

Dispersive Estimates for Charge transfer models

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Motivation for this talk about the *linear Schrödinger equation*: NLS with initial data = sum of several solitons (plus some small perturbation) which are far apart and do not approach each other at later times. We address the *asymptotic stability problem*:

$$(*) \quad i\partial_t\psi + \frac{1}{2}\Delta\psi + \beta(|\psi|^2)\psi = 0$$

in \mathbb{R}^d , $d \geq 3$. Initial data

$$\psi_0(x) = \sum_{j=1}^k w_j(0, x) + R_0(x).$$

Here w_j are solitons

$$w_j(t, x) = w(t, x; \sigma_j(0)) = e^{i\theta_j(t, x)} \phi(x - x_j(t), \alpha_j(0))$$

$$\theta_j(t, x) = v_j(0) \cdot x - \frac{1}{2}(|v_j(0)|^2 - \alpha_j^2(0))t + \gamma_j(0)$$

$$x_j(t) = v_j(0)t + D_j(0).$$

It is well-known that if $\phi = \phi(\cdot, \alpha)$ satisfies

$$(**) \quad \frac{1}{2} \Delta \phi - \frac{\alpha^2}{2} \phi + \beta(|\phi|^2) \phi = 0$$

then w_j satisfies (*) with arbitrary constant parameters $\sigma_j(0) = (v_j(0), D_j(0), \gamma_j(0), \alpha_j(0))$. Solutions of (**) are known to exist under suitable conditions on β . We need the *nonlinear stability condition* $\langle \partial_\alpha \phi, \phi \rangle > 0$.

Theorem 1. *Assume separation, nonlinear stability, and some spectral conditions. Then $\exists \epsilon > 0$ such that for R_0 satisfying*

$$\sum_{k=0}^s \|\nabla^k R_0\|_{L^1 \cap L^2} < \epsilon$$

for some integer $s > \frac{d}{2}$, there exists a path $\sigma(t)$ in \mathbb{R}^{2d+2} such that

$$\left\| \psi(t) - \sum_{j=1}^k w_j(t, x; \sigma_j(t)) \right\|_{\infty} \lesssim (1+t)^{-\frac{d}{2}}$$

as $t \rightarrow \infty$. The parameters $\sigma_j(t)$ converge as $t \rightarrow \infty$.

The proof is perturbative. Linearizing leads to the system for $\vec{R} = \begin{pmatrix} R \\ \bar{R} \end{pmatrix}$ ($R = R(t) = \text{perturbation}$)

$$(\dagger) \quad i\partial_t \vec{R} + \begin{pmatrix} \frac{1}{2}\Delta & 0 \\ 0 & -\frac{1}{2}\Delta \end{pmatrix} \vec{R} + \sum_{j=1}^k V_j(t, \cdot - \vec{v}_j t) \vec{R} = F$$

$$\vec{R}|_{t=0} = \vec{R}_0, \quad \text{where}$$

- \vec{v}_j are distinct vectors in \mathbb{R}^d (not the velocities of the initial solitons)

- V_j are matrix potentials

$$V_j(t, x) = \begin{pmatrix} U_j(x) & -e^{i\theta_j(t,x)} W_j(x) \\ e^{-i\theta_j(t,x)} W_j(x) & -U_j(x) \end{pmatrix},$$

- $\theta_j(t, x) = (|\vec{v}_j|^2 + \alpha_j^2)t + 2x \cdot \vec{v}_j + \gamma_j$

- U_j, W_j are exponentially decaying

- F contains all the rest: interaction terms between different solitons $O(w_i w_j)$ for $i \neq j$, fully nonlinear term $|R|^p$ (at least for nonlinearity $|\psi|^{p-1}\psi$), mixed terms, terms due to difference between straight path and time-dependent path (better considered as a *small* potential that perturbs the left-hand side).

Each moving matrix potential in (†) has a stationary counterpart

$$H_j = \begin{pmatrix} \frac{1}{2}\Delta - \frac{1}{2}\alpha_j^2 + U_j & -W_j \\ W_j & -\frac{1}{2}\Delta + \frac{1}{2}\alpha_j^2 - U_j \end{pmatrix}$$

via a modulation and Galilei transform. Scalar Galilean transformation:

$$(1) \quad \mathfrak{g}_{v,D}(t) = e^{-i\frac{|v|^2}{2}t} e^{-ix \cdot v} e^{i(D+tv)p}$$

where $p = -i\nabla$; analogue of $x \mapsto x - D - tv$, $p \mapsto p - v$.

Vector-valued version

$$\mathcal{G}_{v_j}(t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \mathfrak{g}_{v_j,0}(t) f_1 \\ \mathfrak{g}_{v_j,0}(t) f_2 \end{pmatrix}.$$

Suppose $\mathcal{S}_j(t)$ solves $i\partial_t \mathcal{S}_j(t) + V_j(t, \cdot - t\vec{v}_j) \mathcal{S}_j(t) = 0$,

$\mathcal{S}_j(0) = Id$. Then

$$\mathcal{S}_j(t) = \mathcal{G}_{v_j}(t)^{-1} \mathcal{M}_j(t)^{-1} e^{itH_j} \mathcal{M}_j(0) \mathcal{G}_{v_j}(0),$$

where for some linear function $\omega_j(t)$

$$\mathcal{M}_j(t) = \begin{pmatrix} e^{i\omega_j(t)} & 0 \\ 0 & e^{-i\omega_j(t)} \end{pmatrix}.$$

Our goal: Prove $L^1 \rightarrow L^\infty$ estimates for the system (†). Obstacle: “Traveling bound states”.

- If $H_\ell \vec{\psi} = E\vec{\psi}$, then $e^{itH_\ell} \vec{\psi} = e^{itE} \vec{\psi}$ and $\vec{R}(t) := \mathcal{G}_{v_\ell}(t)^{-1} \mathcal{M}_\ell(t)^{-1} e^{itE} \vec{\psi}$ solves

$$i\partial_t \vec{R} + \begin{pmatrix} \frac{1}{2}\Delta & 0 \\ 0 & -\frac{1}{2}\Delta \end{pmatrix} \vec{R} + \sum_{j=1}^k V_j(t, \cdot - \vec{v}_j t) \vec{R} = o(1).$$

So need to impose the condition that $\mathcal{G}_{v_\ell}(t) \mathcal{M}_\ell(t) \psi(t)$ has no component (relative to a suitable projection) on the eigenfunctions of H_j .

- Note that H_j is not selfadjoint. It turns out (because of symmetries of the nonlinear equation) that H_j has a nontrivial root-space at zero: $H_j \psi = \phi \neq 0$, $H_j \phi = 0$. Then $e^{itH_j} \psi = \psi + it\phi$. So we do not even have L^2 -boundedness here.

- spectral issues related to systems will be discussed later in more detail.

Start with the scalar homogeneous case in \mathbb{R}^d , $d \geq 3$:

$$(\ddagger) \quad i\partial_t\psi + \frac{1}{2}\Delta\psi + [V_1 + V_2(\cdot - \vec{v}t)]\psi = 0.$$

Graf proved *asymptotic completeness* of these **charge transfer models** around 1990 (more on this later). Here V_j are nice enough so that

$$(JSS) \quad \left\| e^{it(\frac{1}{2}\Delta + V_j)} P_j f \right\|_{\infty} \leq C |t|^{-\frac{d}{2}} \|f\|_1,$$

$1 - P_j$ = orthogonal projection onto the bound states of $\frac{1}{2}\Delta + V_j$. Estimate (JSS) was obtained by Journé, Soffer, Sogge. For (‡) with evolution $U(t)$ define $\psi_0 \in L^2(\mathbb{R}^d)$ to be a *scattering state* if

$$\|(1 - P_1)U(t)\psi_0\|_2 + \|(1 - P_2)g_{\vec{v}}(t)U(t)\psi_0\|_2 \rightarrow 0.$$

Theorem 2. *For such ψ_0 one has the decay*

$$\|U(t)\psi_0\|_{L^2 + L^\infty} \lesssim \langle t \rangle^{-d/2} \|\psi_0\|_{L^1 \cap L^2}.$$

Here $\langle t \rangle = (1 + t^2)^{-\frac{1}{2}}$ and $L^2 + L^\infty = (L^1 \cap L^2)^*$.

Norm in $L^2 + L^\infty$ is $\|f\|_{2+\infty} = \inf\{\|f_1\|_2 + \|f_2\|_\infty : f_1 + f_2 = f\}$.

Two easy estimates: (1) Decay for a single *small time-dependent* potential: $H_0 = \frac{1}{2}\Delta$,

$$e^{itH} = e^{itH_0} + i \int_0^t e^{i(t-s)H_0} V(s) e^{isH} ds$$

Clearly, $\|e^{itH_0} f\|_{2+\infty} \leq C_0 \langle t \rangle^{-\frac{d}{2}} \|f\|_{1\cap 2}$. Assume that $\|e^{itH} f\|_{2+\infty} \leq C_1 \langle t \rangle^{-\frac{d}{2}} \|f\|_{1\cap 2}$. Then from Duhamel (for times $t \gg 1$) with $\sup_s \|V(s)\|_{1\cap\infty} < \varepsilon$

$$\begin{aligned} \langle t \rangle^{\frac{d}{2}} \|e^{itH} f\|_{2+\infty} &\leq C_0 \|f\|_{1\cap 2} \\ &+ \langle t \rangle^{\frac{d}{2}} \int_1^{t-1} C_0 |t-s|^{-\frac{d}{2}} \|V(s)\|_{1\cap 2} C_1 \langle s \rangle^{-\frac{d}{2}} ds \|f\|_{1\cap 2} \\ &+ \langle t \rangle^{\frac{d}{2}} \int_{t-1}^t \|V(s)\|_{\infty\cap 2} C_1 \langle s \rangle^{-\frac{d}{2}} ds \|f\|_{1\cap 2} \\ &+ \langle t \rangle^{\frac{d}{2}} \int_0^1 C_0 |t-s|^{-\frac{d}{2}} \|V(s)\|_2 ds \|f\|_2 \\ &\lesssim [C_0 + (1 + C_0)C_1\varepsilon] \|f\|_{1\cap 2}. \end{aligned}$$

Thus if ε small, then $C_1 \lesssim C_0$. This argument needs $d \geq 3$ also for non-technical reasons: In dimensions $d = 1, 2$ small potentials can have bound states.

(2) From local decay to global decay (Ginibre):
 Before J-S-S both Rauch (exponentially decaying potentials) and Jensen, Kato (power like decay) proved estimates of the form

$$(LD) \quad \|\chi e^{itH} P_c \chi f\|_2 \lesssim \langle t \rangle^{-\frac{d}{2}} \|f\|_2$$

for some decaying weight χ . Let $H_0 = \frac{1}{2}\Delta$, $H = H_0 + V$:

$$(3) \quad \begin{aligned} e^{itH} P_c &= e^{itH_0} P_c + i \int_0^t e^{i(t-s)H_0} V e^{isH} P_c ds \\ &= e^{itH_0} P_c + i \int_0^t e^{i(t-s)H_0} V P_c e^{isH} ds \end{aligned}$$

$$(4) \quad \begin{aligned} P_c e^{isH_0} &= P_c e^{isH} - i \int_0^s P_c e^{i(s-\tau)H} V e^{i\tau H_0} d\tau \\ &= P_c e^{isH} - i \int_0^s e^{i(s-\tau)H} P_c V e^{i\tau H_0} d\tau. \end{aligned}$$

Inserting (4) into (3) yields

$$\begin{aligned} e^{itH} P_c &= e^{itH_0} P_c + i \int_0^t e^{i(t-s)H_0} V P_c e^{isH_0} ds \\ &\quad - \int_0^t \int_0^s e^{i(t-s)H_0} V e^{i(s-\tau)H} P_c V e^{i\tau H_0} d\tau. \end{aligned}$$

Apply the local decay estimate to the middle piece of the last term:

$$V e^{i(s-\tau)H} P_c V$$

with V playing the role of χ . **Conclusion:** Local L^2 -decay (LD) implies that

$$\|e^{itH} P_c f\|_{2+\infty} \lesssim \langle t \rangle^{-\frac{d}{2}} \|f\|_{1 \cap 2}.$$

J-S-S removed L^2 under the additional assumption that $\|\widehat{V}\|_1 < \infty$ since then

$$\sup_{1 \leq p \leq \infty} \left\| e^{-it\frac{1}{2}\Delta} V e^{it\frac{1}{2}\Delta} \right\|_{p \rightarrow p} \leq \|\widehat{V}\|_1.$$

In the dispersive estimates for charge transfer models as in Theorem 2 one can remove L^2 from the left-hand side. This passage from local to global proved to be very useful for systems, since one can adapt an old technique of Rauch for exponentially decaying potentials to the case of systems.

Proof of Theorem 2:

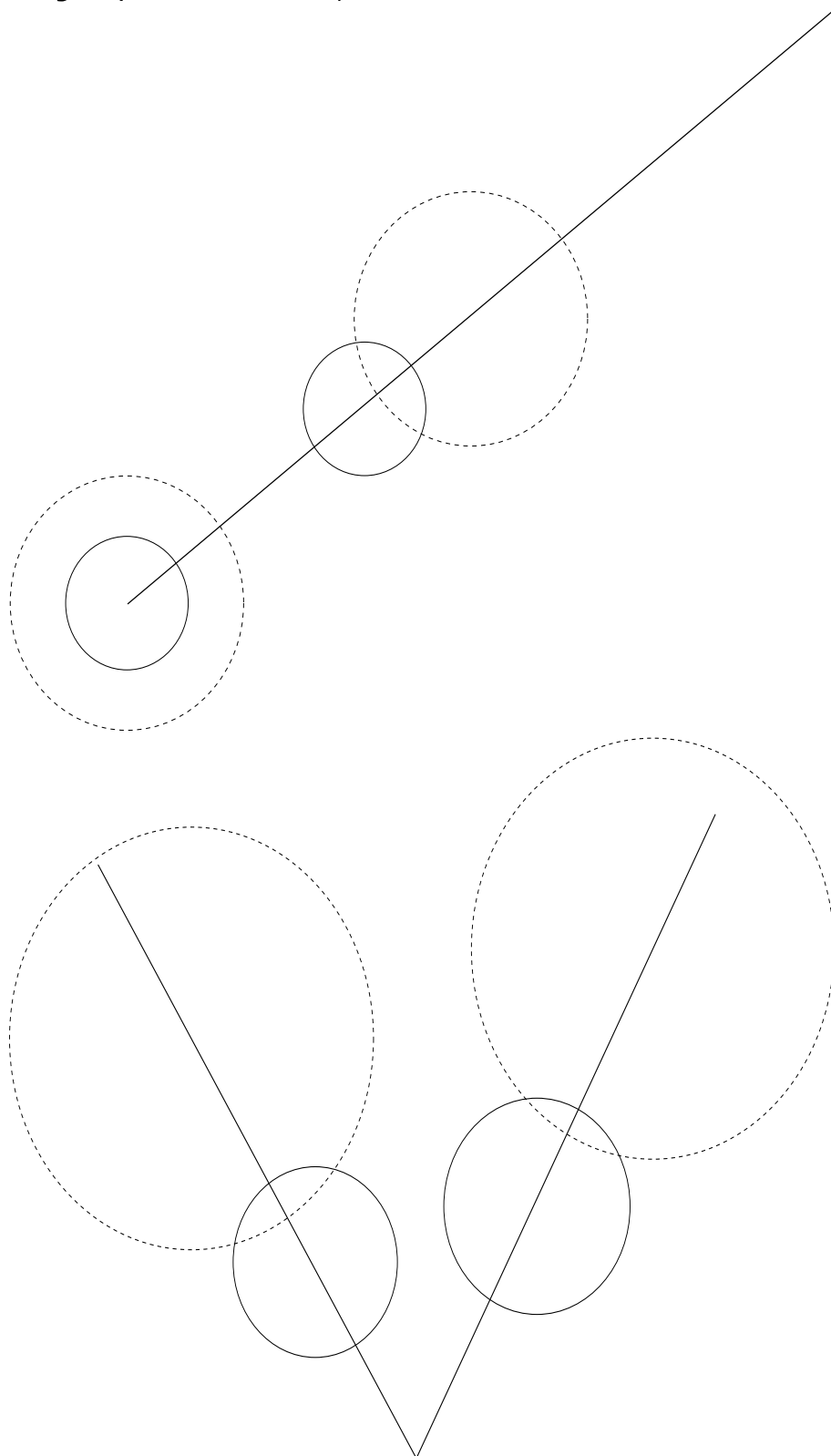
- for *scattering states* projections onto (moving) bound states decay exponentially due to the exponential decay of eigenfunctions of $\frac{1}{2}\Delta + V$ with negative eigenvalues (Agmon's estimate) and the exponential decay of the potentials.
- reduce to the dispersive bound for individual potentials by means of a "channel decomposition", i.e., split evolution

$$U(t) = \chi_1(t, \cdot)U(t) + \chi_2(t, \cdot)U(t) + \chi_3(t, \cdot)U(t)$$

with cut-offs $\chi_1(t, x) = \chi(x/(\delta t))$, $\chi_2(t, x) = \chi((x - t\vec{v})/(\delta t))$. Use Duhamel to compare $\chi_1(t, \cdot)U(t)$ to $\chi_1(t, \cdot)e^{itH_1}$ etc., where $H_1 = \frac{1}{2}\Delta + V_1$. One can factor in the projections P_1 for free from the previous comment: if ψ_0 is scattering, then

$$\|\chi_1(t, \cdot)(1 - P_1)U(t)\psi_0\| \lesssim e^{-\alpha t} \|\psi_0\|_2.$$

χ_1, χ_2 at two times (first with one stationary potential, and then with two moving ones):



As in the easy argument involving small potentials we will use a bootstrap argument. The required smallness for that comes from two sources:

- interaction between channels is small for projections onto small momenta (propagation estimates or low velocity estimates). Easy: For fixed T, M

$$\sup_{|t| \leq T} \|\chi e^{it\Delta} F(|p| \leq M) \chi(\cdot - y)\|_{2 \rightarrow 2} \rightarrow 0$$

as $|y| \rightarrow \infty$. Harder (also for H): If $F(|p| \leq M)$ is a smooth cut-off that vanishes when $|p| > M$ and $\alpha > 1$ then

$$\begin{aligned} & \|\chi(|x| > \alpha(R + Mt)) e^{it\frac{\Delta}{2}} F(|p| \leq M) \chi(|x| < R)\|_{2 \rightarrow 2} \\ & \leq C_N (R + Mt)^{-N} \quad \text{for all } N > 0. \end{aligned}$$

- for high momenta one obtains a gain via Kato's smoothing estimate

$$\sup_{B_R} \int_0^A \left\| (-\Delta)^{\frac{1}{4}} e^{itH} f \right\|_{L^2(B_R)}^2 dt \leq C_R A \|f\|_2^2$$

A on RHS needed: bound states of H .

More details: • Decay of the bound states. Split

$$U(t)\psi_0 = \psi_2 + \psi_1 = \sum_{i=1}^m a_i(t)u_i + \psi_1(t, \cdot)$$

where $H_1 u_j = \lambda_j u_j$, $\{u_j\}$ orthonormal. Project

$$i\partial_t \psi + \frac{1}{2}\Delta \psi + [V_1 + V_2(\cdot - \vec{v}t)]\psi = 0$$

onto u_j :

$$i\dot{a}_j + \lambda_j a_j + \langle \psi, V_2(\cdot - t\vec{v})u_j \rangle = 0.$$

Since u_j are exponentially decaying, and $a_j(\infty) = 0$, we get $|a_j(t)| \leq C e^{-\alpha t}$.

• Channel decomposition. With $\chi_1 = \chi_1(t, \cdot)$,

$$\begin{aligned} \chi_1 P_1 U(t) \psi_0 &= \chi_1 e^{itH_1} P_1 \psi_0 \\ &+ i\chi_1 \int_0^{t-A} e^{i(t-s)H_1} P_1 V_2(\cdot - s\vec{v}) U(s) \psi_0 ds \\ &+ i\chi_1 \int_{t-A}^t e^{i(t-s)H_1} P_1 V_2(\cdot - s\vec{v}) U(s) \psi_0 ds. \end{aligned}$$

Treat first integral as in previous bootstrap argument, get small constant if A large. Second integral more difficult. Expand again, but first project:

$$\begin{aligned}
& \chi_1 \int_{t-A}^t e^{i(t-s)H_1} P_1 V_2(\cdot - s\vec{v}) U(s) \psi_0 ds = \\
& \chi_1 \int_{t-A}^t e^{i(t-s)H_1} P_1 V_2(\cdot - s\vec{v}) (1 - P_2(s)) U(s) \psi_0 ds \\
& + \chi_1 \int_{t-A}^t e^{i(t-s)H_1} P_1 V_2(\cdot - s\vec{v}) P_2(s) U(s) \psi_0 ds.
\end{aligned}$$

Here $P_2(s) = \mathfrak{g}_{-\vec{v}}(t) P_2 \mathfrak{g}_{\vec{v}}(t)$. Integral with $1 - P_2(s)$ is exponentially small. The second one is expanded again by Duhamel \Rightarrow one easy term that does not involve $U(t)$ (we skip it) and another

$$\begin{aligned}
& \chi_1 \int_{t-A}^t \int_0^s e^{i(t-s)H_1} P_1 V_2(\cdot - s\vec{v}) \mathfrak{g}_{-\vec{v}}(s) e^{i(s-\tau)H_2} P_2 \\
& V_1(\cdot + \tau\vec{v}) \mathfrak{g}_{\vec{v}}(\tau) U(\tau) \psi_0 d\tau ds.
\end{aligned}$$

Split integration into $0 \leq \tau \leq s - B$, $s - B \leq \tau \leq s$. The former gives desired small constant if B large. For the latter use low velocity estimate ($\delta > 0$ small)

$$\|\chi_1(t, \cdot) e^{i(t-s)H_1} P_1 F(|\vec{p}| \leq M) V_2(\cdot - s\vec{v})\|_{L^2 \rightarrow L^2} \leq \frac{AM}{\delta t},$$

$\forall |t - s| \leq A$ (\Rightarrow *small factor*) and Kato smoothing.

The remaining Kato term

$$\chi_1 \int_{t-A}^t \int_{s-B}^s e^{i(t-s)H_1} P_1 F(|\vec{p}| > M) V_2(\cdot - s\vec{v}) \\ \mathfrak{g}_{-\vec{v}}(s) e^{i(s-\tau)H_2} P_2 V_1(\cdot + \tau\vec{v}) \mathfrak{g}_{\vec{v}}(\tau) U(\tau) \psi_0 d\tau ds$$

is estimated as follows:

- Estimate entire expression in L^2 , get rid of $\chi_1 e^{i(t-s)H_1} P_1$.
- Move $F(|\vec{p}| > M)$ through $V_2(\cdot - s\vec{v}) \mathfrak{g}_{-\vec{v}}(s)$, which changes $F(|\vec{p}| > M)$ to $F(|\vec{p} - \vec{v}| > M)$, and also introduces the commutator $[V_2, F(|\vec{p}| > M)]$ which has small norm
- Remains to deal with

$$\int_{t-A}^t \int_{s-B}^s \|V_2 F(|\vec{p} - \vec{v}| \geq M) e^{i(s-\tau)H_2} P_2 \\ \mathfrak{g}_{\vec{v}}(\tau) V_1 U(\tau) \psi_0\|_{L^2} d\tau ds.$$

Apply Kato's smoothing estimate to evolution $e^{i(s-\tau)H_2}$ with cut-off V_2 . Gain small factor $M^{-\frac{1}{2}}$.

Asymptotic completeness of charge-transfer models

$$(*) \quad i\partial_t\psi + \frac{1}{2}\Delta\psi + \sum_{j=1}^k V_j(\cdot - t\vec{v}_j)\psi = 0$$

goes back to Yajima (1980) and Graf (1990).

Review of basic scattering for $\frac{1}{2}\Delta + V$:

$$\text{Wave operators } \Omega := s - \lim_{t \rightarrow \infty} e^{-itH} e^{it\frac{\Delta}{2}}$$

exist by Cook's method if $|V(x)| < (1 + |x|)^{-1-\varepsilon}$.

Are isometries with $\text{Ran}(\Omega) \subset \mathcal{H}_{a.c.}(H)$. If

$$\widetilde{\Omega} := s - \lim_{t \rightarrow \infty} e^{-it\frac{\Delta}{2}} e^{itH} P_{a.c.}$$

exists, then $\text{Ran}(\Omega) = \mathcal{H}_{a.c.}(H)$. *Asymptotic com-*

pleteness means this and $L^2 = \mathcal{H}_{a.c.}(H) + \mathcal{H}_{p.p.}(H)$,

where p.p.-spectrum = finite number of negative

bound states. Dynamically: $\forall f \in L^2 \exists g \in L^2$ s.t.

$$e^{itH} f = e^{it\frac{\Delta}{2}} g + \sum_{j=1}^m e^{itE_j} g_j + o(1)_{L^2}$$

where $Hg_j = E_j g_j$, $E_j < 0$.

Suppose V is nice enough that Ω exists and that one has the dispersive bound

$$\|e^{itH} P_b^\perp f\|_{2+\infty} \lesssim \langle t \rangle^{-\frac{d}{2}} \|f\|_{1 \cap 2}$$

where P_b is the projection onto the (negative) bound states. Then asymptotic completeness follows:

$$\begin{aligned} \int_0^\infty \|e^{-itH} V e^{it\frac{\Delta}{2}} f\|_2 dt &\leq \int_0^\infty \|V\|_{2 \cap \infty} \|e^{it\frac{\Delta}{2}} f\|_{2+\infty} dt \\ &\lesssim \int_0^\infty \langle t \rangle^{-\frac{d}{2}} \|f\|_{1 \cap 2} dt < \infty \end{aligned}$$

since $d \geq 3$ (Cook's method). Thus Ω is unitary onto $\text{Ran}(P_b^\perp)$, excludes s.c. spectrum.

Similar logic applies to the charge transfer case. First, meaning of asymptotic completeness for charge transfer models (in Graf's terminology): $\Omega := s - \lim_{t \rightarrow \infty} U(t)^{-1} e^{it\frac{\Delta}{2}}$ and $\Omega_j := s - \lim_{t \rightarrow \infty} U(t)^{-1} \mathbf{g}_{-\vec{v}_j}(t) P_b(H_j) e^{itH_j} \mathbf{g}_{\vec{v}_j}(0)$ exist, and $L^2 = \text{Ran}(\Omega) \oplus \bigoplus_{j=1}^k \text{Ran}(\Omega_j)$ as an orthogonal direct sum. Dynamically, one has the following:

Theorem 3. Let $\{u_{s,j}\}_{s=1}^{S_j}$ be the efs of $H_j = \frac{1}{2}\Delta + V_j$, corresponding to the negative evs $\{\lambda_{s,j}\}_{s=1}^{S_j}$. Then $\forall \psi_0 \in L^2$ the solution $U(t)\psi_0$ of (*) can be written in the form

$$U(t)\psi_0 = \sum_{j=1}^k \sum_{s=1}^{S_j} A_{s,j} e^{it\lambda_{s,j}} \mathbf{g}_{-\vec{v}_j}(t) u_{s,j} + e^{it\frac{\Delta}{2}} \phi_0 + o(1)_{L^2},$$

for some choice of constants $A_{s,j}$ and $\phi_0 \in L^2$.

Write $\psi(t) := U(t)\psi_0$. Project the equation via $P_b(H_j, t) \Rightarrow P_b(H_j, t)\psi(t) = \sum_{s=1}^{S_j} a_{s,j}(t) e^{it\lambda_{s,j}} u_{s,j}$ where $a_{s,j}(t) \rightarrow A_{s,j}$ as $t \rightarrow \infty$. Let $\|\mathbf{g}_{\vec{v}_j}(t)U(t)v_{s,j} - e^{it\lambda_{s,j}} u_{s,j}\|_2 \rightarrow 0$ and define $\phi_1 := \psi_0 - \sum_{j=1}^k \sum_{s=1}^{S_j} A_{s,j} v_{s,j}$. Since $P_b(H_j, t)P_b(H_m, t) \rightarrow 0$ one checks that $P_b(H_j, t)U(t)\phi_1 \rightarrow 0$. Hence Theorem 2 and Cook's method from above imply that $s - \lim_{t \rightarrow \infty} e^{-it\frac{\Delta}{2}} U(t)\phi_1 =: \phi_0$ exists, at least if $\psi_0 \in L^1 \cap L^2$. Hence $U(t)\psi_0 = e^{it\frac{\Delta}{2}} \phi_0 + \sum_{j=1}^k \sum_{s=1}^{S_j} A_{s,j} U(t)v_{s,j} + o(1)$, and we are done.

For the nonlinear applications we need decay estimates for **charge transfer models with matrix potentials**. This requires estimates for single matrix potentials: $\|e^{itA}P_s f\|_{2+\infty} \lesssim \langle t \rangle^{-\frac{3}{2}} \|f\|_{1 \cap 2}$ ($d = 3$) with $A = \begin{pmatrix} \frac{1}{2}\Delta + \mu + U & -W \\ W & -\frac{1}{2}\Delta - \mu - U \end{pmatrix}$, and P_s is a suitable projection. Matrix charge transfer models require both Galilei transforms and modulations (see p. 3) acting on different A 's. Impose the following spectral assumptions on A acting on $\mathcal{H} := L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$:

- U, W decay exponentially
- $\text{spec}(A) \subset \mathbb{R}$ and $\text{spec}(A) \cap (-\mu, \mu) = \{\omega_\ell \mid 0 \leq \ell \leq M\}$, where $\omega_0 = 0$ and all ω_j are distinct eigenvalues
- There are no eigenvalues in $\text{spec}_{ess}(A)$
- For $1 \leq \ell \leq M$, $L_\ell := \ker(A - \omega_\ell)^2 = \ker(A - \omega_\ell)$, and $\ker(A) \subsetneq \ker(A^2) = \ker(A^3) =: L_0$. Moreover, these spaces are finite dimensional.

- $\text{Ran}(A - \omega_\ell)$, $1 \leq \ell \leq M$ and $\text{Ran}(A^2)$ are closed.
- The spaces L_ℓ are spanned by exponentially decreasing functions in \mathcal{H} (say with bound $e^{-\varepsilon_0|x|}$), and their derivatives decay similarly.
- The points $\pm\mu$ are not resonances of A .
- All these assumptions hold as well for the adjoint A^* . We denote the corresponding (generalized) eigenspaces by L_ℓ^* .

There is a direct sum decomposition

$$\mathcal{H} = \sum_{j=0}^M L_j + \left(\sum_{j=0}^M L_j^* \right)^\perp,$$

and let P_s denote the corresponding (nonorthogonal) projection onto the second summand. It plays the role of the continuous subspace from the self-adjoint case. In addition to the previous conditions we need the *linear stability assumption*

$$\|e^{itA} P_s f\|_2 \leq C \|f\|_2.$$

Under all these conditions we prove the desired bound

$$(SysD) \quad \|e^{itA} P_s f\|_{2+\infty} \lesssim \langle t \rangle^{-\frac{3}{2}} \|f\|_{1 \cap 2}$$

by means of an old technique of Rauch, based on analytic continuation of the resolvent across the spectrum. This **only** works for exponentially decreasing potentials. Cuccagna (2001) obtained (SysD) without the L^2 addition/intersection for potentials with power-like decay by proving L^∞ boundedness of the wave operators in this context (Yajima's approach in the scalar case). Rauch's method is much more transparent, albeit more restrictive.

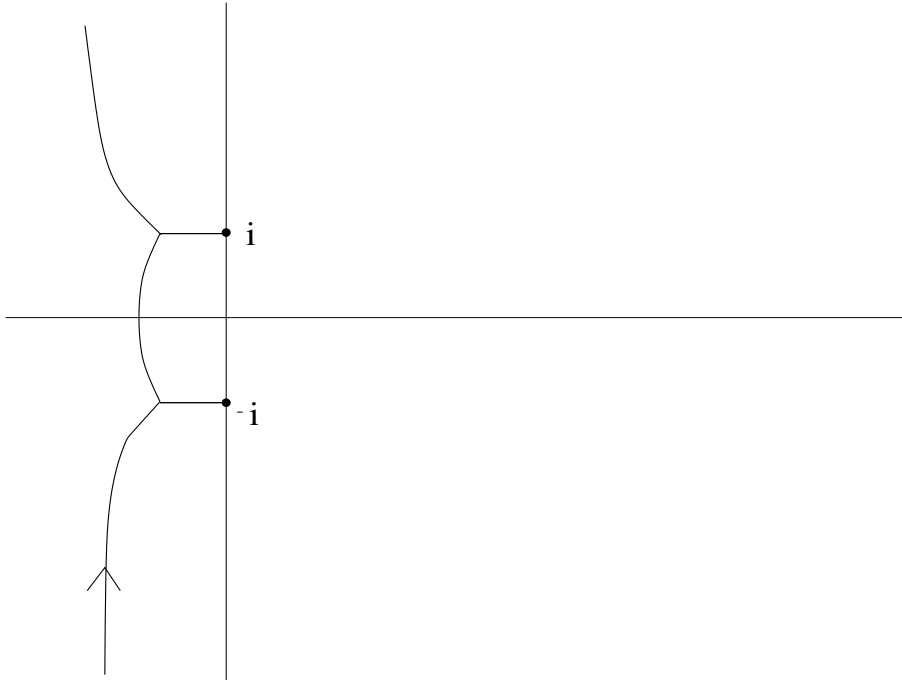
Outline of the method: Laplace transform and inverse ($\operatorname{Re}z > 0$ and $d > 0$)

$$\int_0^\infty e^{itA} P_s e^{-tz} dt = -(iA - z)^{-1} P_s$$

$$-\frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{t\tau} (iA - \tau)^{-1} P_s d\tau = e^{itA} P_s.$$

Note: there is no spectral representation (unlike the selfadjoint case $e^{itH} = \int e^{it\lambda} dE(\lambda)$), but not needed for these formulas. **Idea:** Shift contour in the second integral to the left *across the spectrum of iA* which is $i\mathbb{R}$. **Price:** Resolvent remains bounded only in exponentially weighted L^2 . Reason: $(-\Delta + z)^{-1}(x) = \frac{e^{-|x|\sqrt{z}}}{4\pi|x|}$ for $z \in \mathbb{C} \setminus (-\infty, 0]$. To continue across the slit set $z = \zeta^2$, $\operatorname{Re}\zeta > 0$. Then analytic continuation of $(-\Delta + \zeta^2)^{-1}(x) = \frac{e^{-|x|\zeta}}{4\pi|x|}$ to $\operatorname{Re}\zeta < 0$ has exponential growth. Let $(E_\varepsilon f)(x) = e^{-\varepsilon|x|} f(x)$. Resolvent identity + analytic Fredholm alternative $\Rightarrow E_\varepsilon (iA - \tau)^{-1} E_\varepsilon$ (for some fixed $\varepsilon > 0$)

extends as a *meromorphic* L^2 -operator valued function to the right of the following contour Γ_ε , with finitely many poles that all lie inside some disk. The two points are $\pm i\mu$ (we set $\mu = 1$).



Therefore shifting the contour yields

$$e^{itA} P_s = -\frac{1}{2\pi i} \int_{\Gamma_\varepsilon} e^{t\tau} (iA - \tau)^{-1} P_s d\tau$$

+ contribution from the residues.

Exclude the residues coming from the imaginary axis by means of the following version of von Neumann's ergodic theorem: $\frac{1}{T} \int_0^T e^{-i\omega t} e^{itA} P_s dt \rightarrow 0$ as $T \rightarrow \infty$.

The proof of this result depends strongly on our spectral assumptions as well as the linear stability assumption. The contribution from the curved pieces of the contour Γ_ε is exponentially decaying in time t , whereas the horizontal segments give the main contribution of $|t|^{-\frac{3}{2}}$. The latter requires making sure that the analytically continued, exponentially weighted, resolvent is regular at the points $\pm i\mu$. This is ensured since $\pm\mu$ are neither eigenvalues nor resonances of A .

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