Dispersive estimates for Schrödinger equations

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$$i\partial_t\psi + H(t)\psi = 0 \qquad (t,x)\in\mathbb{R}^{1+n}, n\geq 3,$$
$$H(t) = -\triangle + V \qquad V = V(t,x)$$

Preservation of L^2: $\|\psi(t,\cdot)\|_2 = \|\psi(0,\cdot)\|_2$ as long as V real-valued, but V can be time-dependent.

Decay estimate for $V=0$: $\|e^{-it\triangle}f\|_\infty \lesssim t^{-\frac{n}{2}}\|f\|_1$. This follows from

$$e^{-it\triangle}f(x) = C\,t^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{\frac{|x-y|^2}{4ti}}\,f(y)\,dy.$$

$L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ obtained by interpolation with L^2, $q=p'$, $2\geq p\geq 1$.

Basic problem: Prove these dispersive estimates for $V\neq 0$.

Another way of viewing dispersive estimates (classical scattering): Let $\widehat{f} \in C_0^\infty(\mathbb{R}^n)$. Then

$$e^{it\triangle} f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)}\, \widehat{f}(\xi)\, d\xi.$$

Critical point of the phase is $x - 2t\xi = 0$, stationary phase gives that an observer sitting at x at time t basically sees $\widehat{f}(\frac{x}{2t})\, t^{-\frac{n}{2}}$. This fits together with L^2 preservation, since the solution decays rapidly outside a ball of radius Ct, and the L^∞ norm is $t^{-\frac{n}{2}}$.

Consider first $V = V(x)$. Obstacle for dispersive estimate: Eigenfunctions $H\psi = E\psi$. Then $e^{itH}\psi = e^{itE}\psi$, so no decay.

Refined question: Is $\|e^{itH}P_c\psi\|_\infty \lesssim t^{-\frac{n}{2}}\|\psi\|_1$, where P_c is the projection onto the continuous spectrum?

Some evidence for this: Let $B \subset \mathbb{R}^n$ be compact, $f \in L^2(\mathbb{R}^n)$. Then

$$\|e^{itH}P_{a.c.}f\|_{L^2(B)} \;\to\; 0 \qquad (1)$$

$$\frac{1}{T}\int_0^T \|e^{itH}P_{s.c.}f\|_{L^2(B)}^2\,dt \;\to\; 0 \qquad (2)$$

(2) is the "RAGE"-theorem (Ruelle-Amrein-Gorgescu-Enns). (1) reduces to the Riemann-Lebesgue theorem, whereas (2) amounts to Wiener's test for atoms.

Theorems of Rauch (3), Jensen–Kato (4):

$$\left\| e^{-\varepsilon|x|} e^{itH} P_c \, e^{\varepsilon|x|} \right\|_{2\to 2} \lesssim (1+t)^{-\frac{1}{2}} \tag{3}$$
$$\text{provided} \quad e^{2\varepsilon|x|} V(x) \in L^\infty(\mathbb{R}^3)$$

$$\left\| \langle x\rangle^{-s} e^{itH} P_c \, \langle x\rangle^{s} \right\|_{2\to 2} \lesssim (1+t)^{-\frac{1}{2}} \tag{4}$$
$$\text{provided} \quad |V(x)| \, \langle x\rangle^{\beta} \in L^\infty(\mathbb{R}^3), \quad \beta > 3, s > 5/2.$$

Remarks:

- These weighted estimates amount to decay of L^2–mass on compact sets for compactly supported data (one cannot take general L^2 data).

- The decay of $(1+t)^{-\frac{1}{2}}$ instead of $(1+t)^{-\frac{3}{2}}$ is due to *resonances at zero energy*: $H\psi = 0$ where $\psi \in \bigcap_{\sigma > \frac{1}{2}} L^{2,-\sigma}(\mathbb{R}^n)$ but $\psi \notin L^2(\mathbb{R}^n)$. Thus resonances are another obstacle for the dispersive bound.

Theorem of Journé, Soffer, Sogge: Let $n \geq 3$, $\widehat{V} \in L^1(\mathbb{R}^n)$, $|V(x)| \lesssim (1+|x|)^{-n-4}$, plus some regularity of V. If zero is neither a resonance nor an eigenvalue, then $\|e^{itH}P_c f\|_\infty \lesssim t^{-\frac{n}{2}}\|f\|_1$. Strichartz: $\|e^{itH}P_c f\|_{L^p(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$, where $p = 2 + \frac{4}{n}$ (mixed norm Strichartz up to endpoint, inhomogeneous Strichartz if no bound states at all).

- Here $\sigma(H) = [0,\infty) \cup \{\lambda_j | j = 1, \ldots, N\}$, $[0,\infty) = \sigma_{a.c.}(H)$ (Weyl), $\lambda_N < \lambda_{N-1} < \ldots < \lambda_1 \leq 0$ eigenvalues of finite multiplicity (Birman-Schwinger), $\sigma_{p.p.}(H) \cap (0,\infty) = \emptyset$ (Kato), $\sigma_{s.c.}(H) = \emptyset$ (Agmon-Kato-Kuroda)

- Proof: Split into high and low energies, reduction to $V = 0$ via perturbative argument (Duhamel), use smoothing bound

$$\left\|\chi_B \, (-\triangle)^{\frac{1}{4}} e^{-it\triangle} f\right\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|f\|_2$$

  for high energies to get smallness. For low energies use Jensen-Kato asymptotic expansion.

Yajima's approach via wave operators:

Let $W = s - \lim_{t\to\infty} e^{-itH}e^{itH_0}$ be the usual wave-operator, $H_0 = -\triangle$, $H = H_0 + V$. It exists if $|V(x)| \lesssim (1+|x|)^{-1-\varepsilon}$: Let $\widehat{f} \in C_0^\infty(\mathbb{R}^n)$, $\widehat{f} = 0$ close to the origin. Then

$$\begin{aligned}
&\int_1^\infty \Big\| \frac{d}{dt} e^{-itH} e^{itH_0} f \Big\|_2 \, dt \\
= \; &\int_1^\infty \Big\| e^{-itH} V e^{itH_0} f \Big\|_2 \, dt \\
\lesssim \; &\int_1^\infty \Big(\Big\| V e^{itH_0} f \Big\|_{L^2(|x|\sim t)} + O(t^{-N}) \Big) \, dt \\
\lesssim \; &\int_1^\infty t^{-\frac{n}{2}} \|V\|_{L^2(|x|\sim t)} \, dt + 1 \\
\lesssim \; &\int_1^\infty t^{-\frac{n}{2}} \, t^{\frac{n}{2}} \, t^{-1-\varepsilon} \, dt + 1 < \infty
\end{aligned}$$

So for a dense family of functions the limit $\lim_{t\to\infty} e^{-itH}e^{itH_0} f$ exists, and so the strong limit exists since $\sup_t \|e^{-itH}e^{itH_0}\| < \infty$.

The wave operators are isometries, they have the intertwining property

$$e^{itH}W = We^{itH_0},$$

Range$(W) \subset \mathcal{H}_{a.c.}(H)$, $W^*W = id_{L^2}$, $WW^* = P_{a.c.}$ (for the last property need completeness). Thus

$$e^{itH}P_{a.c.} = We^{itH_0}W^*.$$

If $\|W\|_{L^\infty \to L^\infty}$ is finite, then

$$\|e^{itH}P_{a.c.}\|_{1\to\infty} = \|We^{itH_0}W^*\|_{1\to\infty} \lesssim t^{-\frac{n}{2}}.$$

Yajima's theorem: Let $n = 3$ (for simplicity), $\langle x\rangle^\sigma V \in L^2(\mathbb{R}^3)$ for some $\sigma > 1$. If V small or $|V(x)| \lesssim (1+|x|)^{-5-\varepsilon}$, and zero neither eigenvalue nor resonance, then $\|W\|_{p\to p} < \infty$ for $1 \le p \le \infty$.

- Thus the dispersive and Strichartz estimates as in [J-S-S] hold under much weaker assumptions on V.

- The proof is based on Kato's stationary representation of the wave operators.

- Results in $n = 1$ by Wermer, $n = 2$ by Yajima.

A digression on resonances:

Agmon's argument on absence of positive energy resonances. Limiting absorption principle:

$$\|(-\triangle - \lambda - i0)^{-1}\|_{L^{2,\sigma} \to L^{2,-\sigma}} < \infty \qquad (5)$$

for $\forall\, \sigma > \frac{1}{2}$, $\lambda > 0$. Suppose $|V(x)| \lesssim (1+|x|)^{-1-\varepsilon}$, and

$$(-\triangle + V - \lambda)f = 0 \quad \text{for } f \in \bigcap_{\sigma > \frac{1}{2}} L^{2,-\sigma}(\mathbb{R}^n)$$

and some $\lambda > 0$. Then $Vf \in L^{2,\frac{1}{2}+\varepsilon-}$, and thus

$$g := -Vf = (-\triangle - \lambda)f$$

implies that $\widehat{g}$ has a restriction to $S_{\sqrt{\lambda}} = [|\xi| = \sqrt{\lambda}]$ which is in $L^2(S_{\sqrt{\lambda}})$. This restriction has to vanish. By (5) one has $\widehat{f} = \frac{\widehat{g}}{|\xi|^2 - \lambda}$. This amounts to one (radial) derivative of $\widehat{g}$. Hence $f \in L^{2,-\frac{1}{2}+\varepsilon-}$; repeating this argument shows that $f \in L^{2,-\frac{1}{2}+n\varepsilon-}$ for all $n = 1, 2, \ldots$ as desired.

Theorem A: Let $n = 3$. Suppose

$$\int_{\mathbb{R}^6} \frac{|V(x)||V(y)|}{|x-y|^2}\,dxdy \;<\; (4\pi)^2 \qquad (6)$$

$$\|V\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|}\,dy \;<\; 4\pi. \qquad (7)$$

Then $\|e^{itH}f\|_\infty \lesssim t^{-\frac{3}{2}}\|f\|_1$.

- Theorem A is a "small potential" result. Those are best stated in terms of norms that are invariant under the scaling $V \to R^2V(R\cdot)$, as $L^{\frac{n}{2}}(\mathbb{R}^n)$, or the norms in (6) and (7). If $V \in L^{\frac{3}{2}+}(\mathbb{R}^3) \cap L^{\frac{3}{2}-}(\mathbb{R}^3)$ and small, then these conditions apply. In particular $|V(x)| \le c_0(1+|x|)^{-2-\varepsilon}$, c_0 small, suffices. Kato's theorem (1965): If (6) holds, then wave operators are unitary. In dimensions $n = 1, 2$ arbitrarily small potentials can have negative eigenvalues.

Theorem B: Let $n \geq 3$. Suppose $|V(x)| \lesssim (1 + |x|)^{-2-\varepsilon}$. Then one has mixed norm Strichartz estimates (up to endpoints), provided zero is neither a zero eigenvalue nor a resonance (exclude all bound states for the inhomogeneous case):

$$\|e^{itH} P_c f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}$$

where

$$\frac{2}{q} = n(\frac{1}{2} - \frac{1}{r}) \qquad 2 \leq r < \frac{2n}{n-2}.$$

- In Theorem B can have negative bound states, but nothing else: If $|V(x)| \lesssim (1 + |x|)^{-1-\varepsilon}$, then $\sigma_{p.p.} \cap (0, \infty) = \emptyset$ and $\sigma_{s.c.} = \emptyset$.

- The conjecture that Theorem B should hold appears to be due to [J-S-S].

- Both Theorem A and B rely on Kato's smoothing theory.

Theorem C: Let $n = 3$. Suppose

$$\sup_t \|V(t,\cdot)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\|V(\widehat{\tau},x)\|_{\mathcal{M}(\tau)}}{|x-y|}\,dx < c_0. \tag{8}$$

Then for every $\psi_s \in L^2(\mathbb{R}^3)$

$$i\partial_t\psi - \triangle\psi + V(t,\cdot)\psi = 0, \qquad \psi(s,\cdot) = \psi_s(\cdot)$$

has a unique weak solution that satisfies

$$\|\psi(t,\cdot)\|_\infty \lesssim |t-s|^{-\frac{3}{2}}\|\psi(s,\cdot)\|_1.$$

- $V(\widehat{\tau},x)$ denotes the partial Fourier transform in the time variable (in the distributional sense), and $\mathcal{M}$ are the measures. (8) is again scaling invariant. Note: V is not assumed to decay in time.

- Take $V(t,x) = V_0(x)T(t)$ with $T(t)$ is (quasi) periodic: $T(t) = \sum_\nu c_\nu e(t\nu\cdot\omega)$ where $\sum_\nu |c_\nu| < \infty$. Thus $\|V(\widehat{\tau},x)\|_{\mathcal{M}(\tau)} \lesssim |V_0(x)|$. So (8) holds if $|V_0(x)| \lesssim (1+|x|)^{-2-\varepsilon}$ and V_0 small, say.

Recall Kato's smoothing theory: Formally, wave operators are given by

$$\begin{aligned} Wf &= f - i\int_0^\infty e^{-itH}Ve^{itH_0}f\,dt \\ \langle Wf, g\rangle &= \langle f, g\rangle - i\int_0^\infty \langle Ve^{itH_0}f, e^{itH}g\rangle\,dt \\ &= \langle f, g\rangle - i\int_{-\infty}^{\infty} \langle VR_0(\lambda)f, R_V(\lambda)g\rangle\,d\lambda. \quad (9) \end{aligned}$$

Here

$$\begin{aligned} R_V(\lambda) &= (-\triangle + V - \lambda + i0)^{-1} \\ &= -\lim_{\varepsilon\to 0+} i\int_0^\infty e^{itH}e^{-it\lambda-\varepsilon t}\,dt. \end{aligned}$$

Write $V = AB$, $A = |V|^{\frac{1}{2}}, B = |V|^{\frac{1}{2}}\mathrm{sign(V)}$. Then need

$$\int_{-\infty}^{\infty} \|AR_0(\lambda)f\|_2^2\,d\lambda < \infty, \int_{-\infty}^{\infty} \|BR_V(\lambda)f\|_2^2\,d\lambda < \infty.$$

Main insight of Kato here is that these conditions, which are difficult to check, follow from

$$\|V\|_K := \sup_{\Im z>0} \|AR_0(z)B\| < \infty.$$

This can be seen by a simple TT^* argument: Let $Tf = AR_0(\lambda)f$. Since $T : L^2(\mathbb{R}^3) \to L^2_\lambda L^2_x \Longleftrightarrow TT^* : L^2_\lambda L^2_x \to L^2_\lambda L^2_x$, suffices to prove the latter.

$$\begin{aligned} T^*F &= \int_{-\infty}^{\infty} R_0(\lambda)^* \, A\, F(\lambda)\, d\lambda \\ TT^*F &= \int_{-\infty}^{\infty} AR_0(\mu)R_0(\lambda)^* \, A\, F(\lambda)\, d\lambda \\ &= \int_{-\infty}^{\infty} A\frac{R_0(\mu) - R_0(\lambda)^*}{\lambda - \mu} \, A\, F(\lambda)\, d\lambda. \end{aligned}$$

Using the L^2 boundedness of the vector–valued Hilbert transform one obtains that

$$\|TT^*F\|_{L^2_\mu L^2_x} \lesssim \|V\|_K \, \|F\|_{L^2_\lambda L^2_x} \Rightarrow \|T\| \lesssim \|V\|_K^{\frac{1}{2}}.$$

Using Hilbert-Schmidt norms one gets

$$\|V\|_K \le \|V\|_R := \Big(\int_{\mathbb{R}^6} \frac{|V(x)||V(y)|}{|x-y|^2}\, dxdy\Big)^{\frac{1}{2}}$$

since $R_0(\lambda) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|}$.

We need the same also for $R_V(\lambda)$. Let

$$Q(\lambda) = AR_0(\lambda)B.$$

Then

$$\begin{aligned} R_V(\lambda) &= R_0(\lambda) + R_0(\lambda)VR_V(\lambda) \\ AR_V(\lambda) &= AR_0(\lambda) + Q(\lambda)AR_V(\lambda) \\ AR_V(\lambda) &= (1 - Q(\lambda))^{-1}AR_0(\lambda), \end{aligned}$$

where the last step is valid provided

$$\sup_\lambda \|Q(\lambda)\| < 1.$$

Thus

$$\|Q(z)\| \leq \frac{1}{4\pi}\Big(\int_{\mathbb{R}^6} \frac{|V(x)||V(y)|}{|x-y|^2}\, dxdy\Big)^{\frac{1}{2}} < 1,$$

is sufficient.

Conclusion: Under this condition, the wave operators exist and are unitary, i.e.,

$$-\triangle + V \simeq -\triangle \quad \text{(unitary equivalence)}.$$

Proof of Theorem A:

$$
\begin{aligned}
&\sup_{L\ge 1}\Big|\Big\langle e^{itH}\psi(\sqrt{H}/L)f,g\Big\rangle\Big| \\
\le\ &\sup_{L\ge 1}\Big|\int_0^\infty e^{it\lambda}\,\psi(\sqrt{\lambda}/L)\langle E'(\lambda)f,g\rangle\,d\lambda\Big| \\
=\ &\sup_{L\ge 1}\Big|\int_0^\infty e^{it\lambda}\,\psi(\sqrt{\lambda}/L)\Im\langle R_V(\lambda)f,g\rangle\,d\lambda\Big| \\
=\ &\sup_{L\ge 1}\Big|\int_0^\infty e^{it\lambda}\psi(\sqrt{\lambda}/L)\sum_{k=0}^\infty \Im\langle R_0(\lambda)(VR_0(\lambda))^k f,g\rangle\,d\lambda\Big| \\
\le\ &\sum_{k=0}^\infty \int_{\mathbb{R}^6}|f(x_0)||g(x_{k+1})|\int_{\mathbb{R}^{3k}}\frac{\prod_{j=1}^k|V(x_j)|}{\prod_{j=0}^k 4\pi|x_j-x_{j+1}|}\,\cdot \\
&\cdot\sup_{L\ge 1}\Big|\int_0^\infty e^{it\lambda}\psi(\sqrt{\lambda}/L)\sin\Big(\sqrt{\lambda}\sum_{\ell=0}^k|x_\ell-x_{\ell+1}|\Big)\,d\lambda\Big|\,dx.
\end{aligned}
$$

Use the oscillatory integral bound:

$$
\Big|\int_0^\infty e^{it\lambda}\chi(\sqrt{\lambda}/L)\sin(\sqrt{\lambda}a)\,d\lambda\Big|\lesssim t^{-\frac{3}{2}}|a|.
$$

Hence continue as follows:

$$\leq \; Ct^{-\frac{3}{2}} \sum_{k=0}^{\infty} \int_{\mathbb{R}^6} |f(x_0)||g(x_{k+1})|$$

$$\int_{\mathbb{R}^{3k}} \frac{\prod_{j=1}^{k} |V(x_j)|}{(4\pi)^{k+1} \prod_{j=0}^{k} |x_j - x_{j+1}|} \sum_{\ell=0}^{k} |x_\ell - x_{\ell+1}| \, dx$$

$$\leq \; Ct^{-\frac{3}{2}} \sum_{k=0}^{\infty} \int_{\mathbb{R}^6} |f(x_0)||g(x_{k+1})| \; (k+1)\Big(\frac{\|V\|_{\mathcal{K}}}{4\pi}\Big)^k \, dx$$

$$\leq \; Ct^{-\frac{3}{2}} \, \|f\|_1 \|g\|_1,$$

since we assumed that $\|V\|_{\mathcal{K}} < 4\pi$. As an example for the "collapsing" that appears above take $k = 2$:

$$\int_{\mathbb{R}^6} \frac{|V(x_1)||V(x_2)|}{|x_0 - x_1||x_1 - x_2||x_2 - x_3|} |x_2 - x_3| \, dx_1 dx_2$$

$$\leq \int_{\mathbb{R}^3} \frac{|V(x_1)|}{|x_0 - x_1|} \, dx_1 \; \sup_{\widetilde{x}_1} \int_{\mathbb{R}^3} \frac{|V(x_2)|}{|\widetilde{x}_1 - x_2|} \, dx_2.$$

Some remarks about the time-dependent case, Theorem C:

- The proof is motivated by the previous one, although much more involved. Instead of the spectral theorem, use an expansion in terms of an iterated Duhamel expansion (Dyson series). Since we are working with the $L^\infty_t L^{\frac{3}{2}}_x$-norm, we need the endpoint Strichartz bound of Keel-Tao to show that (weak) solutions exist in the space $X = L^\infty_t(L^2_x(\mathbb{R}^3)) \cap L^2_t(L^6_x(\mathbb{R}^3))$.

- Using functional calculus, the individual terms of the Dyson series are expressed in terms of the potential and certain oscillatory integrals, whose phases can vanish to third order:

$$\left| \int_0^\infty e^{\frac{1}{2}i\lambda^2}\, e^{\pm i \sum_{j=1}^m b_j \sqrt{\lambda^2+\sigma_j}}\, \omega(\lambda)\, d\lambda \right| \leq C_0\,(1+\sigma_1)^{\frac{1}{4}}$$

  for any $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$ and $b_j > 0$. Here $|\omega^{(j)}(\lambda)| \leq a_0\,\lambda^{-j}$ for $j = 0, 1, 2$ and all $\lambda > 0$.

It is much easier to derive dispersive estimates for small time-dependent potentials under stronger conditions. For example, suppose

$$\sup_t (\|\widehat{V}(t,\cdot)\|_1 + \|V(t,\cdot)\|_1) < c_0$$

where c_0 is sufficiently small. Then it follows quite easily from Duhamel's formula that the flow $U(t)$ of $i\partial_t\psi - \triangle\psi + V(t)\psi = 0$ satisfies the dispersive bound, $\|U(t)f\|_\infty \lesssim t^{-\frac{3}{2}}\|f\|_1$. The Fourier transform of the potential enters since

$$\|e^{-itH_0}\, g\, e^{itH_0}\|_{p\to p} \le \|\widehat{g}\|_1$$

for $1 \le p \le \infty$. By another application of Duhamel one can show that then also

$$\|e^{-isH_0} V(t+s) U(t+s,t)\|_{p\to p} \le \|\widehat{V}(t+s)\|_1.$$

Jean Bourgain has recently relaxed these conditions somewhat.