Consider $f, g \in L^\infty(\mathbb{T})$ functions on the circle. Suppose $|\hat{f}(n)| \leq \hat{g}(n)$ for all $n \in \mathbb{Z}$ ($g$ majorizes $f$). Then $\|f\|_2 \leq \|g\|_2$. But also $\|f\|_{2k} \leq \|g\|_{2k}$ for all positive integers $k$. Follows from: $\|f\|_{2k}^{k} = \|f^k\|_2$, $\|g\|_{2k}^{k} = \|g^k\|_2$, and

$$|\hat{f}^k(\nu)| = \left| \sum_{\nu_1 + \ldots + \nu_k = \nu} \hat{f}(\nu_1) \hat{f}(\nu_2) \ldots \hat{f}(\nu_k) \right| \leq \sum_{\nu_1 + \ldots + \nu_k = \nu} \hat{g}(\nu_1) \hat{g}(\nu_2) \ldots \hat{g}(\nu_k) = \hat{g}^k(\nu)$$

for all $\nu \in \mathbb{Z}$. Note that one cannot change the condition to $|\hat{f}(n)| \leq |\hat{g}(n)|$. By symmetry, this would amount to saying that putting absolute values on the Fourier coefficients would not change the $L^p$-norm by more than a multiplicative constant. Impossible by Khinchin’s inequality!
Basic Question: For which $1 \leq p \leq \infty$ does $|\hat{f}(n)| \leq \hat{g}(n)$ for all $n \in \mathbb{Z}$ imply that $\|f\|_p \leq C_p \|g\|_p$ with some constant $C_p$ that only depends on $p$?

Easy observation: Answer is “No” for $1 \leq p < 2$. Take $\hat{g} = \chi_{[0,N]}$ and $\hat{f}(n) = \pm 1$ randomly, independently with $\mathbb{P} = \frac{1}{2}$ each, $\hat{f}(n) = 0$ if $n \notin [0,N]$. Then by Khinchin $\|f\|_p \geq c_p \|f\|_2 = c_p \|g\|_2$ with large probability, constant $c_p$ independent of $N$. But $\|g\|_p/\|g\|_2 \to 0$ as $N \to \infty$.

Thus Hardy-Littlewood only asked for the range $2 \leq p \leq \infty$. They also realized that $C_p > 1$ if $p = 3$, later extended to $p > 2$ not an even integer by R. Boas. Finally, Katznelson realized that this implies $C_p = \infty$ if $p \notin 2\mathbb{Z}$. 
Take $p = 3$, $g(z) = 1 + z + z^3$, $f(z) = 1 + z - z^3$, $z = re^{2\pi i \theta} = re(\theta)$. Since

$$(1 + z \pm z^3)^{3/2} = 1 + \binom{3/2}{1}z + \binom{3/2}{2}z^2 + \left[\binom{3/2}{3} \pm \binom{3/2}{1}\right]z^3 + O(z^4).$$

Punch line: \( \binom{3/2}{3} < 0 \). Thus Plancherel implies $\|g(re(\cdot))\|_3 < \|f(re(\cdot))\|_3$ provided $0 < r < 1$ is small enough. Similar argument for all $2 < p \notin 2\mathbb{Z}$.

Katznelson realized that thus $C_p = \infty$ for all $2 < p \notin 2\mathbb{Z}$: Suppose $\|f\|_p > (1 + \delta)\|g\|_p$ for some trig polynomials $f, g \neq 0$, $|\hat{f}(n)| \leq \hat{g}(n)$ for $\forall n \in \mathbb{Z}$. Easy property of Riemann integrals:

$$\|f(\cdot)f(N\cdot)\|_{L^p(\mathbb{T})} \to \|f\|_{L^p(\mathbb{T})}^2$$

as $N \to \infty$, same for $g$. Also, $g(\cdot)g(N\cdot)$ majorizes $f(\cdot)f(N\cdot)$. So increase $(1 + \delta)$ to $(1 + \delta)^2$. Continue to get $C_p = \infty$. 

3
Quantified Majorant Problem (due to Gerd Mockenhaupt): For $p \geq 2$ and $E \subset [1, N]$ let

$$B_p(N, E) := \sup_{|a_n| \leq 1} \left\| \sum_{n \in E} a_n e(n\theta) \right\|_p / \left\| \sum_{n \in E} e(n\theta) \right\|_p$$

$$B_p(N) := \sup_{E \subset [1, N]} B_p(N, E).$$

We know that $B_p(N) \to \infty$ as $N \to \infty$ if $p \not\in 2\mathbb{Z}^+$. Find $\gamma_p := \limsup_{N \to \infty} \frac{\log B_p(N)}{\log N} \geq 0$. In particular, is $\gamma_p = 0$? If $\gamma_p = 0$, then $\forall \epsilon > 0 \exists C_\epsilon$ such that

$$(\ast) \quad \sup_{|a_n| \leq 1} \left\| \sum_{n \in E} a_n e(n\theta) \right\|_p \leq C_\epsilon N^\epsilon \left\| \sum_{n \in E} e(n\theta) \right\|_p$$

for all $E \subset [1, N]$.

$$(\ast) \text{ appears to be a hard problem!}$$

Mockenhaupt’s lower bound: $B_p(N) > \exp \left( C \frac{\log N}{\log \log N} \right)$, and there is an easy bound $\gamma_p \leq (1 - \frac{p}{4})(1 - \frac{2}{p})$ if $2 \leq p \leq 4$ by “interpolation”.
One can (and should) formulate the same question on finite Abelian groups $G$ instead of $[1, N]$, with $E \subset \hat{G}$ an arbitrary set of characters. Since $B^p(G \times H) \geq B^p(G)B^p(H)$, one has for $G = (\mathbb{Z}/5\mathbb{Z})^N$, say, $B^p(G) \geq |G|^\varepsilon_p$ with some $\varepsilon_p > 0$ for $2 < p < 4$. This uses that $B^p(\mathbb{Z}_5) > 1$ (by explicit example). In other words, $\gamma_p > 0$ for $2 < p < 4$ in this case!

**Significance of $N^\varepsilon$:** If true, then this would imply the restriction and therefore the Kakeya conjectures in harmonic analysis. For $n \geq 2$, $E \subset \mathbb{R}^n$ is a **Kakeya set** (or **Besicovitch set**) provided for every $\vec{v} \subset S^{n-1}$ there exists $y = y(\vec{v})$ so that

$$y + t\vec{v} \in E \quad \text{for} \quad \forall \ 0 \leq t \leq 1.$$

**Classical fact:** $\exists E$ with meas$(E) = 0$. If $n = 2$, then dim$(E) = 2$. For $n \geq 3$ it is unknown if dim$(E) = n$, only partial results known. More on this later.
• By Hausdorff-Young (since $p \geq 2$):

$$\sup_{|a_n| \leq 1} \left\| \sum_{n \in E} a_n e(n\theta) \right\|_p \leq |E|^{\frac{1}{p}},$$

whereas clearly

$$\left\| \sum_{n \in E} e(n\theta) \right\|_p \geq c|E|N^{-\frac{1}{p}}.$$  

Hence

$$B_p(N, E) \leq C \left( \frac{N}{|E|} \right)^{\frac{1}{p}}.$$  

So (*) only interesting for $|E| \ll N$. Hausdorff-Young also settles the case of any arithmetic progression (AP) \{b + aj : 0 \leq j < L \} \subset [1, N].

• Let $E := \{j^2 : 1 \leq j \leq N\}$, $S_N(\theta) = \sum_{j=1}^{N} e(j^2\theta)$. Then $\sqrt{N} \leq \|S_N\|_p \leq C_\varepsilon N^\varepsilon \sqrt{N}$ if $2 \leq p \leq 4$ and $\|S_N\|_p \sim N^{1-\frac{2}{p}}$ (up to $N^\varepsilon$) if $4 \leq p \leq \infty$. Reason for the “kink” at $p = 4$:

$$S_N(\theta)^2 = N + \sum_{\ell \neq 0} \sum_{j,k=1}^{N} \chi_{[j^2-k^2=\ell]} e(\ell\theta).$$

Inner sum is no bigger than $C_\varepsilon N^\varepsilon$. Interpolation between $2 \leq p \leq 4$, and squaring+H.-Y. in the range $4 \leq p \leq \infty$ establish (*).
• Suppose $p > 2$ fixed and $E \subset [1, N]$ satisfies

$$(\dagger) \quad \| \sum_{n \in E} a_n e(n\theta) \|_p \leq K \left( \sum_{n \in E} |a_n|^2 \right)^{\frac{1}{2}}.$$ 

Then this implies that

$$\sup_{|a_n| \leq 1} \| \sum_{n \in E} a_n e(n\theta) \|_p \leq K \| \sum_{n \in E} e(n\theta) \|_p.$$ 

But this property is rare: An set $\Lambda \subset \mathbb{Z}$ such that $\|f\|_p \leq C\|f\|_2$ for all $f$ with $\text{spec}(f) \subset \Lambda$ is called $\Lambda_p$-set. They satisfy

$$|\Lambda \cap [1, N]| \lesssim N^{\frac{2}{p}}, \quad \text{and also} \quad |E| \lesssim K N^{\frac{2}{p}}.$$ 

Bourgain (about 1989) showed that most random subsets of $[1, N]$ of size about $N^{\frac{2}{p}}$ satisfy $(\dagger)$. Let $E_j \subset [2^j, 2^{j+1})$ be such a random set and $\Lambda := \bigcup_j E_j$. Littlewood-Paley $\implies$ $\Lambda$ is $\Lambda_p$ but not $\Lambda_q$ for $q > p$. The only examples for general $q > p > 2$ are these random ones. If $p \in 2\mathbb{Z}$, W. Rudin found explicit examples.
A special case of (*) is the following: For $\forall \epsilon > 0$ there is $C_\epsilon > 0$ such that

\[ (*\star) \quad \sup_{B \subseteq E} \left\| \sum_{n \in B} e(n\theta) \right\|_p \leq C_\epsilon N^\epsilon \left\| \sum_{n \in E} e(n\theta) \right\|_p \]

for $\forall E \subseteq [1, N]$. For fixed $E$ a typical subset $B \subseteq E$ will satisfy (*\star): Take $\{r_n\}_{n \in E}$ to be i.i.d. with $\mathbb{P}[r_n = 1] = \mathbb{P}[r_n = -1] = \frac{1}{2}$ and set $B = \{n \in E : r_n = 1\}$. By Khinchin and $p \geq 2$,

\[
\left\| \sum_{n \in B} e(n\theta) \right\|_p \leq 2 \left\| \sum_{n \in E} e(n\theta) \right\|_p + \left\| \sum_{n \in E} r_n e(n\theta) \right\|_p \\
\leq 2 \left\| \sum_{n \in E} e(n\theta) \right\|_p + C_p \left\| \sum_{n \in E} r_n e(n\theta) \right\|_2 \\
\leq C \left\| \sum_{n \in E} e(n\theta) \right\|_p .
\]

The main result of this talk will be to show that (*\star) holds for typical $A$ and all $B$.

(*) has number theoretic implications that will not be discussed here, see Montgomery’s book.
Main result of this talk (joint work with Gerd Mockenhaupt): **Random subsets** $E \subset [1, N]$ **satisfy the majorant property** $(\ast)$ **and thus** $(\ast\ast)$.

**Theorem 1.** Let $0 < \delta < 1$ be fixed. For every positive integer $N$ we let $\xi_j = \xi_j(\omega)$ be i.i.d. variables with $\mathbb{P}[\xi_j = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$ where $\tau = N^{-\delta}$. Define a random subset

$$S(\omega) = \{j \in [1, N] \mid \xi_j(\omega) = 1\}.$$

Then for every $\varepsilon > 0$ and $2 \leq p \leq 7$ one has

$$(\dagger) \quad \mathbb{P}\left[ \sup_{|a_n| \leq 1} \left\| \sum_{n \in S(\omega)} a_n e(n\theta) \right\|_{L^p(\mathbb{T})} \geq N^\varepsilon \left\| \sum_{n \in S(\omega)} e(n\theta) \right\|_{L^p(\mathbb{T})} \right] \to 0$$

as $N \to \infty$. Moreover, under the additional restriction $\delta \leq \frac{1}{2}$, $(\dagger)$ holds for all $p \geq 7$. 
Comments:

- Bourgain’s theorem on the $\Lambda_p$ property of random sets implies Theorem 1 if $\delta \geq 1 - \frac{2}{p}$ but not for smaller $\delta$. Theorem 1 applies also to random subsets of certain “base-sets” whose Dirichlet kernel satisfy a reverse interpolation inequality.

- Point of Theorem 1: The connection with restriction suggests that the quantitative majorant property with $N^\varepsilon$, i.e., property ($\star$) holds for certain arithmetic sets $E \subset [1, N]$. It is conceivable that it fails for disordered sets $E$. Theorem 1 shows that this is not so.

- It turns out that the majorant property is also stable under random perturbations:
Let $\mathcal{P} \subset [1, N]$ be an arithmetic progression of length $L$ and set $S(\omega) := \{j + \xi_j(\omega) \mid j \in \mathcal{P}\}$ where $\xi_j$ are integer valued i.i.d. variables, uniform in $[-s, s]$.

**Theorem 2.** For every $\varepsilon > 0$ and $4 \geq p \geq 2$ one has

$$
P \left[ \sup_{|a_n| \leq 1} \left\| \sum_{n \in S(\omega)} a_n e(n\theta) \right\|_{L^p(\mathbb{T})} \right] \geq N^{\varepsilon} \left\| \sum_{n \in S(\omega)} e(n\theta) \right\|_{L^p(\mathbb{T})} \rightarrow 0$$

as $N \to \infty$. Moreover, under the additional restriction $L \geq s$, this holds for all $p \geq 4$.

The restrictions $L \geq s$ in Theorem 2 and $\delta < \frac{1}{2}$ in Theorem 1 appear to be of a purely technical nature. The results should be true in general. The proofs of both theorems are based on a technique that Bourgain developed to construct random $\Lambda_p$ sets. Before discussing the proof, we return to the connection with the restriction problem.
Let $\sigma_d$ be surface measure on $S^{d-1}$, $d \geq 2$. Then

$$|\widehat{\sigma_d}(\xi)| \leq C|\xi|^{-\frac{d-1}{2}} \implies \widehat{\sigma_d} \in L^p(\mathbb{R}^d)$$

for $p > \frac{2d}{d-1}$. **Stein’s restriction conjecture:** $\widehat{\phi \sigma_d} \in L^p(\mathbb{R}^d)$ for $p > \frac{2d}{d-1}$ and all $\phi \in L^\infty(S^{d-1})$. This would imply the **Kakeya conjecture:** $\dim(E) = d$ for any Besicovitch set $E \subset \mathbb{R}^d$. Case $d = 2$ easy:

$$\|\widehat{\phi \sigma_2}\|_{L^4(B_R)}^2 \lesssim \|\widehat{\chi_R} * \phi \sigma_2 * \phi \sigma_2\|_2$$

$$\leq \|\widehat{\chi_R} * \sigma_2 * \sigma_2\|_2 \lesssim \|\sigma_2\|_{L^4(B_R)}^2 \lesssim \sqrt{\log R},$$

where $\chi_R$ is a suitable cut-off ($\overline{\chi_R} \geq 0$). This is already enough to show that $\dim(E) = 2$! Take $\phi = \sum_j \pm e^{2\pi i \xi_j} \chi_j$ where $\chi_j$ are smooth, disjoint $R^{-\frac{1}{2}}$-caps on $S^1$ and $\pm$ are random signs. By a result of Tao’s one can remove the log $R$-term by raising 4 to $p > 4$. Observe $d = 2$ uses the **majorant argument** for $p = 4$. If $d > 2$ we would need the majorant property at $2 < p = \frac{2d}{d-1} < 4$. 


Deduction of the higher-dimensional problem in $\mathbb{R}^d$ from the one-dim. majorant property $(\ast)$:

$$\int_{S^{d-1}} |\hat{f}(x)| \, d\sigma_d(x) \leq C \|f\|_{L^p(\mathbb{R}^d)} \iff (Res) \sum_{R<|m|<R+1} |\hat{f}(m)| \leq C R^{d-1} \|f\|_{L^p(\mathbb{T}^d)}$$

as $R \to \infty$. Let $Q(x) = \sum_{0 \leq m_i < N_i} a_m e^{2\pi i m \cdot x}$ be a trigonometric polynomial on $\mathbb{T}^d$, $Q_N$ restriction of $Q$ to $\{(m_1/N_1, \ldots, m_d/N_d) : 0 \leq m_i < N_i\}$; function on $G = \mathbb{Z}^d/(N_1 \mathbb{Z} \times \ldots N_d \mathbb{Z})$. Classical fact:

for $1 < p < \infty$: $\|Q_N\|_{L^p(G_N)} \sim \|Q\|_{L^p(\mathbb{T}^d)}$.

If $N_1, \ldots, N_d$ relatively prime, then $G \asymp \mathbb{Z}_N$, $N = N_1 \ldots N_d$. Let $E \subset \hat{G}$, $E^* = \psi^*(E)$ the dual image:

$$\sup_{|a_n| \leq 1} \left\| \sum_{m \in E} a_m e^{2\pi i m \cdot x} \right\|_{L^p(\mathbb{T}^d)} \sim \sup_{|c_n| \leq 1} \left\| \sum_{k \in E^*} c_k e^{2\pi i k t} \right\|_{L^p(\mathbb{T})}.$$

Let $R$ be large, $N_i \sim R$ relatively prime,

$$E := \{m \in \hat{G} : R < |m| < R + 1\}.$$
Dual of \((\text{Res})\): For \(p > \frac{2d}{d-1}\)

\[
\sup_{|a_n| \leq 1} \left\| \sum_{m \in E} a_m e^{2\pi i m \cdot x} \right\|_{L^p(\mathbb{T}^d)} \lesssim R^{\frac{d}{p}-1}.
\]

Switch to \(E^*\): for \(p > \frac{2d}{d-1}\)

\[
\sup_{|c_n| \leq 1} \left\| \sum_{n \in E^*} c_n e^{2\pi i n t} \right\|_{L^p(\mathbb{T})} \lesssim R^{\frac{d}{p}-1}.
\]

If the majorant property \((\ast)\) holds, then this will follow (up to \(R^\varepsilon\)) from

\[
\left\| \sum_{n \in E^*} e^{2\pi i n t} \right\|_{L^p(\mathbb{T})} \lesssim R^{\frac{d}{p}-1},
\]

which is true! Indeed: By reversing the previous steps this is the same as

\[
(ES) \quad \left\| \sum_{R < |m| < R+1} e^{2\pi i m \cdot x} \right\|_{L^p(\mathbb{T}^d)} \lesssim R^{\frac{d}{p}-1},
\]

which can be verified explicitly for \(p > \frac{2d}{d-1}\). Conclusion: If \((\ast)\) holds, then \(\| \widehat{\phi \sigma_d} \|_{L^p(B_R)} \leq C_\varepsilon R^\varepsilon\) if \(\phi \in L^\infty(S^{d-1})\), \(p > \frac{2d}{d-1}\). Tao’s \(\varepsilon\)-removal argument allows one to get rid of \(R^\varepsilon\).
As far as the estimate (ES) on the exponential sum is concerned, we motivate it by means of the following: Let \( \eta \geq 0 \) be a smooth bump adapted to the unit ball in \( \mathbb{R}^d \), \( \sigma_R \) the surface measure on \( R S^{d-1} \), and \( f_R := \sigma_R * \eta \). Then by Poisson summation

\[
\sum_{m \in \mathbb{Z}^d} \hat{f}_R(m)e(mx) = \sum_{\nu \in \mathbb{Z}^d} f_R(x - \nu)
\]

(1)

\[
= \sum_{\nu \in \mathbb{Z}^d} \overline{\sigma}_R(x - \nu) \hat{\eta}(x - \nu)
\]

(2)

Note that the left-hand side of (1) is a “smoothed out version” of \( \sum_{R^d}^{1} e(mx) \). Recall the decay estimate \( |\overline{\sigma}_R(x)| \lesssim R^{d-1}(1 + R|x|)^{-\frac{d-1}{2}} \). The most significant term in (2) is \( \nu = 0 \) which gives the contribution

\[
R^{d-1} R^{-\frac{d}{p}} + R^{\frac{d-1}{2}}
\]

to the \( L^p(\mathbb{T}^d) \) norm of (1). These terms balance at \( p = \frac{2d}{d-1} \), as claimed.
Main steps in the proof of Theorem 1:

1) Estimate on the expected value of suprema of certain random processes: Given i.i.d. Bernoulli variables \( \{\xi_j\}_{j=1}^N \) control \( \mathbb{E}\left[\sup_{x \in \mathcal{E}} \xi_j x_j \right] \) where \( \mathcal{E} \subset \mathbb{R}_+^N \). It turns out that this expectation depends on the \( L^2 \)-entropy of \( \mathcal{E} \), i.e., the size of \( \varepsilon \)-nets for \( \mathcal{E} \) in \( \ell_2^N \) for all \( \varepsilon > 0 \).

2) Decoupling of random variables: \( \sum_{j=1}^N \xi_j x_j \) and \( x_j \) itself depends on \( \xi_1, \ldots, \xi_N \). Need to decouple in order to apply the previous step.

3) Entropy estimates via the dual Sudakov inequality: If \( B^n \) is the unit ball in \( \ell_2^n \) and \( B_X \) is the unit-ball of a (semi)norm \( \| \cdot \|_X \) in \( \mathbb{R}^n \), estimate the smallest number of \( tB_X \) balls needed to cover \( B^n \). Need to get the optimal dependence on the dimension \( n \).
Lemma 1. Let $\mathcal{E} \subset \mathbb{R}^N_+$, $B = \sup_{x \in \mathcal{E}} |x|_{\ell^2_N}$, and $\xi_j$ be selector variables as above with $\mathbb{P}[\xi_j = 1] = \tau$, $\mathbb{P}[\xi_j = 0] = 1 - \tau$, and $0 < \tau < 1$ arbitrary. Let $1 \leq m \leq N$. Then

$$
\mathbb{E} \sup_{x \in \mathcal{E}, |A| = m} \left[ \sum_{j \in A} \xi_j x_j \right] \lesssim (\tau m + 1)^{\frac{1}{2}} B + 
\int_0^B \sqrt{\log N_2(\mathcal{E}, t)} \, dt
$$

where $N_2$ refers to the $L^2$ entropy.

Related to the Dudley-Fernique estimate for suprema of Gaussian processes, see book by Ledoux, Talagrand.

As motivation consider two extreme choices of $\mathcal{E}$ with $m = \tau N$, which is the most important case:

1) $\mathcal{E}$ consists of a single constant sequence. Then the first term dominates the entropy integral ($= 0$),

$$
\mathbb{E} \sup_{|A| = m} \sum_{j \in A} \xi_j x_j \sim \frac{B}{\sqrt{N} m} = \tau \sqrt{N} B.
$$
2) $\mathcal{E} = \{\chi_A : \vert A \vert = m, A \subset [1, N]\}$. Then $B = \sqrt{m}$, $N_2(\mathcal{E}, 1) \sim \binom{N}{m} \sim N^m$, $N_2(\mathcal{E}, B) \sim 1$. So integral contributes $\sqrt{m \log N} \sqrt{m} \sim m \sqrt{\log N}$. On the other hand,

$$\mathbb{E} \sup_{|A| = m, x \in \mathcal{E}} \sum_{j \in A} \xi_j x_j \sim m,$$

whereas $(\tau m + 1)^{\frac{1}{2}} B \sim \tau m \ll m$. So here the entropy integral dominates the first term.

**Entropy estimates, dual of Sudakov:**

$E(B^n, B_X, t) =$ smallest number of $tB_X$ balls needed to cover the Euclidean ball $B^n$ in $\mathbb{R}^n$. Then the dual Sudakov inequality is

$$(DS) \quad \log E(B^n, B_X, t) \leq C n \left( \frac{M_X}{t} \right)^2$$

with an absolute constant $C$. The *Levy mean* $M_X$ is the average $\| \cdot \|_X$-size of a normalized Gaussian vector:
\[ M_x = \int_{S_{d-1}} \|x\| \, d\sigma_{d-1}(x) = \alpha_n \int_{\Omega} \left\| \sum_{i=1}^{n} g_i(\omega) \tilde{e}_i \right\|_X \, d\mathbb{P}(\omega), \]

where \( \alpha_n \asymp n^{-\frac{1}{2}} \) is some explicit constant. The point of \((DS)\) is that the dimensional dependence is optimal! But as \( t \to 0 \) this is a terrible estimate.

Easy to get the correct behavior \( \log \frac{1}{t} \) by means of “rescaling” from \((DS)\).

Most relevant special case for us: \( X = \ell^q_n \). Then from Khinchin one obtains that \( M_X \sim q \), so that \((DS)\) “plus rescaling” yields in this case

\[ \log E(B^n, B_{\ell^q_n}, t) \leq C nq \left( 1 + \log \frac{1}{t} \right) \text{ if } 0 < t < \frac{1}{2}. \]

**Strategy of the proof of Theorem 1:** First one shows (easily) that with large probability

\[ \left\| \sum_{n=1}^{N} \xi_j e(j\theta) \right\|_p^p \sim \tau^p N^{p-1} + (\tau N)^{\frac{p}{2}}. \]
Second, one needs to prove that 
\[ \mathbb{E} \sup_{|a_n| \leq 1} \left\| \sum_{n=1}^{N} a_n \xi_n e(n\theta) \right\|_p^p \] does not exceed this up to possibly \( N^\varepsilon \). Take \( p = 3 \) and use that 
\[ |z|^3 = \bar{z}z|z|. \] Thus, if \( |a_n| \leq 1 \),
\[
\int_0^1 \left| \sum_{n=1}^{N} a_n \xi_n e(n\theta) \right|^3 d\theta = \\
\int_0^1 \sum_{n=1}^{N} \bar{a}_n \xi_n e(-n\theta) \sum_{k=1}^{N} a_k \xi_k e(k\theta) \left| \sum_{\ell=1}^{N} a_{\ell} \xi_{\ell} e(\ell\theta) \right| d\theta \\
\leq \sum_{n=1}^{N} \xi_n x_n, \quad \text{where } x_n = |\langle e(-n\theta), \bar{g}|g\rangle| \\
\] and \( g(\theta) = \sum_{k=1}^{N} a_k \xi_k e(k\theta) \). Hence use Lemma 1 to control \( \mathbb{E} \sup_{|a_n| \leq 1} \) of the first line. Problem: \( x_n \) are random and depend on the \( \xi_j \). One needs to apply a relatively standard decoupling scheme here that splits \( \xi_n = \xi_n(\omega) \) into two sets of independent renditions of the \( \xi \)'s, say \( \xi_n(\omega_1) \) and \( \xi_n(\omega_2) \).
This yields with $m = \tau N$:

$$
\ldots \lesssim m^3 + \mathbb{E}_{\omega_2} \mathbb{E}_{\omega_1} \sup_{x \in \mathcal{E}(\omega_2)} \sup_{|A| = m} \sum_{n \in A} \xi_n(\omega_1) x_n
$$

where

$$
\mathcal{E}(\omega_2) := \left\{ \left( \left| \langle e(n \cdot), \sum_{k=1}^{N} \overline{b_k} \xi_k(\omega_2) e(-k \cdot) \rangle \right| \right)_{n=1}^{N}, \sup_{1 \leq n \leq N} |b_n| \leq 1 \right\} \subset \mathbb{R}_+^N.
$$

Now we can apply Lemma 1. For starters, need to find the diameter of $\mathcal{E}(\omega_2)$. By Plancherel, this is the same as

$$
\mathbb{E}_{\omega_2} \sup_{|b_n| \leq 1} \left\| \sum_{n=1}^{N} b_n \xi_n(\omega_2) e(n\theta) \right\|_4^2
$$

$$
\leq \mathbb{E}_{\omega_2} \left\| \sum_{n=1}^{N} \xi_n(\omega_2) e(n\theta) \right\|_4^2 \sim \tau^2 N^\frac{3}{2} + \tau N,
$$

where we used the **majorant property on** $L^4$!

Thus the first term in Lemma 1 gives a contribution of $(\tau \sqrt{N} + 1)(\tau^2 N^\frac{3}{2} + \tau N) \leq \tau^3 N^2 + (\tau N)^\frac{3}{2}$ which is the desired bound!
Now for the entropy integral in Lemma 1. By Plancherel, the $\ell^2_N$ entropy of $\mathcal{E}(\omega_2)$ is the same as the $L^2(\mathbb{T})$ entropy of the set

$$\left\{ g | g | : g(\theta) = \sum_{\ell=1}^N b_\ell \xi_\ell(\omega_2) e(\ell \theta), \ |b_\ell| \leq 1 \right\}.$$

With large probability

$$\|g|g| - h|h|\|_2 \lesssim \|g - h\|_\infty (\|g\|_2 + \|h\|_2)$$

$$\lesssim N^\varepsilon \|g - h\|_q \sqrt{\tau N}$$

where we used Bernstein’s inequality to pass to a very large $q$ at the cost of $N^\varepsilon$. Moreover, we used that the Fourier support of $g, h$ is of size $\lesssim \tau N$ with large probability. This means that $N_2(\mathcal{E}(\omega_2), t)$ is controlled by the number of $L^q$ balls of radius $t/(N^\varepsilon \sqrt{\tau N})$ needed to cover the set of $g$ arising in $\mathcal{E}$. This is precisely what the dual Sudakov inequality gives us! One checks that the correct bound does indeed follow from this.


