Non-generic blow up solutions for the critical focusing NLS in 1-d.

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$L^2$-critical focusing Schroedinger equation in 1-d:

\[ i \partial_t \psi + \partial_x^2 \psi = -|\psi|^4 \psi, \quad i = \sqrt{-1}, \quad \psi = \psi(t, x) \]

More generally, we have the sub-critical focusing NLS:

\[ i \partial_t \psi + \partial_x^2 \psi = -|\psi|^{2\sigma} \psi, \quad \sigma < 2 \]

as well as super-critical focusing NLS

\[ i \partial_t \psi + \partial_x^2 \psi = -|\psi|^{2\sigma} \psi, \quad \sigma > 2 \]
Focusing case implies ambiguity of sign for the Hamiltonian, preserved on interval of existence:

\[
H(\psi) = \frac{1}{2} \int_{-\infty}^{\infty} |\nabla \psi|^2(t, x)dx - \frac{1}{2\sigma + 2} \int_{-\infty}^{\infty} |\psi|^{2\sigma + 2}(t, x)dx
\]

\(L^2\)-criticality index indicates global well-posedness behavior:

- \(L^2\)-sub-critical: globally well-posed in \(H^1\). Reason: Gagliardo-Nirenberg inequality, which implies

\[||\psi||_{\dot{H}^1} \lesssim H(\psi)\]

- \(L^2\)-critical and super-critical: \(H^1\)-illposedness in the sense that certain \(H^1\) initial data (even smooth initial data) lead to finite time blow up.
Glassey argument for blow up in critical/super-critical case: **virial identity**

\[
\frac{d^2}{dt^2} \left[ \int_{-\infty}^{\infty} |x|^2 |\psi|^2 (t, x) \, dx \right] \lesssim H(\psi)
\]

Hence if \(H(\psi) < 0\) obtain concavity of virial functional and the solution has to cease to exist in the smooth sense.

Problem with this argument: no info on exact location of blow up nor blow up profile.

At this level of generality hard to make more precise statements. Thus we exploit a special feature of the focusing case (in sub-critical, critical and super-critical case), existence of standing wave solutions: \(\phi_0\) called **ground state**

\[
\psi(t, x) = e^{i\alpha t} \phi_0(x, \alpha), \quad \phi_0(x, \alpha) = \frac{\alpha^2 (\frac{3}{2})^{\frac{1}{4}}}{\sqrt{\cosh(\frac{\alpha}{2}x)}}
\]
Now in the critical case, we have a large group of symmetries acting on solutions:

**Galilei Transforms:**

\[ \psi(t, x) \to e^{i(\gamma+vx-v^2t)}e^{-i(2tv+\mu)p} \psi(t, x), \]
\[ p = -i \frac{d}{dx} \]

**pseudo-conformal transformations:**

\[ \psi(t, x) \to (a + bt)^{-\frac{1}{2}} e^{\frac{ibx^2}{4(a+bt)}} \psi\left(\frac{c + dt}{a + bt}, \frac{x}{a + bt}\right) \]

Combining standing waves with pseudo-conformal transformations:

**explicit blow up solutions** in \( L^2 \)-critical case, e. g.

\[ \psi(t, x) = (a + bt)^{-\frac{1}{2}} e^{i\frac{c+dt}{a+bt}} e^{\frac{ibx^2}{4(a+bt)}} \phi_0\left(\frac{x}{a + bt}, 1\right) \]
Natural questions arise: is this the only blow-up type? Is this blow-up stable?

First questions still open at this level of generality, while 2nd is false as stated with only weak type stability expected. First question partially answered for data close to ground state (Perelman, Merle-Raphael). Progress on 2nd question by Bourgain-Wang. Quick summary of some fundamental results pertaining to first issue:

-(Merle Raphael): For initial data in small neighborhood of ground state, two types of blow-up possible: either at least as fast as explicit rate (i.e. $H^1$-norm blows up at least like $\frac{1}{T-t}$), or else like the minimal possible speed, with log log correction:

$$||\psi(t, x)||_{H^1} \sim \sqrt{\frac{\log \log |T - t|}{T - t}}$$
-(Perelman, Merle-Raphael): For initial data in a small neighborhood of ground state with negative energy, one gets the slow blow up type.

-(Raphael): The slow blow-up is stable in a small neighborhood around the ground state.

Thus we are justified to call slow blow up type 'generic', and fast blow up 'non-generic'.

Issue remains as to stability of the non-generic blow-up in certain directions:
(Bourgain-Wang): let \( z_\phi \) solve the problem
\[
 i \partial_t \psi + \partial_x^2 \psi + |\psi|^4 \psi = 0, \psi(0,.) = \phi(.)
\]
on interval \([-\delta, \delta]\). Then provided \( \phi \) smooth and vanishes sufficiently fast at \( x = 0 \), i. e. \( |\phi(x)| \lesssim |x|^A \), \( A \) large enough, there exists smooth \( w(t,x) \) with \( w(0,x) = 0 \) and such that
\[
 \psi(t,x) = |t|^{-\frac{1}{2}} e^{\frac{x^2 - 4}{4it}} \phi_0 \left( \frac{x}{t}, 1 \right) + z_\phi(t,x) + w(t,x)
\]
solves NLS on \([-\delta, 0)\).

Quick summary of proof:

- De-singularize the equation:
\[
 C^{-1} \psi(t,x) := |t|^{-\frac{1}{2}} e^{\frac{x^2}{4it}} \phi \left( -\frac{1}{t}, -\frac{x}{t} \right)
\]
\[
 C^{-1} \psi(t,x) = e^{it} \phi_0(x) + u(t,x)
\]

- Equation for \( u(t,x) \): 
\[
 \left( i \partial_t + \mathcal{H} \right) \begin{pmatrix} u(t,x) \\ \bar{u}(t,x) \end{pmatrix} = N(U), \quad U = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}
\]
\[ \mathcal{H} = \begin{pmatrix} \partial_x^2 - 1 + 3\phi_0^4(x, 1) & 2\phi_0^4(x, 1) \\ -2\phi_0^4(x, 1) & -\partial_x^2 + 1 - 3\phi_0^4(x, 1) \end{pmatrix} \]

Spectral properties of this are well-understood after work of Weinstein, Buslaev-Perelman, Perelman. \( \mathcal{H} \) has algebraic instabilities only, due to 6-dimensional generalized root space.

\[ \|e^{it\mathcal{H}}\Phi(x)\|_{L^2} \lesssim (1 + t^3)\|\Phi(x)\|_{L^2_x} \]

Use \( e^{-it}u(t, x) = C^{-1}z\phi e^{-it} + \tilde{w}(t, x) \), and fast local decay of \( C^{-1}z\phi e^{-it} \) at infinity allows one to counterbalance loss due to algebraic instability of \( \mathcal{H} \) in equation for \( \tilde{w}(t, x) \).

**Question:** can one enlarge this set of initial data?
**Motivation:** Precise structure of the generalized root space of $\mathcal{H}$. Five modes are in 1-1 correspondence with the internal symmetries of the equation, 1 is an *exotic mode*.

\[
\begin{align*}
\eta_1,\text{proper}(z) & := \left( i\phi_0(z,1) \right. \\
& \quad \left. -i\phi_0(z,1) \right), \\
\eta_2,\text{proper}(z) & := \left( z\phi_0'(z,1) + \phi_0(z,1)/2 \right) \\
& \quad \left( z\phi_0'(z,1) + \phi_0(z,1)/2 \right), \\
\eta_3,\text{proper}(z) & := \left( \phi_0'(z,1) \right) \\
& \quad \left( \phi_0(z,1) \right), \\
\eta_4,\text{proper}(z) & := \left( iz\phi_0(z,1) \right) \\
& \quad \left( -iz\phi_0(z,1) \right), \\
\eta_5,\text{proper}(z) & := \left( iz^2\phi_0(z,1) \right) \\
& \quad \left( -iz^2\phi_0(z,1) \right), \\
\eta_6,\text{proper}(z) & := \left( \rho(z) \right) \\
& \quad \left( \rho(z) \right) \\
\end{align*}
\]

\[(-\partial_x^2 + 1 - 5\phi_0^4(x,1))\rho = x^2\phi_0(x,1)\]
This motivates the following Conjecture

**Conjecture** (e. g. Perelman) There exists a co-dimension one stable manifold of initial data resulting in the non-generic blow-up.

To understand this, need some more refined background concerning $\mathcal{H}$. Basic fact: every $L^2$ vector valued function $\left( \begin{array}{c} U \\ V \end{array} \right)$ decomposes as

$$\left( \begin{array}{c} U \\ V \end{array} \right)(x) = \left( \begin{array}{c} U \\ V \end{array} \right)_{\text{dis}}(x) + \sum_{i=1}^{6} \lambda_i \eta_i, \text{proper}(x)$$

Then we have

$$\|e^{it \mathcal{H}} \left( \begin{array}{c} U \\ V \end{array} \right)_{\text{dis}} \|_{L^2_x} \lesssim \| \left( \begin{array}{c} U \\ V \end{array} \right) \|_{L^2_x},$$

$$\|e^{it \mathcal{H}} \left( \begin{array}{c} U \\ V \end{array} \right)_{\text{dis}} \|_{L^\infty_x} \lesssim t^{-\frac{1}{2}} \| \left( \begin{array}{c} U \\ V \end{array} \right) \|_{L^1_x}.$$
\[ \| \langle x \rangle^{-1} e^{i t \mathcal{H}} \begin{pmatrix} U \\ V \end{pmatrix} \|_{L_x^\infty, dis} \lesssim t^{-\frac{3}{2}} \| \langle x \rangle \begin{pmatrix} U \\ V \end{pmatrix} \|_{L_x^1} \]

**Idea behind conjecture:**
Instead of the 'static coupling' used in Bourgain-Wang, try a 'dynamic coupling' of transformed soliton plus radiation:

\[ \psi(t, x) = W(t, x) + R(t, x) \]

\[ W(t, x) = e^{i(\gamma(t) + \omega(t)(x-\mu(t)))} e^{-i\frac{\beta}{4} t \lambda^2(t)(x-\mu(t))^2} \times \sqrt{\lambda(t)} \phi_0(\lambda(t)(x-\mu(t)), 1) \]

-de-singularize the equation by applying suitable transformation \( T_\infty \) which is composition of Galilei and pseudo-conformal transformation. This of course requires that the \( \lambda(t) \) etc. are 'well-behaved asymptotically' as we approach blow up time. This transformation replaces the finite interval \([0, \text{blow up time})\) by the interval \([0, \infty)\).
-Then write $\mathcal{T}_\infty \psi = \tilde{W}(t,x) + U(t,x)$. Now control the root part of $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(t,x)$ by choosing the $\lambda(t)$ etc. suitably. Basically, this amounts to imposing an orthogonality condition

$$\langle \begin{pmatrix} U \\ \bar{U} \end{pmatrix}, \xi_{i,\text{proper}} \rangle = 0, \text{span}\{\xi_{i,\text{proper}}\} \subset \ker \mathcal{H}^*$$

$$i = 2, \ldots, 6$$

Further motivation for conjecture:

**Co-dimension 1 stable manifolds for super-critical NLS in 1-d** (Schlag-K), where essentially this program was carried through, but with reduced symmetry group and root space, of course.

We tried direct attack on the conjecture. What we can prove:
**Theorem** Fix parameters $\lambda, \beta \sim 1$, $\omega, \gamma, \mu \lesssim 1$. Given a vector valued function $\left( \frac{U}{\bar{U}} \right) (0, x)$ with $\langle \left( \frac{U}{\bar{U}} \right) (0, x), \xi_i \text{ proper} \rangle = 0$, $i = 1, \ldots, 6$, and also $|||U(0, x)||| < \delta$ for small enough $\delta > 0$, there exist $\tilde{\lambda}_i \in \mathbb{R}$ with $|\tilde{\lambda}_i| \lesssim |||U(0, x)|||^2$, as well as $\{a_\infty, b_\infty, v_\infty, y_\infty, \gamma_\infty\}$ all within a fixed interval of length $\sim |||U(0, x)|||^2$, such that

$$\psi(0, x) = W(0, x) + T^{-1}_\infty [U(0, x) + \sum_{i=1}^6 \tilde{\lambda}_i \eta_i, \text{proper}]$$

leads to non-generic blow up. We have

$$W(0, x) = e^{i(\gamma + \omega(x-\mu))} e^{-i\frac{\beta}{4} \lambda^2 (x-\mu)^2} \sqrt{\lambda} \phi_0 (\lambda(x-\mu), 1)$$

$$T_\infty = e^{-i(v_\infty^2 s + \gamma_\infty + v_\infty y)} e^{i(2v_\infty s + y_\infty)} p \begin{pmatrix} a_\infty & b_\infty \\ 0 & a_\infty^{-1} \end{pmatrix}$$
Difficulties one encounters when carrying out the above heuristics:

- **A: Control of the modulation parameters**, i. e. $\lambda(t)$, $\beta(t)$ etc.

- **B: Control of the radiation part**

**A:** In order to mimic the Bourgain-Wang 'Change of Gauge', i. e. de-singularization of the equation, need to impose precise asymptotics on the modulation parameters $\lambda(t)$, $\beta(t)$ etc., which in turn depend on the parameters determining $T_\infty$, i. e. $\{a_\infty, b_\infty, y_\infty, v_\infty, y_\infty\}$. For example, we impose

$$|\lambda(t)(a_\infty^{-1} - b_\infty t) - 1| \lesssim (a_\infty^{-1} - b_\infty t)^{\frac{1}{2}-\delta_1}$$

These asymptotic conditions have to be compatible with the modulation equations flowing from the orthogonality condition. Careful analysis of these equations forces one to obtain quite delicate estimates on the root part of the radiation.
B: Recall $T_\infty \psi = \tilde{W}(t, x) + U(t, x)$. Write

$$\begin{pmatrix} U \\ \bar{U} \end{pmatrix} = \begin{pmatrix} U \\ \bar{U} \end{pmatrix}_{dis} + \sum_{i=1}^{6} \lambda_i \eta_i, \text{proper}$$

The coefficients $\lambda_i$, $i = 1, \ldots, 5$ corresponding to the 'good modes' are determined by the orthogonality condition

$$\left\langle \begin{pmatrix} U \\ \bar{U} \end{pmatrix}, \xi_j \right\rangle = 0, \ j = 2, \ldots, 6$$

The coefficient $\lambda_6$ in turn satisfies a certain ODE, and the requirement that $\lambda_6(s)$ vanish at $s = +\infty$ forces a suitable initial condition for $\lambda_6(0)$. **Main issue: control over** $\lambda_6$.

Note that the analogous issue for the super-critical NLS, namely controlling the exponentially unstable mode, is much simpler due to an exponential damping. In present context, analysis of the modulation equations and the ODE determining $\lambda_6$ shows the need to control quantities of the form
\[
\int_T^\infty t \int_t^\infty \langle U^2(s) - \bar{U}^2(s), \phi \rangle \, ds \, dt
\]
for suitable Schwartz function \( \phi \). This suggests that we need the optimal local decay for the radiation as dictated by linear theory, namely

\[
\| U(t, x) \phi \|_{L_\infty x} \lesssim \langle t \rangle^{-\frac{3}{2}}
\]

This is one of the main issues: How to deduce the optimal local decay from the non-linear PDE controlling \( U \).

\[
i \partial_t \left( \frac{U}{\bar{U}} \right) + \mathcal{H}U = V \left( \frac{U}{\bar{U}} \right) + \left( \frac{|U|^4 U}{- |U|^4 \bar{U}} \right)
\]

Naive argument using Duhamel:

\[
\|\langle x \rangle^{-1} \left( \frac{U}{\bar{U}} \right)_{\text{dis}}(t, x)\|_{L_\infty x} \lesssim \int_0^t (t - s)^{-\frac{3}{2}} \|\langle x \rangle \left( \frac{|U|^4 U}{- |U|^4 \bar{U}} \right)(s, x)\|_{L_1 x} \, ds
\]

On the other hand \( \| \langle x \rangle U(t, x) \|_{L_2 x} \lesssim \langle t \rangle \| U(t, x) \|_{L_2 x} \).
If one combines this with $\|U(s,.)\|_{L^\infty_x} \lesssim \langle s \rangle^{-\frac{1}{2}}$ expected from linear theory, one gets the integral

$$\int_0^t (t - s)^{-\frac{3}{2}} s^{-\frac{1}{2}} ds \sim t^{-\frac{1}{2}}$$

It appears hopeless to recover strong local decay!

The way out of the dilemma is the primary novelty of our approach:

exploit algebraic fine structure of non-linearity, i.e. a 'null-structure' of sorts.

But even the strong local decay estimate is not enough to control

$$\int_T^\infty t \int_t^\infty \langle U^2(s,.) - \tilde{U}^2(s,.),\phi \rangle dsdt$$

What saves the day is additional symplectic 'null-structure' in this term.
So why can’t we prove the Perelman conjecture? Issue is that while we can deduce the necessary a priori estimates required for an iteration, we cannot compare different iterates due to a growing phase function. Thus we cannot compare iterates. We can prove the theorem by appealing to a **topological fixed point argument:**

Schauder-Tychochonoff: A compact convex subset of a Banach space has the fixed point property.