d)} &\leq C_\delta \| |x|^{-\tau} \Delta u \|_{L^q(\mathbb{R}^d)}, \quad q = \frac{2d}{d+2}, \quad q' = \frac{2d}{d-2} \\ \tau &\geq 0, \quad \text{dist}\left(\tau, \mathbb{Z} + \frac{d-2}{2}\right) > \delta > 0 \\ u &\in C_{\text{comp}}^\infty(\mathbb{R}^d \setminus \{0\}) \end{aligned}$$

*Proof.* (Using Sogge)

$$r = e^t, \quad -\infty < t < \infty, \quad w \in S^{d-1}$$

$$\Delta = \partial_r^2 + \frac{d-1}{r} \partial_r + \frac{1}{r^2} \Delta_{S^{d-1}}, \quad \partial_r = \frac{1}{r} \partial_t, \quad \partial_r^2 = \frac{1}{r^2} \partial_t^2 - \frac{1}{r^2} \partial_t$$

$$\Delta = e^{-2t} [\partial_t^2 + (d-2) \partial_t + \Delta_{S^{d-1}}], \quad dx = e^{dt} dt dw$$

$$\| |x|^{-\tau} u \|_{L^{q'}(dx)} = \| e^{-t\tau} d^{\frac{dt}{q'}} u \|_{L^{q'}(dt dw)} = \| e^{-t\tau} e^{\frac{d-2}{2}t} u \|_{L^{q'}(dt dw)}$$

$$\| |x|^{-\tau} \Delta u \|_{L^q(dx)} = \| e^{-t\tau} e^{\frac{d+2}{2}t} e^{-2t} Q u \|_{L^q(dt dw)} \quad Q = \partial_t^2 + (d-2) \partial_t + \Delta_{S^d}$$

$$\| e^{-t(\tau - \frac{d-2}{2})} u \|_{L_{t,w}^{q'}} \stackrel{?}{\leq} C \| e^{-t(\tau - \frac{d-2}{2})} Q u \|_{L_{t,w}^q}$$

$$\| u \|_{L_{t,w}^{q'}} \stackrel{?}{\leq} C \| Q_\tau u \|_{L_{t,w}^q} \quad \text{where } Q_\tau = e^{-t(\tau - \frac{d-2}{2})} Q e^{t(\tau - \frac{d-2}{2})}$$

supp  $u(\cdot, w)$  is compact, so (there  $u$  suddenly becomes  $f$ ):

$$\begin{array}{ccc} \text{in } L_w^2 & & \text{Fourier and inv. Fourier (basically } \hat{\delta}_{t-s} = e^{i\eta(t-s)}) \\ \downarrow & & \downarrow \end{array}$$

$$f(t, w) = \sum_{k \geq 0} (H_k f(t, \cdot))(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k \geq 0} (H_k f(s, \cdot))(w) e^{i\eta(t-s)} ds d\eta$$

$$\begin{aligned} (Q_\tau f)(t, w) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k \geq 0} e^{-t[\tau - \frac{d-2}{2}]} Q \{ (H_k f(s, \cdot))(w) e^{t[i\eta + \tau - \frac{d-2}{2}]} \} e^{-i\eta s} ds d\eta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k \geq 0} e^{i\eta t} \left( -k(k+d-2) + \left( i\eta + \tau - \frac{d-2}{2} \right)^2 \right) \end{aligned}$$

$$\begin{aligned}
& + (d-2) \left( i\eta + \tau - \frac{d-2}{2} \right) H_k f(s, w) e^{-i\eta s} ds d\eta \\
& = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k \geq 0} \left( (i\eta + \tau)^2 - \left( k + \frac{d-2}{2} \right)^2 \right) H_k f(s, w) e^{i\eta(t-s)} ds d\eta
\end{aligned}$$

$$\begin{aligned}
(Q_t^{-1} f)(t, w) & = (T_\tau f)(t, w) \\
& = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k \geq 0} \left( i\eta + \tau + k + \frac{d-2}{2} \right)^{-1} \left( i\eta\tau - k - \frac{d-2}{2} \right)^{-1} H_k f(s, w) e^{i\eta(t-s)} ds d\eta
\end{aligned}$$

Need to show  $\| T_\tau f \|_{L_{t,w}^{q'}} \leq C \| f \|_{L_{t,w}^q}$

$$\begin{aligned}
\| T_\tau f(t, \cdot) \|_{L_w^{q'}} & \leq \int_{-\infty}^{\infty} \sum_{k \geq 0} |m_k(\tau, t, s)| \| H_k f(\delta, \cdot) \|_{L_w^{q'}} ds \\
& \stackrel{\text{Sogge}}{\leq} C \int_{-\infty}^{\infty} \sum_{k \geq 0} |m_k(\tau, t, s)| (1+k)^{1-\frac{2}{d}} \| f(s, \cdot) \|_{L_w^q} ds \quad \text{where} \\
m_k(\tau, t, s) & = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\eta(t-s)} d\eta}{(i\eta + \tau + k + \frac{d-2}{2})(i\eta + \tau - k - \frac{d-2}{2})} \quad (\text{note: } \left| \tau \pm \left( k + \frac{d-2}{2} \right) \right| > \delta)
\end{aligned}$$

so  $|m_k(\tau, t, s)| \lesssim (1+k)^{-1} \exp(-|t-s| - |\tau \pm k \pm \frac{d-2}{2}|)$  by a residue calculation. So

$$\begin{aligned}
\| T_\tau f(t, \cdot) \|_{L_w^{q'}} & \leq C \int_{-\infty}^{\infty} \underbrace{\sum_{k \geq 0} (1+k)^{-\frac{2}{d}} \exp(-|t-s| \left| \tau \pm k \pm \frac{d-2}{2} \right|)}_{\leq C_\delta |t-s|^{-1+\frac{2}{d}}} \| f(s, \cdot) \|_{L_w^q} ds \\
& \leq C \int_{-\infty}^{\infty} C_\delta |t-s|^{-1+\frac{2}{d}} \| f(s, \cdot) \|_{L_s^q} ds \\
& \leq C_\delta \| f \|_{L_{t,w}^q} \quad \text{by fract. integration}
\end{aligned}$$

□

### Derrick-Pohozaev Identity:

We have

$$-\Delta u + mu = f(u) = |u|^{p-1}u, \quad m \geq 0, \quad 1 < p < \frac{d+2}{d-2} \quad (*)$$

Proceed formally in  $\mathbb{R}^d$

$$I[u] = \int \left\{ \frac{1}{2} |\nabla u|^2 + \frac{m}{2} |u|^2 - F(u) \right\} dx$$

has crit. point at  $u$

$$u_r(x) = u(rx), \text{ so } \frac{d}{dr}I[u_r]|_{r=1} = 0$$

$$I[u_r] = r^{-d} \int \left\{ \frac{1}{2}r^2 |\nabla u|^2 + \frac{m}{2}|u|^2 - F(u) \right\} dx$$

$$\frac{d}{dr} \Big|_{r=1} I[u_r] = -\frac{d-2}{2} \int |\nabla u|^2 - \frac{d}{2} \int mu^2 + d \int F(u) = 0 \quad (= \text{Derrick Pohozaev})$$

Also, we have  $\int |\nabla u|^2 + \int mu^2 - \int |u|^{p+1} = 0$  (integrate (\*) by parts). So

$$0 = \left( \frac{d}{p+1} - \frac{d-2}{2} \right) \int |\nabla u|^2 + \left( \frac{d}{p+1} - \frac{d}{2} \right) m \int u^2 \quad (< 0 \text{ if } p > \frac{d+2}{d-2} \text{ or } m = 0)$$

So if

$$\left. \begin{array}{l} m > 0 \quad \& \quad p \geq \frac{d+2}{d-2} \\ m \geq 0 \quad \& \quad p > \frac{d+2}{d-2} \end{array} \right\} \Rightarrow u \equiv 0$$

Now, what about in a bdd domain  $\Omega$ ?

**Proposition:** Let  $\Omega$  be bdd domain in  $\mathbb{R}^d$ , star-shaped w.r.t. some interior point. Let  $u \in C^2(\Omega)$ . Solve

$$\begin{cases} -\Delta u + mu = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

If  $p > \frac{d+2}{d-2}$ ,  $m \geq 0$ , or  $p \geq \frac{d+2}{d-2}$ ,  $m > 0$ , then  $u \equiv 0$ .

*Proof.*

$$\int (-\Delta u + mu)x_j \partial_j u = \int |u|^{p-1} u x_j \partial_j u \quad \circledast$$

↓ summed over  $l, j$

$$\int_{\Omega} -\Delta u x_j \partial_j u = \int_{\partial\Omega} -\frac{\partial u}{\partial n} x_j \partial_j u + \int_{\Omega} (\partial_j u)^2 + \partial_l u x_j \partial_{jl} u. \quad \frac{\partial u}{\partial n} = \vec{\nu} \cdot \nabla u$$

$\vec{\nu}$  = outer norm

$$\begin{aligned} \int_{\Omega} \partial_l u x_j \partial_{jl} u &= \int_{\Omega} x_j \partial_j \frac{1}{2} |\nabla u|^2 = \frac{1}{2} \int_{\partial\Omega} \vec{\nu} \cdot \vec{x} |\nabla u|^2 - \frac{d}{2} \int_{\Omega} |\nabla u|^2 \\ \int_{\partial\Omega} \frac{\partial u}{\partial n} \vec{x} \cdot \vec{\nabla} u &= \int_{\partial\Omega} \vec{x} \cdot \vec{\nu} |\nabla u|^2 \end{aligned}$$

So

$$\begin{aligned} \int_{\Omega} -\Delta u x_j \partial_j u &= -\frac{1}{2} \int_{\partial\Omega} \vec{x} \cdot \vec{\nu} |\nabla u|^2 + \int_{\Omega} |\nabla u|^2 \left( \frac{2-d}{2} \right) \\ \int_{\Omega} u x_j \partial_j u &= \int_{\Omega} x_j \partial_j \frac{1}{2} u^2 = \int_{\partial\Omega} \vec{x} \cdot \vec{\nu} \frac{1}{2} u^2 - \frac{d}{2} \int_{\Omega} u^2 = -\frac{d}{2} \int_{\Omega} u^2 \end{aligned}$$

$$\int_{\Omega} |u|^{p-1} u x_j \partial_j u = \int_{\Omega} x_j \partial_j \left( \frac{|u|^{p+1}}{p+1} \right) = -\frac{d}{p+1} \int_{\Omega} |u|^{p+1}$$

So  $\circledast$  is

$$\frac{d-2}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\partial, \Omega} \vec{x} \cdot \vec{\nu} |\nabla u|^2 + \frac{md}{2} \int_{\Omega} u^2 = \frac{d}{p+1} \int_{\Omega} |u|^{p+1}$$

and

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 + m \int_{\Omega} u^2 &= \int_{\Omega} |u|^{p+1} \\ \left( \frac{d-2}{2} - \frac{d}{p+1} \right) \int_{\Omega} |\nabla u|^2 + m \left( \frac{d}{2} - \frac{d}{p+1} \right) \int_{\Omega} u^2 &\leq 0 \end{aligned}$$

because  $\vec{x} \cdot \vec{\nu} \geq 0$  if 0 is the “star” point □

**Variational approach** (for  $p < \frac{d+2}{d-2}$ ).

$$\begin{aligned} -\Delta u + mu &= |u|^{p-1} u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{aligned} \quad p < \frac{d+2}{d-2}$$

$$E[u] = \int \frac{1}{2} |\nabla u|^2 + \frac{m}{2} u^2$$

$$G[u] = \frac{1}{p+1} \int |u|^{p+1}$$

**Step 1:**  $\inf\{E[u] | u \in H_0^1(\Omega), G[u] = 1\}$  is attained. Let  $\{u_j\}$  be a minimizing sequence in  $H_0^1(\Omega) \Rightarrow u_j \rightarrow u_\infty$  in  $H_0^1(\Omega)$ . So  $u_j \rightarrow u_\infty$  in  $L^{p+1}(\Omega)$  (since  $p+1 < 2^*$ )  $\Rightarrow G[u_\infty] = 1$ ,  $E[u_\infty] \leq \inf E[u_j]$ . So  $u_\infty$  is a minimizer.

**Step 2:**  $u_\varepsilon = (u_\infty + \varepsilon v)(h(\varepsilon))^{-1}$  where  $(h(\varepsilon))^{p+1} = G[u_\infty + \varepsilon v]$ . Then  $G[u_\varepsilon] = 1$  and

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} E[u_\varepsilon] = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} h(\varepsilon)^{-2} E[u_\infty + \varepsilon v] \\ &= -2h'(0)h(0)E[u_\infty] + \langle E'[u_\infty], v \rangle \\ &= -2 \frac{\langle G'[u_\infty], v \rangle}{p+1} E[u_\infty] + \langle E'[u_\infty], v \rangle \end{aligned}$$

So

$$\int \nabla u_\infty \nabla v + m u_\infty v = \underbrace{\frac{2}{p+1} E[u_\infty]}_{\lambda > 0} \int |u_\infty|^{p-1} u_\infty v$$

So  $u_\infty$  is a weak solution of

$$\begin{aligned} -\Delta u + mu &= \lambda |u|^{p-1} u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{aligned}$$

**Step 3:**

$$u = \lambda^{-\beta} u_\infty \text{ for } \beta = -\frac{1}{p-1} \Rightarrow \begin{cases} -\Delta u + mu = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Suppose  $u_\infty = u_\infty^+ - u_\infty^-$  where  $u_\infty^+ \not\equiv 0$ ,  $u_\infty^- \not\equiv 0$ .

$$1 = G[u_\infty] = \alpha_+^{p+1} G\left[\frac{u_\infty^+}{\alpha_+}\right] + \alpha_-^{p+1} G\left[\frac{u_\infty^-}{\alpha_-}\right] \quad (\alpha\text{'s s.t. the } G\text{'s are 1})$$

$$E[u_\infty] = \alpha_+^2 E\left[\frac{u_\infty^+}{\alpha_+}\right] + \alpha_-^2 E\left[\frac{u_\infty^-}{\alpha_-}\right] \geq (\alpha_+^2 + \alpha_-^2) E[u_\infty] \quad (u_\infty \text{ is minimizer})$$

Here  $1 = \alpha_+^{p+1} + \alpha_-^{p+1}$  and  $0 < \alpha_\pm < 1 \Rightarrow \alpha_+^2 + \alpha_-^2 < 1 \Rightarrow$  contradiction.

Now  $u_\infty^+ \equiv 0$  or  $u_\infty^- \equiv 0$  (assume  $u_\infty \geq 0$ )

Assume  $p > 2 \Rightarrow u_\infty \in C^2(\bar{\Omega})$

$$-\Delta u_\infty + mu_\infty \geq 0$$

By strong max principle:  $u_\infty > 0$  in  $\Omega$

Evans — Ch. 9  $\Rightarrow u_\infty$  radial

So we have a solution  $u_\infty > 0$ , radial (same for  $u$ ).

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Solve  $-\Delta u + mu = g(u)$  in  $\mathbb{R}^d$ .

$m > 0$   $g$  nondegenerate ( $g \neq 0$ ) subcritical (Derrick-Pohozaev)

$$|g'(u)| \lesssim 1 + |u|^{p-1} \quad 1 < p < \frac{d+2}{d-2}$$

$$G(u) = \int_0^u g(v) dv$$

$$G(u) > \frac{1}{2} mu^2 \text{ for some } u > 0$$

$$G(u) = o(u^2) \text{ as } u \mapsto 0$$

$$I[u] = \frac{1}{2} \int |\nabla u|^2$$

$$V[u] = \int \left( G(u) - \frac{1}{2} mu^2 \right) dx \quad \text{functionals on } H^1.$$

$$|G(u)| \leq \varepsilon u^2 + C_\varepsilon |u|^{p+1} \quad \text{by Gagliardo } -N - S$$

We know that  $\|u\|_{p+1} \leq C \|\nabla u\|_2^{1-\theta} \|u\|_2^\theta$  for some  $0 < \theta < 1$ .

**Theorem.**  $\mathcal{A} := \{u \in H^1(\mathbb{R}^d) : V(u) = 1\} \neq \emptyset$  and  $I_\infty = \inf_{u \in \mathcal{A}} I[u]$  is attained at some  $u_0 \in \mathcal{A}$  which is radial, non-increasing and non-negative. Moreover,  $\exists \lambda > 0$ , s.t.  $-\Delta u = \lambda[g(u_0) - mu_0]$  in the weak sense.

**Corollary.**  $d = 3$   $\otimes$  has a classical solution ( $\in C^2(\mathbb{R}^d)$ ) which is positive, radial, and non-increasing.

*Proof.*  $u(x) = u_0(\lambda^{-\frac{1}{2}}x)$

$$\begin{aligned} -\Delta u &= -\lambda^{-1}(\Delta u_0)(\lambda^{-\frac{1}{2}}x) = (g(u_0) - mu_0)(\lambda^{-\frac{1}{2}}x) \\ &= g(u) - mu = f(u) \end{aligned}$$

Recall interior regularity theory  $\Rightarrow u \in H_{\text{loc}}^2(\mathbb{R}^d)$ .

Idea:  $-\Delta \partial_j u = f'(u) \partial_j u$

$$\begin{aligned} \int \nabla \partial_j u \cdot \nabla (x^2 \partial_j \cdot u) &= \int f'(u) \partial_j u x^2 \partial_j u \\ \int \chi^2 |\nabla \partial_j u|^2 &\leq 2 \int \chi |\nabla \chi| |\nabla \partial_j u| |\partial_j u| + \int \underbrace{|f'(u)|}_{1+|u|^{p-1}} |\partial_j u|^2 \chi^2 \\ &\leq \frac{1}{4} \int \chi^2 |\nabla \partial_j u|^2 + 4 \int |\nabla x|^2 |\partial_j u|^2 + C \underbrace{\left( \int_{\Omega} (1 + |u|^{p-1})^{\frac{p+1}{p-1}} \right)^{\frac{p-1}{p+1}}}_{\lesssim \|u\|_{H^1}} \\ &\quad \left( \int |\chi \partial_j u|^{p+1} \right)^{\frac{2}{p+1}} \end{aligned}$$

For  $d = 3$   $\frac{1}{2} - \frac{1}{p} = \frac{2}{3} \Rightarrow H_{\text{loc}}^2(\mathbb{R}^3) \hookrightarrow C_{\text{loc}}^{0,\alpha}(\mathbb{R}^3)$

$$-\Delta u = f(u) \in C_{\text{loc}}^\alpha$$

Schauder estimates  $\Rightarrow u \in C_{\text{loc}}^{2,\alpha}$

$$[\Delta^2 \phi]_\alpha \leq C[\Delta \phi]_\alpha \text{ if } \phi \in C_0^2$$

↑

↙

In harmonic analysis

compactly supported

last year

In order to get rid of the compact support assumption  $\rightarrow$  extra technical complications.

We know  $u$  is a radial, non-increasing &  $u \geq 0$ .

Suppose  $u(x_0) = 0$  for some  $x_0 \Rightarrow u(x)$  if  $|x| \geq x_0$ .

$$\begin{aligned}
-\Delta u + mu &= q(x) \underbrace{u}_{\geq 0} & q(x) &= \frac{g(u(x))}{u(x)} \\
-\Delta u &\geq 0 & \parallel & \\
&& (q^+ - q^-)u & \\
-\Delta u + \underbrace{(m + q^-)}_{\geq 0} u &= q^+ u \geq 0
\end{aligned}$$

If  $u = 0$  anywhere  $\Rightarrow u \equiv 0$  □

### Proof of theorem

1)  $\mathcal{A} \neq \emptyset$  we know  $\mu_0 = G(w) - \frac{1}{2}mw^2 > 0$  for some fixed  $w > 0$ .

$$u(x) = \begin{cases} w & \text{if } |x| < R \\ w(R+1-|x|)_+ & \text{of } |x| > R \end{cases}$$

$$V[u] \geq C_d \mu_s R^d - CR^{d-1} > 0 \text{ if } R \text{ large}$$

$$V[u(\sigma \cdot)] = \sigma^{-d} V(u) = 1$$

2) Pick a minimizing sequence  $\{u_n\}$

$$I[u_n] \searrow I_\infty \quad V[u_n] = 1$$

make  $u_n$  radial and decreasing. More precisely, let  $v_n = |u_n|^*$

[Digression: decreasing rearrangements, wedding cake principle ...]

$$\begin{aligned}
V[u_n] &= V[|u_n|] = V[|u_n|^*] \\
\int |\nabla u_n|^2 &= \int |\nabla |u_n||^2 \stackrel{\text{Lemma}}{\geq} \int |\nabla |u_n|^*|^2.
\end{aligned}$$

**Lemma.** If  $f \in H^1(\mathbb{R}^d) \Rightarrow |f|^* \in H^1$  and  $\int |\nabla |f|^*|^2 \leq \int |\nabla f|^2$

*Proof.* (Lieb)

$$\begin{aligned}
f_1 g_1 h \geq 0 \text{ measurable} &\Rightarrow \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(x-y) h(y) dx dy \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^*(x) g^*(x-y) h^*(y) dx dy
\end{aligned}$$

$$g_t(x) = \frac{C}{t^{\frac{d}{2}}} e^{-\frac{\pi x^2}{t}} \quad \hat{g}_t(\xi) = e^{-\pi t \xi^2}$$

$$\begin{aligned}
f\phi \in L^2 I_t[\phi] &= \frac{1}{t} \left\{ \int_{\mathbb{R}^d} |\phi|^2 - \int \Phi(x)\Phi(y)g_t(x-y)dx dy \right\} \\
&= \frac{1}{t} \int |\hat{\phi}|^2(\xi) \underbrace{(1 - e^{\pi t \xi^2})}_{\leq \pi t \xi^2} d\xi
\end{aligned}$$

If  $f \in H^1$

$$\begin{array}{ccc}
I_t[f] & \geq & I_t[f^*] \\
\text{DCT } \downarrow & & \downarrow \text{Fatou} \\
f^* \in L^2 & & \\
C_0 \int |\nabla f|^2 & & C_0 \int |\nabla f^*|^2
\end{array}$$

□

WLOG  $u_n = v_n \quad \sup_n \|u_n\|_{H^1} < \infty$

$$\begin{aligned}
L^2 \text{ part} \quad 1 + \frac{1}{2} \int m u_n^2 &\leq \int G(u_n) \\
&\leq \varepsilon \int u_n^2 + C_\varepsilon \int |u_n|^{2^*} \leq \varepsilon \int u_n^2 + C_\varepsilon \int |\nabla u_n|^2
\end{aligned}$$

WLOG  $u_n \rightarrow u_\infty$  in  $H^1$   
 $u_n \rightarrow u_\infty$  in  $L_{\text{loc}}$  for any  $1 \leq q \leq p+1$   
 $u_n \rightarrow u_\infty$  a.e.

$$\begin{aligned}
I[u_\infty] &\leq \lim I[u_n] = I_\infty \\
\int_{\mathbb{R}^d} G(u_n) &\mapsto \int_{\mathbb{R}^d} G(u_\infty)
\end{aligned}$$

Need to prevent the bulk of  $u_n$  to dmsh off to  $\infty$ .

**Lemma.** Let  $f \in H^1(\mathbb{R}^d)$   $d \geq 3$  and radial. Then  $\exists R = R(d) > 0$  and a continuous function  $F(x)$  s.t.

$$F(x) = f(x) \text{ for a.e. } |x| > R \text{ and } |F(x)| \leq C_d |x|^{\frac{1-d}{\alpha}} \|f\|_{H^1}$$

$$\begin{aligned}
\lim_{\mathbb{R}^d} \left| \int (G(u_n) - G(u_\infty)) \right| &\leq \overbrace{\lim_{|*| < R} \int |G(u_n) - G(u_\infty)|}^{=0} \\
&\quad + 2 \lim_{|x| > R} \int |G(u_n)| + \int_{|x| > R} |G(u_\infty)| \\
&\leq \varepsilon \sup_n \|u_n\|_2^2 \leq \varepsilon C
\end{aligned}$$

$$1 = \overline{\lim} V[u_n] = \overline{\lim} \int (G(u_n) - \frac{1}{2}mu_n^2) = \int G(u_\infty) - \frac{1}{2}m \underline{\lim} \int u_n^2 \leq V[u_\infty]$$

If

$$V[u_\infty] > 1 \quad V[u_\infty(\sigma \cdot)] = \sigma^{-d}V[u_\infty] = 1$$

$$I[u_\infty(\sigma \cdot)] = \sigma^{(2-d)} \cdot I[u_\infty] < I[u_\infty]$$

$$V[u_\infty] = 1, u_\infty \in \mathcal{A} \quad I[u_\infty] = I_\infty$$

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$$-\Delta u + mu = g(u) \quad \text{in } \mathbb{R}^d, d \geq 3. \quad \textcircled{*}$$

$m > 0, g$  odd,  $g \in C^1(\mathbb{R})$ .

$$|g'(u)| \leq C(1 + |u|)^{p-1}, \quad 1 < p < \frac{d+2}{d-2}$$

$G(u) = \int_0^u g(v) dv$  satisfies

$$G(w) > \frac{1}{2}mw^2 \quad \text{for some } w > 0.$$

$$G(u) = \sigma(u^2) \quad \text{as } u \rightarrow 0.$$

$$I[u] = \int \frac{1}{2}|\nabla u|^2, \quad V[u] = \int G(u) - \frac{1}{2}mu^2.$$

**Theorem.** (Bereslycki-Lions,  $\sim 80$ )

Let  $\mathcal{A} = \{u \in H^1(\mathbb{R}^d) ; V[u] = 1\}$ . Then  $\inf_{u \in \mathcal{A}} I[u]$  is attained at some  $u_\infty \in \mathcal{A}$  where  $u_\infty$  is radial, decreasing, nonnegative, and  $\exists \lambda > 0$  s.t.  $u_\infty$  is a weak solution of  $-\Delta u = \lambda(g(u) - mu)$ .

*Proof.* We found the minimizer  $u_\infty$  as a weak limit in  $H^1$  of the symmetric, nonincreasing, rearrangement of some other sequence of minimizers.  $I[u_\infty] = I_\infty, u_\infty \geq 0$ .

$$V[(u_\infty + \varepsilon\theta)(\sigma(\varepsilon)\cdot)] = 1 = \sigma(\varepsilon)^{-d}V[u_\infty + \varepsilon\theta]$$

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} I[(u_\infty + \varepsilon\theta)(\sigma(\varepsilon)\cdot)] = 0$$

$$\underbrace{V[w + \varepsilon v]} = V[w] + \varepsilon \langle V'[w], v \rangle + \sigma(\varepsilon).$$

$$\int G(w + \varepsilon v) - \frac{1}{2}m(w + \varepsilon v)^2$$

$$= V[w] + \int (G(w + \varepsilon v) - G(w)) - \varepsilon m w v + O(\varepsilon^2).$$

$$= V[w] + \underbrace{\varepsilon \langle V'[w], v \rangle}_{\substack{\text{by definition,} \\ f(g(w)v - mvv)}} + \int (G(w + \varepsilon v) - G(w)) - \varepsilon g(w)v + O(\varepsilon^2)$$

with

$$\begin{aligned} & \left| \int G(w + \varepsilon v) - G(w) - \varepsilon g(w)v \right| \\ &= \left| \int_0^1 \int \frac{d}{dt} G(w + \varepsilon tv) dt - \varepsilon g(w)v \right| \\ &\leq \int_0^1 \int_{\mathbb{R}^d} |g(w + \varepsilon tv) - g(w)| \varepsilon |v| dx dt \\ &\leq \int_0^1 \int_0^1 \int_{\mathbb{R}^d} |g'(w + \varepsilon tsv)| \varepsilon^2 t |v|^2 dx dt ds \\ &\leq C \varepsilon^2 \int_{\mathbb{R}^d} (1 + |w|^{p-1} + |v|^{p-1}) |v|^2 \leq C \varepsilon^2 \end{aligned}$$

$$\begin{aligned} v, w \in H^1 &\hookrightarrow L^{p+1}(\mathbb{R}^d) \\ &\hookrightarrow L^{2^*}(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \end{aligned}$$

For  $|w|^{p-1}|v|^2$ , interpolate by Gagliardo - N-S.

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} I[(u_\infty + \varepsilon\phi)(\sigma(\varepsilon)\cdot)] \\ &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \sigma(\varepsilon)^{2-d} I[u_\infty + \varepsilon\phi] \\ \text{N.B. } &\begin{cases} \sigma'(0) \text{ exists because } -d\sigma'(0) + \langle V'[u_\infty], \phi \rangle = 0 \\ \sigma(0) = 1 \end{cases} \\ &= (2-d)\sigma'(0)I[u_\infty] + \langle I'[u_\infty], \phi \rangle \\ &\Rightarrow \langle I'[u_\infty], \phi \rangle = \frac{\lambda}{\frac{d-2}{d}I[u_\infty] > 0} \langle V'[u_\infty], \phi \rangle \end{aligned}$$

And then rescale

$$-\Delta u_\infty = \lambda(g(u_\infty) - mu_\infty)$$

to get

$$-\Delta u + mu = g(u)$$

□

**Decay lemma** for radial  $H^1$  functions.

Let  $f \in H^1(\mathbb{R}^d)$ ,  $d \geq 3$ , be radial. Then  $f(x) = F(x)$  a.e., where  $F$  is continuous away from 0 (Jacobian factor,  $f$  is actually Hölder sth along rays by Sobolev embedding), and  $|F(x)| \leq C|x|^{\frac{1-d}{2}} \|f\|_{H^1}$  for  $|x| > R(d)$ .

*Proof.* Assume first that  $f \in C_0^\infty$ .  $m = \frac{d-1}{2}$

$$\begin{aligned} \frac{d}{dr}(r^m f(r))^2 &= 2(r^m f(r))' r^m f(r) \\ &\leq \left( (r^m f(r))' \right)^2 + (r^m f(r))^2 \end{aligned}$$

Integrate  $\int_0^r$ :

$$\begin{aligned} (r^m f(r))^2 &\leq \int_0^r \left[ (\rho^m f(\rho))' \right]^2 d\rho \\ &\quad + \int_0^r \rho^{2m} f(\rho)^2 d\rho \\ &= \int_0^r \rho^{2m} f'(\rho) d\rho + 2m \int_0^r \rho^{2m-1} f'(\rho) \overbrace{f(\rho)}^A d\rho \\ &\quad + m^2 \int_0^r \rho^{2m-2} \underbrace{f(\rho)^2}_B d\rho + \underbrace{\int_0^r \rho^{2m} f(\rho)^2 d\rho}_{\leq \|f\|_{H^1}^2} \end{aligned}$$

$$A = \int_0^r \rho^{2m-1} \frac{1}{2} \left( (f(\rho))^2 \right)' d\rho = -\frac{2m-1}{2} \int_0^r \rho^{2m-2} f(\rho)^2 d\rho + \frac{r^{2m-1}}{2} f(r)^2$$

(no contrib. at 0 because  $2m-1 = d-2 \geq 1$ ).

$$(r^m f(r))^2 \leq C \|f\|_{H^1}^2 + \underbrace{[m^2 - (2m-1)m]}_{=m-m^2 \leq 0} \int_0^r \rho^{2m-2} f^2 d\rho + mr^{2m-1} f(r)^2$$

$$r^{2m} \left( 1 - \frac{m}{r} \right) f(r)^2 \leq C \|f\|_{H^1}^2$$

Approximate  $H^1$  radial by  $C^\infty$  radial. □

Recall the main corollary:  $\otimes$  has a  $C^2$  solution  $u$  which is radial, nonincreasing and positive. Moreover,  $u$  decays exponentially,  $\forall \varepsilon > \exists C_\varepsilon : |u(x)| \leq C_\varepsilon e^{(-\sqrt{m-\varepsilon})|x|}$ .

*Proof.*

$$\begin{aligned}
u(x) &= u_\infty(\lambda^{-\frac{1}{2}}x), \quad \text{check that this is a weak solution of } \circledast \\
-\Delta u &= f(u) = g(u) - mu \\
u &\in H_{loc}^2(\mathbb{R}^d) \stackrel{d=3}{\Leftrightarrow} C_{loc}^{0,\alpha} \Rightarrow f(u) \in C_{loc}^\alpha \text{ (because } g \in C^1)
\end{aligned}$$

Recall Schauder estimates,  $0 < \alpha < 1$

$$\begin{aligned}
[D^2g]_\alpha &\leq C_{\alpha,d}[\Delta g]_\alpha \quad \forall g \in C_0^\infty \\
\Rightarrow [D^2u]_\alpha &< \infty \\
-\Delta u &= f(u) \text{ a.e.}
\end{aligned}$$

Observe that  $\partial_j u$  is a weak solution of

$$\begin{aligned}
-\Delta(\partial_j u) &= \underbrace{f'(u)}_{\in L^\infty} \partial_j u \\
\text{because } u &\in \underbrace{\left\{ \begin{array}{l} H^1(\mathbb{R}^d) \\ C_{loc}^{0,\alpha} \end{array} \right\}}_{\in L^2(\mathbb{R}^d)} \text{ and decays by the lemma}
\end{aligned}$$

$$\Rightarrow \partial_j u \in H^2(\mathbb{R}^d) \Rightarrow u \in H^3(\mathbb{R}^d) \stackrel{d=3}{\Leftrightarrow} C^{1,\alpha}$$

In order to apply Schauder, mollify,

$$\begin{aligned}
u_\varepsilon &= u * \eta_\varepsilon \\
-\Delta u_\varepsilon &= f(u) * \eta_\varepsilon \\
-\Delta(\chi u_\varepsilon) &= \chi(f(u) * \eta_\varepsilon) - \Delta \chi u_\varepsilon - 2\nabla \chi \nabla u_\varepsilon \\
[D^\alpha(\chi u_\varepsilon)]_\alpha &\leq C[\chi \cdot (f(u) * \eta_\varepsilon)]_\alpha + C \cdot [\Delta \chi u_\varepsilon]_\alpha + C[\nabla \chi \nabla u_\varepsilon]_\alpha.
\end{aligned}$$

Note that

$$\begin{aligned}
\sup_{1 > \varepsilon > 0} [\chi \cdot f(u) * \eta_\varepsilon]_\alpha &< \infty \\
\sup_{0 < \varepsilon < 1} [\Delta \chi \cdot u_\varepsilon]_\alpha &< \infty \\
\sup_{0 < \varepsilon < 1} [\nabla \chi \cdot \nabla u_\varepsilon]_\alpha &< \infty \\
\Rightarrow \sup_{0 < \varepsilon < 1} [D^2(\chi u_\varepsilon)]_\alpha &< \infty
\end{aligned}$$

$$\chi u \in H^2, \quad |D^2(\chi u_\varepsilon)(x) - D^2(\chi u_\varepsilon)(y)| \leq C|x - y|^\alpha$$

$$|D^2(\chi u_\varepsilon)(x) - \underbrace{\int_{B(0,1)} D^2(\chi u_\varepsilon)(y) dy}_{\downarrow \varepsilon \rightarrow 0}} \leq C(1+|x|)^\alpha$$

$$\int_{B(0,1)} D^2(\chi u)(y) dy \text{ by LDCT.}$$

$\Rightarrow \left\{ \begin{array}{l} \{D^2(\chi u_\varepsilon)\}_{\varepsilon>0} \text{ is locally uniformly bdd} \\ \text{and equicontinuous} \end{array} \right.$

$$D^2(\chi u_{\varepsilon_j}) \xrightarrow{L^\infty} D^2(\chi u) \text{ a.e.}$$

$$\Rightarrow u \text{ is } C^2$$

Actually  $D^2u \in C^{0,\alpha}$

Decay:

$$-\Delta u - \frac{g(u)}{u}u = -mu$$

$$V(x) = \begin{cases} -\frac{g(u(x))}{u(x)} & \text{if } u(x) \neq 0 \\ 0 & \text{else.} \end{cases}$$

$$Hu = -\Delta u - \frac{g(u)}{u}u$$

Apply Agmon.  $V$  is continuous and  $V(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , and

$$\lim_{R \rightarrow \infty} \sup_{|x|>R} |V(x)| = 0.$$

(use  $u$  radial and  $g(u) = \sigma(u)$  at the origin).

$$\int_{\mathbb{R}^d} e^{2(\sqrt{m}-\varepsilon)|x|} |u(x)|^2 \leq C_\varepsilon$$

□

Back to NLS:

$$i\partial_t \psi + \frac{1}{2}\Delta \psi + f(|\psi|^2)\psi = 0$$

$$\psi = 0 \text{ on } \partial\Omega \times [0, \infty)$$

$$\psi = u, \quad t = 0.$$

$$\psi(t) = e^{\frac{i\alpha^2}{2}t} u(x)$$

$$\Rightarrow \begin{cases} \frac{-\alpha^2}{2}u + \frac{1}{2}\Delta u + f(u^2)u = 0 \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

e.g.,  $f(u^2) = |u|^{2\sigma}$ , focusing case,

$$0 < \sigma < \frac{4}{d-2}$$

Stability of solitons.

$$\tilde{\psi}(t, x) = u(x) \left( 1 + \varepsilon r(t, x) + O(\varepsilon^2) \right) e^{i \left( \frac{\alpha^2}{2} t + \varepsilon s(t, x) + O(\varepsilon^2) \right)}$$

(Heuristics here)

$$= \left( u + \varepsilon \underbrace{ru}_w + \varepsilon i \underbrace{su}_v + O(\varepsilon^2) \right) e^{\frac{i}{2} \alpha^2 t}$$

→ Find estimates of boundedness on  $v, w$  when initially of size  $O(1)$ .

Answer: stable for  $\sigma d < 2$ , unstable for  $\sigma d > 2$ .

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$$i \partial_t \psi + \frac{1}{2} \Delta \psi + |\psi|^{2\sigma} \psi = 0 \quad (\text{or with } f(|\psi|^2) \psi) \quad \sigma < \frac{2}{d-2}$$

$$\psi(t, x) = e^{i \frac{\alpha^2}{2} t} u(x; \alpha)$$

$$- \frac{\alpha^2}{2} u + \frac{1}{2} \Delta u + u^{2\sigma+1} = 0 \quad (\text{or with } f(u^2) u)$$

$u < 0$ , radial, etc. (in  $d = 3$ )

$$d = 1 \quad \& f(u) = u^\sigma \Rightarrow \phi(x) = \frac{(\sigma + 1)^{\frac{1}{2\sigma}}}{\cosh^{\frac{1}{\sigma}}(\sigma x)} \quad (0 < \sigma < \infty) \quad \text{solves on } \mathbb{R}:$$

$$\phi'' - \phi + |\phi|^{2\sigma} \phi = 0$$

**Heuristic discussion of stability:**

$$\psi(t, x) = u(x, \alpha) \left( 1 + \varepsilon r(t, x) + O(\varepsilon^2) \right) e^{i \left( \frac{\alpha^2}{2} t + \varepsilon \beta(t, x) + O(\varepsilon^2) \right)}$$

$$\approx \left( u + \varepsilon \underbrace{ur}_w + i \varepsilon \underbrace{u\beta}_v \right) e^{i \frac{\alpha^2}{2} t}$$

$$\begin{aligned} 0 &= i(\varepsilon \dot{w} + i \varepsilon \dot{v}) + \frac{1}{2} (\Delta u + \varepsilon \Delta w + i \varepsilon \Delta v) + f(|u + \varepsilon w + i \varepsilon v|^2) (u + \varepsilon w + i \varepsilon v) - \frac{\alpha^2}{2} (u + \varepsilon w + i \varepsilon v) \\ &\approx [f(u^2) + f'(u^2) 2 \varepsilon u w] (u + \varepsilon w + i \varepsilon v) \end{aligned}$$

$$0 = -\dot{v} + \frac{1}{2} \Delta w + f(u^2) w + 2 f'(u^2) u^2 w - \frac{\alpha^2}{2} w$$

$$0 = \dot{w} + \frac{1}{2} \Delta v + f(u^2) v - \frac{\alpha^2}{2} v$$

So:

$$\begin{pmatrix} \dot{w} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & L_0 \\ -L_1 & 0 \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} \quad \text{with} \quad \begin{aligned} L_0 &= \left( -\frac{1}{2}\Delta + \frac{\alpha^2}{2} \right) - f(u^2) \\ L_1 &= \left( -\frac{1}{2}\Delta + \frac{\alpha^2}{2} \right) - f(u^2) - 2f'(u^2)u^2 \end{aligned}$$

or

$$\ddot{w} = -L_0 L_1 w$$

$$\dot{v} = -L_1 L_0 v$$

$L_0 u = 0 \Rightarrow u$  is e-func. of  $L_0$ .

$\sigma(L_0) = [\frac{\alpha^2}{2}, \infty) \cup \{\text{e-vals below } \frac{\alpha^2}{2}\}$  (since  $f(u^2)$  is rapidly decaying).

Ground state of  $L_0$  is simple and positive (and orthogonal to  $u$  if different from  $u$ ). Since  $u > 0 \Rightarrow \inf \sigma(L_0) = 0$  and  $u$  is the ground state.

**Theorem.**  $H = -\Delta + V$  and  $V(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Assume  $\langle Hf, f \rangle < 0$  for some  $f \in \text{Dom}(H) = H^2$ . Then  $\inf_{\substack{\|f\|_2=1 \\ f \in H^2}} \langle Hf, f \rangle$  is attained at some  $f_0 > 0$ ,  $\|f_0\|_2 = 1$  and

$\langle Hf_0, f_0 \rangle = \lambda_0 < 0$  is a simple e-val.

*Proof.* (Sketch):

$$\left. \begin{aligned} \langle Hf_n, f_n \rangle \searrow \lambda_0 \\ \int |\nabla f_n|^2 + V f_n^2 = I(f_n) \end{aligned} \right\} \& \left. \begin{aligned} \|f_n\|_2 = 1 \\ (V \in L^\infty) \end{aligned} \right\} \sup_n \|f_n\|_{H^1} < \infty$$

So

$$\left. \begin{aligned} f_n \xrightarrow{H^1} f_\infty \quad \& \quad \left. \begin{aligned} f_n \xrightarrow{L^2_{\text{loc}}} f_\infty \\ f_n \xrightarrow{\text{a.e.}} f_\infty \end{aligned} \right\} \end{aligned} \right\} \xrightarrow{\text{(Fatou)}} \int f_\infty^2 \leq 1 \quad \& \quad \int_{|x|>R} V f_n^2 \rightarrow \int_{|x|>R} V f_\infty^2$$

Unit in  $n$ . □

**Claim:**

$$\lim_{R \rightarrow \infty} \sup_n \int_{|x|>R} f_n^2 \rightarrow 0$$

*Proof.* Let

$$\tilde{f}_n(x) = \frac{f_n(x)\chi(x/R_n)}{\| \cdot \|_2} \quad \text{if} \quad \int_{|x|>R_n} f_n^2 > \delta > 0 \quad \text{for some } n \rightarrow \infty$$

$$I(\tilde{f}_n) \geq \| \cdot \|_2^{-2} \int \left\{ |\nabla f_n|^2 \chi^2 \left( \frac{x}{R_n} \right) - C R_n^{-2} |f_n|^2 - \|V\|_{L^\infty(|x|>R_n)} |f_n|^2 \right\} \text{ etc.}$$

Actually

$$\left. \begin{aligned} \|f_\infty\|_{H^1} \leq \underline{\lim} \|f_n\|_{H^1} \\ \int V f_n^2 \rightarrow \int V f_\infty^2 \quad (\text{since } V \rightarrow 0) \end{aligned} \right\} \Rightarrow I(f_\infty) \leq \underline{\lim} I(f_n) = \lambda_0$$

Since  $\lambda_0 < 0 \Rightarrow \|f_\infty\|_2 = 1$  and  $f_\infty$  is the minimizer. So

$$L_0 \geq 0 \text{ \& } L_1 \partial_\alpha u = -\alpha u \text{ (from } \partial_\alpha(L_0 u(\cdot; \alpha)) = 0)$$

$$\begin{aligned} L_0^{\frac{1}{2}} \ddot{v} &= -L_0^{\frac{1}{2}} L_1 L_0^{\frac{1}{2}} L_0^{\frac{1}{2}} v \\ L_0^{-\frac{1}{2}} \ddot{w} &= -\underbrace{L_0^{\frac{1}{2}} L_1 L_0^{\frac{1}{2}}}_A L_0^{-\frac{1}{2}} w \end{aligned}$$

□

**Exercise:** Show that  $A$  is self-adjoint with  $\text{Dom}(A) = H^4$

$$L_0^{\frac{1}{2}} v = C_0 \cos(\sqrt{A}t)\phi_0 + C_1 \sin(\sqrt{A}t)\phi_1$$

If  $\sigma(A) \subset [0, \infty)$ , then “stability.” If not, then “instability.”

$$\inf_{g \in H^4} \frac{\langle g|A|g \rangle}{\langle g|g \rangle} = \inf_{g \in H^4} \frac{\overbrace{\langle L_0^{\frac{1}{2}} g | L_1 | L_0^{\frac{1}{2}} g \rangle}^h}{\langle g|g \rangle} = \inf_{\substack{h \in H^2 \\ h \perp u}} \frac{\langle h|L_1|h \rangle}{\langle h|L_0^{-1}|h \rangle}$$

Since  $L_0 \geq 0$ , the question is  $\langle h|L_1|h \rangle \stackrel{?}{\geq} 0$  for all  $h \in H^2$ ,  $h \perp u$ .

$L_1(\partial_j u) = \partial_j(L_0 u) = 0 \Rightarrow \partial_j u$  is e-func of  $L_1$  with e-val 0.

In  $d = 1$ , we have  $\phi'$  having 1 node  $\Rightarrow$  there exists unique negative e-val.

In all  $d$  expect  $\text{Ker} L_1 = [\partial, u, \dots, \partial_d u]$  and there exists unique and negative e-val.

Solve variational problem  $\inf_{\substack{h \perp u \\ \|h\|_2=1}} \langle h|L_1|h \rangle$  (minimizer  $h_0$  and  $\langle h_0|L_1|h_0 \rangle = \beta$ ).

So  $L_1 h_0 = \alpha u + \beta h_0$  or  $(L_1 - \beta)h_0 \equiv \alpha u$

We know  $\beta \geq \mu_0$  (= negative e-val of  $L_1$ ). Assume  $\beta < 0$ .

First:  $\beta \neq \mu_0$  (otherwise  $h$  is unique “positive” ground state of  $L_1 \Rightarrow u \not\perp h_0 \Rightarrow \Rightarrow \Leftarrow$ )

So:  $0 > \beta > \mu_0 \Rightarrow \alpha \neq 0$  (because we claim  $\sigma(L_1) \cap (\mu_0, 0) = \emptyset$ ).

$$\begin{aligned} h_0 &= \mathcal{L}(L_1 - \beta)^{-1} u \\ 0 &= \langle u|(L_1 - \beta)^{-1}|u \rangle \end{aligned}$$

Let  $w(\lambda) = \langle u|(L_1 - \lambda)^{-1}|u \rangle \Rightarrow$

$w(\lambda) \rightarrow -\infty$  as  $\lambda \downarrow \mu_0$  (because  $u > 0$  and ground state of  $L_1$  is  $> 0$ ).

$$w'(\lambda) = \langle u|(L_1 - \lambda)^{-2}|u \rangle > 0$$

$$\downarrow L_1 \partial_\alpha u = -\alpha u$$

$$w(0) = \langle u | L_1^{-1} | u \rangle = -\frac{1}{\alpha} (u, \partial_\alpha u) = -\frac{1}{2\alpha} \frac{\partial}{\partial \alpha} \| u \|_2^2$$

$$w(0) > 0 \Rightarrow \text{instability}$$

$$w(0) < 0 \Rightarrow \text{stability}$$

$$u(x; 1) = g \text{ solves } -\frac{g}{2} + \frac{1}{2} \Delta g + g^{2\sigma+1} = 0$$

$$\frac{1}{2} \Delta (\alpha^m g(\alpha x)) = \alpha^{2+m} \left( \frac{g}{2} - g^{2\sigma+1} \right) = \alpha^2 \frac{(\alpha^m g)}{2} - (\alpha^m g)^{2\sigma+1} \text{ if } m = \frac{1}{\sigma}$$

So

$$u(x, \alpha) = \alpha^{\frac{1}{\sigma}} g(\alpha x)$$

$$\frac{\partial}{\partial \alpha} \| u(\cdot, \alpha) \|_2^2 = \frac{\partial}{\partial \alpha} \alpha^{\frac{2}{\sigma}-d} \| g \|_2^2 \begin{cases} > 0 & \text{if } 2 > d\sigma \quad (\Rightarrow \beta < 0) \\ < 0 & \text{if } 2 < d\sigma \quad (\Rightarrow \beta > 0) \end{cases}$$

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Last time  $H = -\Delta + V$   $\langle Hf, f \rangle < 0$

$\{f_n\}$   $\| f_n \|_2 = 1$   $\langle Hf_n, f_n \rangle \searrow \mathcal{A}_0 < 0$

$f_n \rightarrow f_\infty$  in  $H^1$   $f_n \rightarrow f_\infty$   $L_{loc}^2$   $\| f_\infty \|_2 \leq 1$

**Claim**  $\limsup_{R \rightarrow \infty} \int_{|x| > R} |f_n|^2 = 0$  otherwise then

$$\exists n \nearrow R_n \nearrow \infty \int_{|x| > R_n} |f_n|^2 > \delta > 0$$

$$\tilde{f}_n(x) = \frac{x \left( \frac{x}{R_n} \right) f_n}{\| \chi_{(\tilde{R}_1)} f_n \sqrt{1 + \delta} \|_{L^2}} \text{supp } \chi \subset B_{(0,2)}$$

$$\begin{aligned} I[\tilde{f}_n] &= \int \frac{1}{2} |\nabla \tilde{f}_n|^2 + \frac{1}{2} V \tilde{f}_n^2 \\ &\leq \frac{1}{1-\delta} \int \frac{1}{2} |\nabla f_n|^2 + \frac{1}{2} V f_n^2 + O(1) = \frac{1}{1-\delta} I(f_n) \\ &\Rightarrow \frac{\lambda_0}{1-\delta} < \lambda_0 < 0 \end{aligned}$$

**Theorem.** (Weinstein '86)  $d\sigma < 2$

$g(\cdot, \alpha)$  the ground state, i.e.,  $-\frac{1}{2} \Delta g + \frac{\alpha^2}{2} g - g^{2\sigma+1} = 0$

$g > 0$  radial

$\varepsilon > 0 \exists \delta > 0$  s.t.

$\forall \psi \in H^1$  s.t.  $\| \phi - e^{i\gamma g(\cdot + y)} \|_{H^1} < \delta$

then  $\sup_t \inf_{\gamma, y} \| \psi(t, \cdot) - e^{i\gamma g(\cdot + y)} \|_{H^1} < \varepsilon$

where

$$\begin{cases} i\partial_t\psi + \frac{1}{2}\Delta\psi + |\psi|^{2\sigma}\psi = 0 \\ \psi|_{t=0} = \phi \end{cases}$$

*Proof.*

$$\begin{aligned} \theta_f &= \{e^{i\gamma}f(\cdot + y) : \gamma \in \mathbb{R} \quad y \in \mathbb{R}^d\} \\ \rho(\psi(t), \theta_g) &= \inf_{\gamma, y} \{ \|\nabla\psi - e^{i\gamma}\nabla g(\cdot + y)\|_2^2 + \alpha^2 \|\psi - e^{i\gamma}g(\cdot + y)\|_2^2 \} \end{aligned}$$

□

**Exercise:** Check that  $\exists \gamma = \gamma(t) \quad y = y(t)$  continuous such that at those values. inf is attained.

$$\begin{aligned} e^{-i\gamma(t)}\psi(t; -y(t)) &= g + \overbrace{u(t, \cdot) + iv(t, \cdot)}^{w(t, \cdot)} \\ \min(\alpha^2, 1) \|w(t)\|_{H^1}^2 &\leq \rho(\psi(t), \theta_g) \leq \max(\alpha^2, 1) \|w(t)\|_{H^1}^2 \end{aligned}$$

action 
$$\zeta[f] = \int \frac{1}{2}|\nabla f|^2 + \frac{\alpha^2}{2}f^2 - \frac{1}{\sigma+1}|f|^{2\sigma+2} dx$$

Fact 1) 
$$\zeta'[g] = 0$$

2) 
$$\zeta[\psi(t)] = \zeta[\phi] = \mathcal{H}[\psi(t)] + \frac{\alpha^2}{2} \int |\psi(t)|^2$$

$$\langle \zeta'[g], h \rangle = \int \frac{1}{2}2\nabla g \cdot \nabla h + \frac{\alpha^2}{2}2gh - 2g^{2\sigma+1}h = 0$$

3) 
$$\delta\zeta[\phi] = \zeta[\phi] - \zeta[g] = \zeta[e^{-i\gamma(0)}\phi(\cdot - y(0))] - \zeta[g] < \varepsilon$$

$$\delta\zeta[\psi] = \zeta[g + w(t)] - \zeta[g] \underset{goal}{>} C_0 \underbrace{\|W(t)\|_{H^1}^2 - C(\|W(t)\|_{H^1}^{2+\mu})}_{h(\|W(t)\|_{H^1})} \|W\|_{H^1}^\sigma$$

$$\Rightarrow \|W(t)\|_{H^1} < \varepsilon$$

$$\begin{aligned} &\zeta[g+w] - \zeta[g] \\ &= \int \frac{1}{2}|\nabla(g+w)|^2 + \frac{\alpha^2}{2}|g+w|^2 - \frac{1}{\sigma+1}|g+w|^{2\sigma+2} - \int \frac{1}{2}|\nabla g|^2 + \frac{\alpha^2}{2}|g|^2 - \frac{|g|^{2\sigma+2}}{\sigma+1} \\ &\quad g \text{ real } w = u + iv \\ &= \int \frac{1}{2}2\nabla g \nabla u + \frac{\alpha^2}{2}2gu - 2g^{2\sigma+1}u dx + \\ &\quad \int \frac{1}{2}|\nabla u|^2 + \frac{\alpha^2}{2}|u|^2 + \int \frac{1}{2}|\nabla v|^2 + \frac{\alpha^2}{2}v^2 \\ &\quad - \int \frac{1}{\sigma+1}(|g+w|^{2\sigma+2} - |g|^{2\sigma+2} - (2\sigma+2)g^{2\sigma+1}u) \end{aligned}$$

$\sigma = 1$  non-linear term

$$\begin{aligned}
& |g+w|^4 - |g|^4 - 4g^3u \\
&= |g+u|^4 + v^4 + 2(g+u)^2v^2 - g^4 - 4g^3u \\
&= 6g^2u^2 + 4gu^3 + u^4 + 2g^2v^2 + 4guv^2 + 2u^2v^2 + v^4 \\
&= \int \underbrace{\left(\frac{1}{2}|\nabla u|^2 + \frac{\alpha^2}{2}u^2 - 3g^2u^2\right)}_{(L_1u,u)} + \int \underbrace{\left(\frac{1}{2}|\nabla v|^2 + \frac{\alpha^2}{2}v^2 - g^2v^2\right)}_{(L_0v,v)} + O(\|w\|_3^3 + \|w\|_4^4) \\
&\quad \langle L_1u, u \rangle + \langle L_0v, v \rangle - C(\|w\|_{H^1}^3 + \|w\|_{H^1}^4) \\
L_0 &= -\frac{\Delta}{2} + \frac{\alpha^2}{2} - g^{2\sigma} \quad L_1 = -\frac{\Delta}{2} + \frac{\alpha^2}{2} - (1+2\sigma)g^{2\sigma}
\end{aligned}$$

Find the constraint on  $\gamma(t)y(t)$

$$\begin{aligned}
& \|\nabla\psi - e^{i\gamma}\nabla g(\cdot+y)\|_2^2 + \alpha^2 \|\psi - e^{-i\gamma}g(\cdot+y)\|_2^2 \\
&= \|\nabla\psi\|_2^2 + \|\psi\|_2^2 + \|\nabla g\|_2^2 + \alpha^2 \|g\|_2^2 \\
&\quad - 2\operatorname{Re}[\langle \nabla\psi, e^{i\gamma}\nabla g(\cdot+y) \rangle + \alpha^2 \langle \psi, e^{i\gamma}g(\cdot+y) \rangle] \\
&\quad \frac{\partial}{\partial\gamma} \operatorname{Re}[\dots] = 0 = \operatorname{Im}[\langle \nabla\psi, e^{i\gamma}\nabla g(\cdot+y) \rangle + \alpha^2 \langle \psi, e^{i\gamma}g(\cdot+y) \rangle] \\
&= \operatorname{Im}[\langle \nabla g + \nabla w, \nabla g \rangle + \alpha^2 \langle g+w, g \rangle] \\
&= \langle \nabla v, \nabla g \rangle + \alpha^2 \langle v, g \rangle = \langle v, -vg + \alpha^2 g \rangle \\
&\quad = 2\langle v, g^{2\sigma+1} \rangle
\end{aligned}$$

$$\frac{\partial}{\partial y_i} \operatorname{Re}[\dots] = 0 \Rightarrow \int g^{2\sigma} \frac{\partial g}{\partial \chi_i} \cdot u = 0$$

**Lemma.**  $\forall v \perp g^{2\sigma+1} \Rightarrow (L_0v, v) \geq C_0 \|v\|_{H^1}^2 \quad C_0 > 0$

*Proof.*

$$(L_0v, v) \geq 0 \quad \forall v \in H^2$$

$$\inf_{f \in H^1} (L_0f, f) = \beta_0 \geq 0$$

$$\|f\|_2 = 1 \quad f \perp g^{2\sigma+1} \quad f_n \text{ minimizer sequence } (L_0f_n, f_n) \searrow \beta_0$$

$$\text{if } \beta_0 = 0 \quad \forall \eta > 0 \text{ small}$$

$$\begin{aligned}
& \int \frac{1}{2} |\nabla f_n|^2 + \frac{\alpha^2}{2} |f_n|^2 < \int g^{2\sigma} |f_n|^2 + \eta \\
f_n &\xrightarrow{H^1} f_\infty \quad f_n \rightarrow f_\infty \quad L_{loc}^2 \text{ a.c. } \int |f_\infty|^2 \leq \infty \\
&\Rightarrow \frac{\alpha^2}{2} < \int g^{2\sigma} |f_\infty|^2 + \eta \Rightarrow f_\infty \neq 0
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_a &= \frac{f_\infty}{\|f_\infty\|_2} \perp g^{2\sigma+1} \\
\langle L_0 \tilde{f}_\infty, \tilde{f}_\infty \rangle &\leq 0 \quad \Rightarrow \langle L_0 \tilde{f}_\infty, \tilde{f}_\infty \rangle = 0 \\
\tilde{f}_\infty \in \ker L_0 = \text{span}\{g\} &\Rightarrow \Leftarrow \Rightarrow \tilde{f}_\infty \perp g^{2\sigma+1} \text{ positive function} \\
&\Rightarrow \langle L_0 v, v \rangle \geq C_0 \|v\|_2^2 \\
(L_0 v, v) &= (1 - \delta)(L_0 v, v) + \delta \int |\nabla v|^2 + \frac{\delta \alpha^2}{2} \int |v|^2 - \delta \int g^{2\sigma} |v|^2 \\
\delta \text{ small} \quad |g^{2\sigma}|_\infty \cdot \delta &< \frac{\alpha_0}{2}(1 - \delta) \\
&\geq C\delta \int |\nabla v|^2 + \delta \frac{\alpha^2}{2} \int |v|^2
\end{aligned}$$

□

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**Weinstein's orbital stability theorem** ( $\sim 86$ )Let  $g = g(x; \alpha)$  be the ground state of

$$-\frac{1}{2}\Delta g + \frac{\alpha^2}{2}g - g^{2\sigma+1} = 0$$

Suppose  $d\sigma < 2$ . Then the soliton  $\Psi_0(t, x) = e^{i\frac{\alpha^2}{2}t}g(x; \alpha)$  is *orbitally stable*: i.e.,  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$  s.t.

if  $\otimes : \inf_{\substack{\theta \in \mathbb{R}, \\ y \in \mathbb{R}^d}} \|e^{i\theta}\phi(\cdot + y) - g\|_{H^1} < \delta$ , then

$$\sup_t \int_{\substack{\theta \in \mathbb{R} \\ y \in \mathbb{R}^d}} \|e^{i\theta}\Psi(t, \cdot + y) - g\|_{H^1} < \varepsilon$$

where

$$\begin{cases} i\partial_t \Psi + \frac{1}{2}\Delta \Psi + |\Psi|^{2\sigma}\Psi = 0 \\ \Psi|_{t=0} = \phi \end{cases}$$

*Proof.*

$$\theta_F = \{e^{i\gamma}f(\cdot + y) \mid \gamma \in \mathbb{R}, y \in \mathbb{R}^d\}$$

$$\rho(\Psi(t), g) := \inf_{\theta, y} \{ \|e^{i\theta}\nabla\Psi(t, \cdot + y) - \nabla g\|_2^2 + \alpha^2 \|e^{i\theta}\Psi(t, \cdot + y) - g\|_2^2 \}.$$

□

**Check:** Since  $t \mapsto \Psi(t)$  is  $H^1$ -continuous,  $\exists y(t), \theta(t)$  cont. so that  $\rho(\Psi(t), g)$  is attained at  $y(t)$  and  $\theta(t)$ .

Let

$$\begin{aligned} e^{i\theta(t)}\Psi(t, \cdot + y(t)) &= g + w(t) = g + u(t) + iv(t) \\ \Rightarrow \int g^{2\sigma+1}v \, dx &= 0, \int g^{2\sigma} \frac{\partial g}{\partial x_e} u \, dx = 0, \forall 1 \leq l \leq d. \end{aligned}$$

Action

$$\begin{aligned} \zeta[f] &= \int \left\{ \frac{1}{2} |\nabla f|^2 + \frac{\alpha^2}{2} |f|^2 - \frac{1}{\sigma+1} |f|^{2\sigma+2} \right\} dx \\ &= \mathcal{H}[f] + \frac{\alpha^2}{2} \int |f|^2 \end{aligned}$$

$$\Rightarrow \zeta[\Psi(t)] = \zeta[\Psi(0)] = \zeta[\phi]$$

$$\begin{aligned} \zeta[\Psi(t)] &= \zeta[e^{i\theta(t)}\Psi(t, \cdot + y(t))] \\ &= \zeta[g + w(t)] \end{aligned}$$

$$\delta\zeta[\Psi(t)] = \zeta[\Psi(t)] - \zeta[g] = \zeta[g + w(t)] - \zeta[g]$$

$$\begin{aligned} &\parallel \\ \delta\zeta[\Psi(0)] & \\ \text{small} & \end{aligned}$$

$$\stackrel{\text{Claim}}{>} C_1 \|w(t)\|_{H^1}^2 - C(\|w(t)\|_{H^1}^{2+\mu} + \|w(t)\|_{H^1}^6) \quad \mu > 0$$

$$\begin{aligned} \sigma = 1, d = 1 \quad \zeta[f] &= \int \frac{1}{2} |\nabla f|^2 + \frac{\alpha^2}{2} |f|^2 - \frac{1}{2} |f|^4 \\ \zeta[g + w] - \zeta[g] &\geq (L_0 v, v) + (L_1 u, u) - C(\|w\|_3^3 + \|w\|_4^4) \\ &\quad \uparrow \quad \uparrow \\ &\quad H^1 \quad H^1 \\ &\quad \text{Sob. emb.} \end{aligned}$$

$$L_0 = -\frac{1}{2}\Delta + \frac{\alpha^2}{2} - g^{2\sigma}$$

$$L_0 g = 0$$

$$L_1 = -\frac{1}{2}\Delta + \frac{\alpha^2}{2} - (2\sigma + 1)g^{2\sigma}$$

$$L_1(\vec{\nabla}g) = 0, \quad L_1(\partial_\alpha g) = -\alpha g$$

$\Rightarrow \cdot g$  is the unique positive ground state of  $L_0$

• **Fact:**  $\text{Ker} L_1 = \text{span}\{\partial_j g | 1 \leq j \leq d\}$  and  $L_1$  has unique negative eigenvalue  $\mu_0 < 0$ .

**Lemma.**  $(L_0 v, v) \geq C_0 \|v\|_{H^1}^2, \forall v \perp g^{2\sigma+1}$  (done last time)

**Lemma.**  $(L_1 f, f) \geq 0, \forall f \perp g$

*Proof.* Suppose

$$\inf_{\substack{\|f\|_2^2=1 \\ f \perp g}} (L_1 f, f) = \beta_1 < 0$$

Let  $f_\infty$  be the minimizer:

$$\begin{aligned} \Rightarrow L_1 f_\infty &= \beta_1 f_\infty + \beta_2 g, \quad \mu_0 < \beta_1 < 0 \Rightarrow \beta_2 \neq 0. \\ (L_1 - \beta_1) f_\infty &= \beta_2 g \end{aligned}$$

**Remark:** If  $\beta_1 = \mu_0$ , then  $f_\infty$  is the ground state. If  $L_1 \Rightarrow f_\infty > 0$ : contradicts  $\int f_\infty g = 0$  as  $g > 0$ .

So

$$\begin{aligned} f_\infty &= \beta_2 (L_1 - \beta_1)^{-1} g \\ \Rightarrow 0 &= \omega(\beta_1) = \langle g | (L_1 - \beta_1)^{-1} | g \rangle \\ \text{where } \omega(\lambda) &= \langle g | (L_1 - \lambda)^{-1} | g \rangle \text{ for } \mu_0 < \lambda < 0. \end{aligned}$$

So

$$\begin{aligned} \omega'(\lambda) &> 0 \\ \Rightarrow \omega(0) &> 0 \end{aligned}$$

$$\begin{aligned} \omega(0) &\underset{\uparrow}{=} \langle g | L_1^{-1} | g \rangle \\ \text{as } g &\perp \text{Ker} L_1 = \text{span}\{\vec{\nabla} g\} \\ &= -\frac{1}{\alpha} \langle g | \partial_\alpha g \rangle \\ &= -\frac{1}{2\alpha} \underbrace{\partial_\alpha \|g\|_2^2}_{>0 \text{ as } d\sigma < 2} < 0 \\ &\text{contradiction} \end{aligned}$$

□

**Lemma.**

$$\begin{aligned} \exists C_1 > 0, (L_1 f, f) &\geq C_1 \|f\|_2^2, \forall f \perp g \text{ and} \\ f &\perp g^{2\sigma} \frac{\partial g}{\partial x^l}, 1 \leq l \leq d. \end{aligned}$$

*Proof.*

$$\inf_{\substack{f \in H^1 \\ f \perp g, f \perp \vec{\nabla} g \cdot g^{2\sigma} \\ \|f\|_2^2 = 1}} (L_1 f, f) = \beta_1 \geq 0 \quad \text{By previous lemma.}$$

Assume  $\beta_1 = 0$ ,  $\Rightarrow \exists f_\infty$  minimizer satisfying

$$\begin{aligned} L_1 f_\infty &= \overbrace{\beta_1 f_\infty}^{=0} + \beta_2 g + \vec{\beta}_3 \cdot \vec{\nabla} g g^{2\sigma} \\ (L_1 f_\infty, g) &= \beta_2 (g, g) \\ (f_\infty, L_1 \partial_j g) (L_1 f_\infty, \partial_j g) &= \beta_{3,l} \underbrace{\int \partial_l g g^{2\sigma} \partial_j g}_{A_{jl}} \\ \parallel \\ 0 \end{aligned}$$

$$(A\xi, \xi) = \int g^{2\sigma} |\vec{\nabla} g \cdot \xi|^2 dx > 0 \text{ if } \xi \neq 0$$

So  $\vec{\beta}_3 = 0$

$$\begin{aligned} L_1 f_\infty &= \beta_2 g \\ f_\infty &= -\frac{\beta_2}{\alpha} \partial_\alpha g + \vec{c} \cdot \vec{\nabla} g \\ (f_\infty, g) = 0 &= -\frac{\beta_2}{\alpha} \underbrace{(g, \partial_\alpha g)}_{<0} \Rightarrow \beta_2 = 0 \end{aligned}$$

So  $f_\infty = \vec{c} \cdot \vec{\nabla} g$

$$\begin{aligned} 0 &= \underbrace{(f_\infty, g^{2\sigma} \frac{\partial g}{\partial x^l})}_{\substack{\uparrow \\ f_\infty \perp g^{2\sigma} \vec{\nabla} g}} = c_j \underbrace{\int \frac{\partial g}{\partial x^j} g^{2\sigma} \frac{\partial g}{\partial x^l}}_{\forall j,l} \Rightarrow \vec{c} = 0 \\ &\Rightarrow f_\infty = 0 : \text{contradicts } \|f_\infty\|_2 = 1 \end{aligned}$$

□

So we get

$$\begin{aligned} (L_1 f, f) &\leq C_1 \|f\|_{H^1}^2 \\ \forall f \perp g, f \perp \vec{\nabla} g \cdot g^{2\sigma} \\ (L_1 f, f) &= \int \frac{1}{2} |\nabla f|^2 + \frac{\alpha^2}{2} |f|^2 - (2\sigma + 1) g^{2\sigma} |f|^2 \end{aligned}$$

If  $\| \nabla f \|_2 < \gamma^{-1} \| f \|_2$ , you're okay.

$$(L_1 f, f) \geq \frac{C_1}{2} \| f \|_2^2 + \frac{C_1}{2} \gamma^2 \| \nabla f \|_2^2$$

If  $\| \nabla f \|_2 \geq \gamma^{-1} \| f \|_2$

Then

$$\begin{aligned} (L_1 f, f) &\geq \frac{1}{2} (L_1 f, f) + \frac{C_1}{2} \| f \|_2^2 \\ &\geq \frac{1}{4} \int |\nabla f|^2 dx + \frac{1}{4} \gamma^{-2} \int |f|^2 - \left( \frac{\alpha^2}{2} + (2\sigma + 1) \| g \|_\infty^{2\sigma} \right) \int |f|^2 + \frac{C_1}{2} \| f \|_2^2 \end{aligned}$$

with  $\gamma$  small enough so that

$$\frac{1}{4} \gamma^{-2} > \frac{\alpha^2}{2} + (2\sigma + 1) \| g \|_\infty^{2\sigma}$$

**Observation:** It's enough to consider  $\phi$  so that

$$\| \phi \|_2 = \| g \|_2 .$$

$$\circledast \Rightarrow | \| \phi \|_2 - \| g \|_2 | < \delta$$

Take  $g_\lambda(x, \alpha) := \lambda^{\frac{1}{\sigma}} g(\lambda x, \alpha)$

$$-\frac{1}{2} \Delta g_\lambda - g_\lambda^{2\sigma+1} = \lambda^{\frac{1}{\sigma}} \lambda^2 \left( -\frac{1}{2} (\Delta g)(\lambda x) - g^{2\sigma+1}(\lambda x) \right) =$$

Nonlinearity:

$$\begin{aligned} \frac{2\sigma + 1}{\sigma} &= 2 + \frac{1}{\sigma} \\ &= -\lambda^{\frac{1}{\sigma}} \lambda^2 \frac{\alpha^2}{2} g(\lambda x, \alpha) = -\frac{(\lambda \alpha)^2}{2} g_\lambda(x, \alpha) \end{aligned}$$

So  $g_\lambda(x, \alpha) = g(x, \lambda \alpha)$

$$\| g_\lambda \|_2 = \lambda^{\frac{1}{\sigma}} \lambda^{-\frac{2}{\sigma}} \| g \|_2$$

So  $\exists \lambda$  close to 1 s.t.  $\| \phi \|_2 = \| \lambda^{\frac{1}{\sigma}} g(\lambda \cdot) \|_2$

Assume we've proved theorem for  $\| \phi \|_2 = \| g \|_2$ . That is we've changed  $g$  to  $g_\lambda$  for  $\lambda$  small. But  $\| g - g_\lambda \|_{H^1}$  small  $\Rightarrow$  OK.

**Lemma.**

$$(L_1 u, u) \geq C_1 \| u \|_{H^1}^2 - C(\| w \|_{H^1}^4 + \| w(t) \|_{H^1}^3)$$

with  $u(t)$  as above (plus  $L^2$ -preservation)

*Proof.*

$$\begin{aligned} \| g \|_2^2 &= \| \Psi(t) \|_2^2 = \| g + w(t) \|_2^2 = \| g + u(t) \|_2^2 + \| v(t) \|_2^2 \\ &= \| g \|_2^2 + 2(g, u) + \| w \|_2^2 \\ \Rightarrow (g, u(t)) &= -\frac{1}{2} \| w(t) \|_2^2 \end{aligned}$$

$$u(t) = u_{\parallel}(t) + u_{\perp}(t)$$

$$(\text{wlog } \|g\|_2 = 1) \quad u_{\parallel} = (u, g)g.$$

$$(L_1 u, u) = \underbrace{(L_1 u_{\perp}, u_{\perp})}_{\substack{\geq C_1 \|u_{\perp}\|_{H^1}^2 \\ \uparrow \\ (\text{as } g \perp g^{2\sigma} \vec{\nabla} g \Rightarrow u_{\perp} \perp g^{2\sigma} \vec{\nabla} g)}} + 2(L_1 u_{\perp}, u_{\parallel}) + (L_1 u_{\parallel}, u_{\parallel}).$$

$$(L_1 u_{\parallel}, u_{\parallel}) = (u, g)^2 (L_1 g, g) = \underset{\uparrow}{c} \|w(t)\|_2^4$$

possibly negative

$$(L_1 u_{\parallel}, u_{\perp}) = (u, g)(L_1 g, u) - (u, g)^2 (L_1 g, g)$$

$$\geq -\frac{1}{2} \|w(t)\|_2^2 \|w(t)\|_2 \|L_1 g\|_2 - c \|w(t)\|_2^4$$

$$\|u_{\perp}\|_{H^1}^2 = \|u - (u, g)g\|_{H^1}^2 = \|u\|_{H^1}^2 - \underbrace{2(u, g)_{H^1} (u, g)_{L^2}}_{\substack{\sim \|w\|^3 \\ \uparrow}} + (u, g)^2 \|g\|_{H^1}^2$$

□