

## Blowup for hyperbolic equations

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Consider Cauchy problem in  $\mathbb{R}^n$  with  $\square = \partial_{tt} - \Delta$ ,

$$\square u = N(u), \quad \square u = N(u, \nabla u) \quad (\text{NLW})$$

data  $(u, \partial_t u)(0) = (u_0, u_1) \in H^s \times H^{s-1}$ . Basic question of solvability and uniqueness:

- **Locally well posed (LWP)** in  $H^s \times H^{s-1}$  ?
- If so, solutions **global (GWP)** or **finite time blow-up** ?

**(LWP)** in  $H^s \times H^{s-1}$  if  $\forall (u_0, u_1) \in H^s \times H^{s-1}$  there is  $T > 0$  and a neighborhood  $U$  of  $(u_0, u_1)$  so that for all data in  $U \exists!$  solution  $(u, u_t) \in C((-T, T); H^s) \times C((-T, T); H^{s-1})$ , depending continuously on data. Also, higher regularity preserved.

**Easy:** (LWP) for large  $s$  via energy methods, (GWP) for small data

Recast (**NLW**) as

$$u = S(u_0, u_1) + \square^{-1}N(u)$$

Find **fixed point** for small times in Banach spaces  $X, Y$  with

$$S : H^s \times H^{s-1} \rightarrow X, \quad \square^{-1} : Y \rightarrow X, \quad N : X \rightarrow Y$$

**Energy method** means

$$X = \{u \in L^\infty(H^s), \partial u \in L^\infty(H^{s-1})\}, \quad Y = L^1(H^{s-1})$$

Boundedness of  $S, \square^{-1}$  trivial:

$$u(t) = \cos(\sqrt{-\Delta}t)u_0 + \frac{\sin(\sqrt{-\Delta}t)}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin(\sqrt{-\Delta}(t-t'))}{\sqrt{-\Delta}}f(t') dt'$$

$$\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_0^t \|f(t')\|_{H^{s-1}} dt'$$

**Obstruction** from  $N : s \geq s_0 + 1$  where  $s_0$  is the **scaling exponent**:  
 $s_0 = \frac{n}{2} - \alpha$ , **(NLW)** invariant under  $\mathcal{D}_\lambda : u \rightarrow \lambda^\alpha u(\lambda x, \lambda t)$ ,  $\lambda > 0$ .  
**Point**: norm of  $\dot{H}^{\frac{n}{2}-\alpha}$  unchanged under  $\mathcal{D}_\lambda$ .

- $s < s_0$  Small data, small time equivalent to **large data, large time** result. Expect *local ill-posedness below scaling*.
- $s = s_0$  Small data, small time equivalent to **small data, large time** result. Same for large data *provided time of existence only depends on size of data*.
- $s > s_0$  Small data, large time equivalent to **large data, small time** result.  $T_{\max} \gtrsim \|(u_0, u_1)\|_{H^s \times H^{s-1}}^{s_0-s}$

Other **symmetries** besides scaling also serve as obstructions to  
**(LWP)**: **Lorentz invariance** leads to concentration along light-rays.

**(GWP)**: Suppose energy  $E : H^{s_c} \times H^{s_c-1} \rightarrow \mathbb{R}$  preserved under the flow of (NLW). Require that  $E(u, u_t) \sim \|(u, u_t)\|_{s_c}^2$ .

- **subcritical case**:  $s_c > s_0$ . **(LWP)** at  $s \geq s_c$  implies **(GWP)** at  $s \geq s_c$  since norm  $\|\cdot\|_{s_c}$  does not grow. *Energy penalizes point-singularity formation via scaling.*
- **critical case**:  $s_c = s_0$ . **(LPW)** equivalent to **(GWP)** for small data. Large data: *Energy neutral to point-singularity formation via scaling*, exclude concentration of energy in the tip of characteristic cone (**Morawetz** identity).
- **supercritical case**:  $s_c < s_0$ . *Energy rewards point-singularity formation via scaling.* No global large data results known, even for Schwartz data.

Consider (**NLW**) with  $N(u) = -|u|^{p-1}u$  and energy

$$E(u) = \int \frac{1}{2} |\partial u|^2 + \frac{1}{p+1} |u|^{p+1} dx$$

Then  $s_c = 1, s_0 = \frac{n}{2} - \frac{2}{p-1}$ . If  $n=3, p=5$  the problem is **energy critical**, and (**GWP**) known (Grillakis, 90). **Open problem**: for example, prove that  $p = 7$  (**NLW**)

$$\partial_{tt}u - \Delta u + u^7 = 0$$

admits *global smooth solutions for all smooth data*.

Expect **finite time blow-up** in many cases:

$$\partial_{tt}u - \Delta u - |u|^{p-1}u = 0$$

Solution  $u(t, x) = c_p \cdot (T - t)^{-\frac{2}{p-1}}$ , truncate to light cone.

Merle-Zaag: if  $p \leq 3$ , then **blow-up** is **self-similar** with this rate!

## The Wave Maps Equation

$\phi : \mathbb{R}^{n+1} \rightarrow M$ ,  $(M, g)$  Riemann manifold with Lagrangian

$$L(\phi) = \frac{1}{2} \int_{\mathbb{R}^{n+1}} -|\partial_t \phi|_g^2 + |\nabla \phi|_g^2 dx dt$$

and associated Euler-Lagrange equation

$$\mathbf{D}^\alpha \partial_\alpha \phi = 0, \quad \square \phi^i = \Gamma_{jk}^i \partial^\alpha \phi^j \partial_\alpha \phi^k, \quad \square \phi \perp T_\phi M$$

E.g.  $\phi = \gamma \circ u$ ,  $D\dot{\gamma} = 0$ ,  $\square u = 0$ . Conserved energy

$$E(\phi) = \frac{1}{2} \int |\partial_t \phi|_g^2 + |\nabla \phi|_g^2 dx$$

Cauchy Problem  $\phi(0) = \phi_0 \in M$ ,  $\partial_t \phi(0) = \phi_1 \in T_{\phi_0} M$ . (WM)  
 scaling  $\phi(t, x) \mapsto \phi(\lambda t, \lambda x)$ . So  $s_0 = \frac{n}{2}$ : (**LWP**) for  $s > \frac{n}{2}$   
 (Machedon, Klainerman, Selberg), (**GWP**) for  $s = \frac{n}{2}$ , small data,  
 "reasonable"  $M$  (Tataru, Tao, Krieger), and (**LIP**) for  $s < \frac{n}{2}$   
 (d'Ancona, Georgiev). Large data (**GWP**): compare  $s_0 = \frac{n}{2}$ ,  $s_c = 1$

Expect (**GWP**) for  $n = 1$  since  $s_c > s_0$  (known, even down to scaling critical), blow-up for  $n \geq 3$  (**Shatah**): there exist **self-similar** solutions  $u(t, x) = v(x/t)$  with  $u = v$ ,  $\partial_t u = -x \cdot \nabla v$  at  $t = 1$ .

Rewrite the *Lagrangian in self-similar coordinates*  $\tau = \sqrt{t^2 - |x|^2}$ ,  $\xi = \frac{x}{t} = \rho\omega$ ,  $\omega \in S^{n-1}$ . Then  $v$  solves the **elliptic PDE**

$$-v_{\rho\rho} - \left( \frac{n-1}{\rho} + \frac{(n-3)\rho}{1-\rho^2} \right) v_\rho + \frac{1}{\rho^2(1-\rho^2)} \Delta_\omega v \perp T_v M$$

which is an **harmonic map** equation on the unit ball of  $\mathbb{R}^n$  with **hyperbolic metric**

$$\frac{d\rho^2}{(1-\rho^2)^2} + \frac{\rho^2}{1-\rho^2} d\omega^2$$

For  $n \geq 3$  non-constant solutions exist (equivariant harmonic maps) which can be **smoothly continued** beyond  $\rho = 1$  with **target manifolds** given in terms of surfaces of revolution (spheres). **Fails** for  $n = 2$ , since the **only solutions** are  $v = \text{const}$ .

**Critical case:**  $n = 2$ ,  $s_c = s_0 = 1$ . We'll take target  $M$  a *surface of revolution* with metric  $ds^2 = d\rho^2 + g^2(\rho)d\theta^2$ ,  $\theta \in S^1$ ,  $g \in C^\infty(\mathbb{R})$ ,  $g(0) = 0$ ,  $g'(0) = 1$  and also assume this:  $g$  **odd** and either

$$g(\rho) > 0 \quad \forall \rho > 0, \quad \int_0^\infty |g(\rho)| d\rho = \infty \quad (1)$$

or if  $M$  is **compact**, that  $g$  has first zero  $\rho_1 > 0$ ,  $g'(\rho_1) = -1$  and  $g$  **periodic** with period  $2\rho_1$ . Consider **equivariant wave maps**  $\phi(t, x) : \mathbb{R}^{1+2} \rightarrow M$ : if  $(r, \phi)$  polar coordinates on  $\mathbb{R}^2$  then

$$\rho = u(t, r), \quad \theta = \phi$$

with equation  $\partial_{tt}\mathbf{u} - \partial_{rr}\mathbf{u} - \frac{1}{r}\partial_r\mathbf{u} + \frac{\mathbf{g}(u)\mathbf{g}'(u)}{r^2} = \mathbf{0}$ . **Smooth data** with **finite energy** have **local smooth solutions** which **cannot be continued** beyond  $t = T_*$  iff **energy concentrates**:  $\exists \varepsilon_0 = \varepsilon_0(M) > 0$  so that

$$\liminf_{t \rightarrow T_*} \frac{1}{2} \int_0^{T_* - t} |\partial u|^2 + \frac{g^2(u)}{r^2} dr > \varepsilon_0$$

**Struwe's bubbling off theorem:** Let  $\phi$  be smooth (EQWM) blowing up at  $t_0$ . Then  $\exists r_j \rightarrow 0+, t_j \rightarrow t_0-$  s.t.

$$\phi_j(t, x) := \phi(t_j + r_j t, r_j x) \rightarrow \Phi_\infty(t, x), \quad r_j = o(t_0 - t_j)$$

strongly in  $H_{\text{loc}}^1([-1, 1] \times \mathbb{R}^2)$ , where  $\Phi_\infty$  a nonconstant, time-independent solution generating nonconstant, smooth, (EQHM)  $\Psi : S^2 \rightarrow M$  ( $\Delta_{S^2} \Psi \perp T_\Psi M$ ).

If  $M$  is non-compact as in (1), then harmonic spheres are known not to exist: (GWP) for such targets in the *equivariant case*.

However: for  $M = S^2$  harmonic spheres do exist:  $z^\ell, \bar{z}^\ell : \mathbb{C} \rightarrow \mathbb{C}$ ,  $\ell \geq 1$ , via stereographic projection  $u(r) = 2 \arctan(r^\ell), \theta = \pm \ell \phi$ .

Main result of this talk for wave maps is to show that blow-up does occur for  $S^2$ ! In the *non-equivariant case* also expect dichotomy between (GWP) for large data and blow-up depending on the geometry of the target manifold ( $S^2$  vs.  $\mathbb{H}^2$ ).

**Theorem (K-S-T, 2006):** Let  $\nu > \frac{1}{2}$ ,  $t_0 > 0$ ,  $\lambda(t) = t^{-1-\nu}$ ,  $N$  large.  $\exists$  (WM)  $\phi(t, r, \phi) = (u(r, t), \phi) : [0, t_0] \times \mathbb{R}^2 \rightarrow S^2$

$$u(r, t) = 2 \arctan(\lambda(t)r) + u^e(r, t) + \varepsilon(r, t), \quad 0 \leq r \leq t$$

$$\mathcal{E}_{\text{loc}}(u^e)(t) \lesssim (t\lambda(t))^{-2} |\log t|^2, \quad \mathcal{E}_{\text{loc}}(\varepsilon)(t) \lesssim t^N \text{ as } t \rightarrow 0$$

$$u^e \in C^{\nu+1/2-}(\{t_0 > t > 0, |x| \leq t\}), \quad \varepsilon \in t^N H_{\text{loc}}^{1+\nu-}(\mathbb{R}^2)$$

Also,  $u(0, t) = 0$  for all  $0 < t < t_0$ . The solution  $u(r, t)$  extends as an  $H^{1+\nu-}$  solution to all of  $\mathbb{R}^2$  and the energy of  $u$  concentrates in the cuspidal region  $0 \leq r \lesssim \frac{1}{\lambda(t)}$  leading to blow-up at  $r = t = 0$ .

**Remarks:** i) Expect  $\nu > 0$ , improve on nonlinear part of proof; blow-up *non-generic*, study *conditional stability*

ii) **Rodnianski-Sterbenz 06** independently obtained blow up; bulk term  $Q(\lambda(t)r)$  with  $Q$  (HM) degree  $\ell \geq 4$ , obtained *stable* rate  $\lambda(t) \gtrsim t^{-1} \sqrt{-\log t}$  (prior work **Bizon-Ovchinnikov-Sigal 04**, **Cote**)

Related result for **energy critical (SLWE)** in  $n = 3$ :

$$\partial_{tt}u - \Delta u - u^5 = 0, \quad (t, x) \in \mathbb{R}^{1+3}$$

Stationary solutions  $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$ .

**Theorem (K-S-T, 2006):** Let  $\nu > \frac{1}{2}$ ,  $\delta > 0$ ,  $\lambda(t) = t^{-1-\nu}$ .  $\exists$  energy solution  $u$  of (SLWE) blowing up at  $r = t = 0$  and  $\forall |x| = r \leq t < t_0$  small,

$$u(x, t) = \lambda^{\frac{1}{2}}(t)W(\lambda(t)r) + \eta(x, t)$$

where  $\mathcal{E}_{\text{loc}}(\eta(\cdot, t)) \rightarrow 0$  as  $t \rightarrow 0$  and outside the cone

$$\int_{[|x| \geq t]} [|\nabla u(x, t)|^2 + |u_t(x, t)|^2 + |u(x, t)|^6] dx < \delta$$

for small  $t > 0$ . In particular, the energy of these blow-up solutions can be chosen arbitrarily close to  $\mathcal{E}(W, 0)$ , i.e., the energy of the stationary solution.

**Background on (SLWE):**  $s_0 = s_c = 1$ ,  $u(t, x) \rightarrow \lambda^{\frac{1}{2}} u(\lambda t, \lambda x)$  leaves both  $\dot{H}^1(\mathbb{R}^3)$  and (SLWE) invariant. (LWP) in energy space  $\dot{H}^1 \times L^2(\mathbb{R}^3)$ , solution cannot be continued beyond  $T_* < \infty$  iff  $\|u\|_{L^8([0, T_*] \times \mathbb{R}^3)} = \infty$ . Critical points of Lagrangian

$$L(u) = \int_{\mathbb{R}^{1+3}} \frac{1}{2} (-|\partial_t u|^2 + |\nabla u|^2) - \frac{1}{6} |u|^6 dt dx$$

and conserved energy

$$\mathcal{E}(u) = \int_{\mathbb{R}^3} \frac{1}{2} (|\partial_t u|^2 + |\nabla u|^2) - \frac{1}{6} |u|^6 dx$$

The critical Sobolev imbedding  $\dot{H}^1 \rightarrow L^6(\mathbb{R}^3)$  has extremizers  $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$ . Conformal group generates all extremizers. Associated EL-equation is  $-\Delta W - W^5 = 0$ , hence stationary solutions of (SLWE). Perturb around  $W$ : Seek  $u = W + \psi$  with  $\psi$  small. Then  $(-\Delta - 5W^4(r))\psi(t, r) = N(W, \psi) = O(|\psi|^2)$  has solutions, with  $H = -\Delta - 5W^4(r)$ ,

$$\psi(t, x) = \cos(\sqrt{H}t)\psi_0 + \frac{\sin(\sqrt{H}t)}{\sqrt{H}}\psi_1 + \int_0^t \frac{\sin(\sqrt{H}(t-t'))}{\sqrt{H}} N(t') dt'$$

**Spectrum** of  $H$  restricted to  $L_{\text{rad}}^2$ : since  $H(\partial_\lambda[\lambda^{\frac{1}{2}}W(\lambda r)]) = 0$  and the zero-mode has a unique positive zero: zero is second eigenvalue (actually: resonance), bottom eigenvalue negative,  $Hg_0 = -k_0^2g_0$ ,

$$\cos(\sqrt{H}t)g_0 = \cosh(k_0t)g_0, \quad \frac{\sin(\sqrt{H}t)}{\sqrt{H}}g_0 = \frac{\sinh(k_0t)}{k_0}g_0.$$

Thus, need to project onto  $g_0^\perp$  to have **linear stability**. Nonlinearly, one has a **local center stable mf theorem for radial data (KS 05)**: In  $B_\delta(0) \subset \langle x \rangle^{-\sigma}(H_{\text{rad}}^3 \times H_{\text{rad}}^2) \ni$  co-dim one Lipschitz graph  $\mathcal{G}$  parameterized by tangent plane  $\langle k_0f_1 + f_2, g_0 \rangle = 0$  s.t. data on  $(W, 0) + \mathcal{G}$  have global solutions  $\lambda_\infty^{\frac{1}{2}}W(\lambda_\infty x) + R$  where  $R$  disperses and scatters to a free energy wave and  $|\lambda_\infty - 1| \lesssim \delta$ .

Expect  $\mathcal{G}$  to divide  $(W, 0) + B_\delta(0)$  into **blow-up/scattering** halves. **Karageorgis-Strauss 06** proved that there is blow-up above the tangent plane of  $\mathcal{G}$  (for  $|u|^5$ ). **Graph of energy** at  $(W, 0)$  is a **saddle surface**:

$$\begin{aligned} \mathcal{E}(W + f_1, f_2) &= \mathcal{E}(W, 0) + \langle D\mathcal{E}(W, 0), (f_1, f_2) \rangle + \\ &\quad + \frac{1}{2} \langle D^2\mathcal{E}(W, 0)(f_1, f_2), (f_1, f_2) \rangle + \dots \end{aligned}$$

Euler-Lagrange equation for  $W$  is equivalent to  $D\mathcal{E}(W, 0) = 0$ , whereas the **second variation** is an **indefinite quadratic form**

$$\langle D^2\mathcal{E}(W, 0)(f_1, f_2), (f_1, f_2) \rangle = -k_0^2 \xi_1^2 + \xi_2^2 + \langle H f_1^\perp, f_1^\perp \rangle + \langle f_2^\perp, f_2^\perp \rangle$$

with  $\xi_1 = \langle f_1, g_0 \rangle, \xi_2 = \langle f_2, g_0 \rangle$ .

Compare this to the definition of the **tangent plane** to  $\mathcal{G}$  as well as the recent work of **Kenig-Merle 06** on the regime of energy **below**  $\mathcal{E}(W, 0)$ . Their work implies that for  $\mathcal{E}(W + f_1, f_2) < \mathcal{E}(W, 0)$  the **center stable manifold** is  $\|\nabla f_1\|_2 = \|\nabla W\|_2$ .

**Strategy of proof for (WM):** We seek a solution of

$$-\partial_{tt}u + u_{rr} + \frac{u_r}{r} - \frac{\sin(2u)}{2r^2} = 0$$

of the form  $u(t, r) = Q(\lambda(t)r) + v(t, r)$  inside the light-cone  $r \leq t$ , with  $\mathcal{E}_{\text{loc}}(v(t, \cdot)) \rightarrow 0$  as  $t \rightarrow 0+$ . By Struwe's result,  $t\lambda(t) \rightarrow \infty$ .

Applying  $\partial_{tt}$  to the bulk-term generates

$$\partial_{tt}Q(\lambda(t)r) = \ddot{\lambda}(t)rQ'(\lambda(t)r) + \dot{\lambda}^2(t)r^2Q''(\lambda(t)r)$$

Note:  $rQ'(r) = \frac{r}{1+r^2} \notin L^2(\mathbb{R}^2)$ . To **cancel** at  $r = \infty$  need ODE

$$\lambda(t)\ddot{\lambda}(t) - 2\dot{\lambda}(t)^2 = 0 \Leftrightarrow \lambda(t) = (c_1t + c_2)^{-1}$$

**Not allowed** by Struwe's theorem!

**Way out:** Replace  $Q(\lambda(t)r)$  by  $Q(\lambda(t)r)\chi(t, r/t)$  (a cut-off to light-cone) with  $\chi(t, \cdot) = 1 + o_{H^1}(1)$  as  $t \rightarrow 0+$ .

Ignoring  $\chi_t$ , setting  $a = \frac{r}{t}$ ,  $\lambda(t) = t^{-1-\nu}$  yields ODE in  $a$  with **principal Sturm-Liouville** term

$$(1 - a)^{\nu + \frac{1}{2}} \partial_a [(1 - a)^{-\nu + \frac{1}{2}} \partial_a] \chi(a) = \dots$$

Fundamental system  $1, (1 - a)^{\nu + \frac{1}{2}}$ , gives **energy solutions** as long as  $\nu > 0$ .

**Actual renormalization scheme:**  $R = \lambda(t)r$ ,  $a = r/t$ ,  
 $u_0(R) = Q(R)$ . Seek  $u_k = u_{k-1} + v_k$  **approximate solutions** of  
(WME) in light-cone  $r \leq t$ , improve with  $k$ . **Iterative scheme** of  
**errors** and **corrections**

$$e_k = (-\partial_{tt} + \partial_{rr} + \frac{1}{r} \partial_r) u_k - \frac{\sin(2u_k)}{2r^2}$$

$$v_{2k+1} = \left( \partial_{rr} + \frac{1}{r} \partial_r - \frac{\cos(2u_0)}{r^2} \right) e_{2k}$$

$$v_{2k} = \left( -\partial_{tt} + \partial_{rr} + \frac{1}{r} \partial_r - \frac{1}{r^2} \right) e_{2k-1}$$

Then  $e_{2k} = -N_{2k}(v_{2k})$ ,  $e_{2k+1} = -\partial_t^2 v_{2k+1} - N_{2k+1}(v_{2k+1})$  with

$$\begin{aligned}
N_{2k+1}(v) &= \frac{\cos(2u_0) - \cos(2u_{2k})}{r^2} v + \frac{\sin(2u_{2k})}{2r^2} (1 - \cos(2v)) \\
&\quad + \frac{\cos(2u_{2k})}{2r^2} (2v - \sin(2v)) \\
N_{2k}(v) &= \frac{1 - \cos(2u_{2k-1})}{r^2} v + \frac{\sin(2u_{2k-1})}{2r^2} (1 - \cos(2v)) \\
&\quad + \frac{\cos(2u_{2k-1})}{2r^2} (2v - \sin(2v))
\end{aligned}$$

**Upshot:** Get errors that decay like  $t^m \forall m$ . Indeed,

$$\begin{aligned}
u_{2k-1}(r, t) &= Q(\lambda(t)r) + \frac{c_k}{(t\lambda)^2} R \log(1 + R^2) + O\left(\frac{(\log(1 + R^2))^2}{R(t\lambda)^2}\right) \\
e_{2k-1} &= O\left(\frac{R(\log(2 + R))^{2k-1}}{t^2(t\lambda)^{2k}}\right)
\end{aligned}$$

where correction is  $C^{\nu+\frac{1}{2}-}$  at  $r = t$ .

**Perturbative analysis:** Exact (**WME**) solution  $u = u_{2k-1} + \varepsilon$ .  
 Solve in coordinates  $R = \lambda(t)r$ ,  $d\tau = \lambda(t)dt$ ,  $\tau = \int_t^1 \lambda(s) ds + \nu^{-1}$ .  
 Set  $\tilde{\varepsilon} = R^{\frac{1}{2}}\varepsilon(t(\tau), \lambda^{-1}R)$ . **Main PDE:**

$$\begin{aligned} & \left( -(\partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R)^2 + \frac{1}{4} \left( \frac{\lambda_\tau}{\lambda} \right)^2 + \frac{1}{2} \partial_\tau \left( \frac{\lambda_\tau}{\lambda} \right) \right) \tilde{\varepsilon} - \mathcal{L} \tilde{\varepsilon} \\ & = \lambda^{-2} R^{\frac{1}{2}} \left( N_{2k-1} (R^{-\frac{1}{2}} \tilde{\varepsilon}) + e_{2k-1} \right) \\ & \mathcal{L} := -\partial_R^2 + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2} \end{aligned}$$

**Note:**  $e_{2k-1}$  decays like  $\tau^{-N}$ .

**Two key issues:** Appearance of **generator of dilations**  $R\partial_R$  and the **strongly singular Schrödinger operator**  $\mathcal{L}$  on  $L^2(0, \infty)$  as part of the driving linear hyperbolic operator.

- Spectral, scattering theory, **Fourier transform**  $\mathcal{F}$  of  $\mathcal{L}$
- $\mathcal{F}$  hits (PDE), solve **transport equation** for  $\mathcal{F}\tilde{\varepsilon}$ ,  $R\partial_R \rightarrow \xi\partial_\xi$

**Spectral theory:** Our Sturm-Liouville operator is **singular** at both  $R = 0$  and  $R = \infty$ . Consider first **regular** case at  $R = 0$ :  $\tilde{\mathcal{L}} = -\frac{d^2}{dR^2} + V(R)$  with  $V(R) \in L^1(0, \infty)$  is **self-adjoint** subject to *Dirichlet BC at zero*, say. For  $\text{Im}z > 0 \exists$  **Weyl-Titchmarsh** solution  $\psi(R, z) \in L^2(0, \infty)$  - unique up to scalar multiples (**limit point case** at  $R = \infty$ ). Set  $\psi(0, z) = 1$ , and write

$$\psi(R, z) = \theta(R, z) + m(z)\phi(R, z)$$

where  $\mathcal{L}\theta(\cdot, z) = z\theta(\cdot, z)$ ,  $\mathcal{L}\phi(\cdot, z) = z\phi(\cdot, z)$  and

$$\theta(0, z) = \phi'(0, z) = 1, \quad \theta'(0, z) = \phi(0, z) = 0$$

Then  $m(z)$  *Herglotz*. **Fourier inversion** (where  $\chi_{(0, \infty)}(\mathcal{L})f = f$ )

$$\hat{f}(\xi) := \int_0^\infty f(R)\phi(R, \xi) dR, \quad f(R) = \int_0^\infty \hat{f}(\xi)\phi(R, \xi)\rho(\xi) d\xi$$

with **spectral measure**  $\rho(d\xi) := \pi^{-1}\text{Im}m(\xi + i0)d\xi$ . **Free case:**  $V = 0, \psi = e^{iR\xi^{\frac{1}{2}}}, \theta = \cos(R\xi^{\frac{1}{2}}), \phi = \xi^{-\frac{1}{2}} \sin(R\xi^{\frac{1}{2}}), m(\xi) = i\xi^{\frac{1}{2}}$

**Singular case:**  $\mathcal{L}_0 = -\frac{d^2}{dR^2} + \frac{3}{4R^2}$ ,  $\mathcal{L}_0 R^{\frac{3}{2}} = \mathcal{L}_0 R^{-\frac{1}{2}} = 0$ ,  
 self-adjoint **without BC** at  $R = 0$ , *limit point case*. For  
 $\mathcal{L} = \mathcal{L}_0 - \frac{8}{(1+R^2)^2}$  have  $\mathcal{L}\phi_0 = \mathcal{L}\theta_0 = 0$ ,  $W(\theta_0, \phi_0) = 1$ , where

$$\phi_0(R) := \frac{R^{\frac{3}{2}}}{1 + R^2}, \quad \theta_0(R) := \frac{1 - 4R^2 \log R - R^4}{2R^{\frac{1}{2}}(1 + R^2)}$$

$\forall \text{Im}z > 0 \exists$  **fund. systems**  $\phi(R, z)$ ,  $\theta(R, z)$ , and  $\psi(R, z), \overline{\psi(R, z)}$   
 of  $\mathcal{L}f = zf$ . Former **entire** in  $z$ ,  $W(\theta(\cdot, z), \phi(\cdot, z)) = 1$ , and

$$\phi(R, z) = \phi_0(R) + R^{-\frac{1}{2}} \sum_{j=1}^{\infty} (R^2 z)^j \phi_j(R)$$

**Fourier inversion:** as above with “**Weyl-Titchmarsh**” function

$$m(z) = \frac{W(\theta(\cdot, z), \psi(\cdot, z))}{W(\psi(\cdot, z), \phi(\cdot, z))}$$

Proved by **Gesztesy-Zinchenko 05**

For  $\mathcal{L}_0 = -\frac{d^2}{dR^2} + \frac{n^2-1/4}{R^2}$  this amounts to **Bessel functions**:

$$\phi(R; z) = \frac{\pi}{2} C^{-1} z^{-n/2} R^{1/2} J_n(z^{1/2} R)$$

$$\theta(R; z) = C z^{n/2} R^{1/2} [-Y_n(z^{1/2} R) + \pi^{-1} \log(z) J_n(z^{1/2} R)]$$

$$\psi(R; z) = C z^{n/2} R^{1/2} [-Y_n(z^{1/2} R) + i J_n(z^{1/2} R)]$$

$$= C z^{n/2} R^{1/2} i H_n^{(1)}(z^{1/2} R)$$

$$= \theta(R; z) + m(z) \phi(R; z)$$

$$m(z) = C^2 \frac{2}{\pi} z^n [i - \pi^{-1} \log(z)], \quad z \in \mathbb{C} \setminus \mathbb{R}^+$$

For our  $\mathcal{L} = -\frac{d^2}{dR^2} + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2}$  obtain

$$\rho(\xi) \asymp \frac{\chi_{[\xi \leq 1]}}{\xi \log^2 \xi} + \xi \chi_{[\xi \geq 1]}$$

**Singularity** at  $\xi = 0$  due to zero energy **resonance**.

**Transference identity:** With  $\mathcal{F}$  the Fourier transform of  $\mathcal{L}$ ,

$$\mathcal{F}\left(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R\right) = \left(\partial_\tau + \frac{\lambda_\tau}{\lambda} (-2\xi\partial_\xi + \mathcal{K})\right)\mathcal{F}$$

where  $\mathcal{K} = -\left(\frac{3}{2} + \frac{\eta\rho'(\eta)}{\rho}\right)\delta(\xi - \eta) + \frac{\rho(\xi)}{\xi - \eta}F(\xi, \eta)$ , and

$$|F(\xi, \eta)| \lesssim \langle \xi + \eta \rangle^{-\frac{3}{2}} (1 + |\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}|)^{-N}$$

**Point:**  $(R\partial_R - 2\xi\partial_\xi)e^{iR\xi^{\frac{1}{2}}} = 0$  and  $\mathcal{K}$  is the **error** when applied to  $\phi(R, \xi)$  rather than  $e^{iR\xi^{\frac{1}{2}}}$ . **Boundedness property**  $\forall \alpha$

$$\begin{aligned} \mathcal{K}_0(\xi, \eta) &:= \frac{\rho(\xi)}{\xi - \eta} F(\xi, \eta) : L_\rho^{2,\alpha} \rightarrow L_\rho^{2,\alpha+1/2} \\ [\mathcal{K}_0, \xi\partial_\xi] &: L_\rho^{2,\alpha} \rightarrow L_\rho^{2,\alpha} \end{aligned}$$

with  $\|f\|_{L_\rho^{2,\alpha}}^2 = \int_0^\infty |f(\xi)|^2 \langle \xi \rangle^{2\alpha} \rho(\xi) d\xi$ .

**Final iteration:** Apply **Fourier transform**  $\mathcal{F}$  to main **(PDE)**

$$\begin{aligned}
& - \left( \partial_\tau - 2\beta\xi\partial_\xi \right)^2 x - \xi x = 2\beta\mathcal{K} \left( \partial_\tau - 2\beta\xi\partial_\xi \right) x + \beta^2 (\mathcal{K}^2 + 2[\xi\partial_\xi, \mathcal{K}]) x \\
& \quad - \left( \frac{\beta^2}{4} + \frac{\dot{\beta}}{2} \right) x + \lambda^{-2} \mathcal{F} R^{\frac{1}{2}} (N_{2k-1} (R^{-\frac{1}{2}} \mathcal{F}^{-1} x) + e_{2k-1})
\end{aligned}$$

where  $\beta = \frac{\lambda_\tau}{\lambda}$ . We solve this with **zero terminal conditions**. I.e., let  $H(\tau, \sigma)$  be the **backward fundamental solution** for the operator

$$\left( \partial_\tau - 2\frac{\lambda_\tau}{\lambda} \xi \partial_\xi \right)^2 + \xi$$

and by  $H(\tau, \sigma)$  its kernel,  $x(\tau) = \int_\tau^\infty H(\tau, \sigma) f(\sigma) d\sigma$ . **Then**  
 $\forall \alpha \geq 0 \exists C = C(\alpha)$  **large so that uniformly in**  $\sigma \geq \tau$

$$\begin{aligned}
& \|H(\tau, \sigma)\|_{L_\rho^{2,\alpha} \rightarrow L_\rho^{2,\alpha+1/2}} \lesssim \tau \left( \frac{\sigma}{\tau} \right)^C \\
& \left\| \left( \partial_\tau - \frac{\lambda_\tau}{\lambda} 2\xi \partial_\xi \right) H(\tau, \sigma) \right\|_{L_\rho^{2,\alpha} \rightarrow L_\rho^{2,\alpha}} \lesssim \left( \frac{\sigma}{\tau} \right)^C
\end{aligned}$$

**Main linear estimate, small Lipschitz constant:** Define

$$\|f\|_{L^\infty, N L_\rho^{2, \alpha}} := \sup_{\tau \geq 1} \tau^N \|f(\tau)\|_{L_\rho^{2, \alpha}}$$

*Given  $\alpha \geq 0$  let  $N$  be **large enough**. Then*

$$\|Hb\|_{L^\infty, N-2 L_\rho^{2, \alpha+1/2}} + \left\| \left( \partial_\tau - 2\beta\xi\partial_\xi \right) Hb \right\|_{L^\infty, N-1 L_\rho^{2, \alpha}} \lesssim \frac{1}{N} \|b\|_{L^\infty, N L_\rho^{2, \alpha}}$$

To **close the loop**, it remains to show this:

*Assume that  $N$  is large enough and  $\frac{\nu}{2} + \frac{3}{4} > \alpha > \frac{1}{4}$ . Then the map*

$$x \rightarrow \lambda^{-2} \mathcal{F} R^{\frac{1}{2}} (N_{2k-1} (R^{-\frac{1}{2}} \mathcal{F}^{-1} x))$$

*is **locally Lipschitz** from  $L^\infty, N-2 L_\rho^{2, \alpha+1/2}$  to  $L^\infty, N L_\rho^{2, \alpha}$ .*

In order to **lower**  $\nu > \frac{1}{2}$  to  $\nu > 0$ , say, would need to **improve** on this nonlinear estimate. Need  $\frac{\nu}{2} > \alpha > \frac{1}{4}$  due to **regularity** of  $e_{2k-1}$  from **renormalization step**.

**The semilinear case** Similar scheme, but some major differences:

- No "cuspidal result" as **Struwe's theorem** expected for (**SLWE**); in fact, blow-up could occur on complicated set. Currently **little intuition** why  $\lambda(t) = t^{-1-\nu}$  rates appear. Role of **criticality**? Compare work of **Merle-Zaag** for (**SLWE**) and **Merle-Raphael, G. Perelman** for  $L^2$  critical (**NLS**).
- **Spectrum** of linearized operator  $-\Delta - 5W^4$  (restricted to  $L^2_{\text{rad}}$ ) is  $\{-k_0^2\} \cup [0, \infty)$  with a *zero energy resonance*. Need to kill the *exponentially growing mode*; heuristically (!?), tie eval  $-k_0^2$  to stable, **self-similar** blow-up rate  $\lambda(t) = t^{-1}$  (with corrections), whereas the **resonance (by itself)** leads to all the *slow rates* that we observed. Since for (**WM**),  $\mathcal{L}_Q =$  linearized operator has *purely continuous spectrum* provided  $\deg(Q) = 1$ , perhaps no **generic rate** for (**WM**) relative to the **ground state (HM)**. **Note:** If  $\deg(Q) > 1$ , then  $\mathcal{L}_Q$  has zero eigenvalue!! **Rodnianski-Sterbenz 06** obtain stable rate  $\lambda(t) \gtrsim t^{-1} \sqrt{-\log t}$ .

- In **(WM)** the scaling is  $Q(\lambda r)$  whereas for **(SLWE)** it is  $\lambda^{\frac{1}{2}}W(\lambda r)$ . Difference in the nonlinearity:  $\frac{\sin(2u)}{2r^2}$  versus  $u^5$ . The former comes from  $\square\phi = \phi(|\nabla\phi|^2 - |\phi_t|^2)$ , the latter has no gradient;  $\lambda^{\frac{1}{2}}$  in front of  $W$  produces same scaling as  $r^{-2}$ .
- The role of the **zero energy resonance** in the formation of **blow-up** not understood. It appears in both **(WM)** and **(SLWE)**. In  $4 + 1$ -dim. energy critical **Yang-Mills**, however, the linearized operator has a **zero energy eigenvalue** which appears to destroy the **renormalization procedure**:

$$\partial_{tt}u - \Delta_{\mathbb{R}^2}u - \frac{2}{r^2}u(1 - u^2) = 0$$

$$Q(r) = \frac{1 - r^2}{1 + r^2}, \quad rQ'(r) = -4\frac{r^2}{(1 + r^2)^2}$$

$$\mathcal{L} = -\frac{d^2}{dR^2} + \frac{15}{4R^2} - \frac{24}{(1 + R^2)^2}, \quad \mathcal{L}R^{-\frac{3}{2}} = \mathcal{L}R^{\frac{5}{2}} = 0$$

Perhaps *slow blow-up* does not occur here !?