New Phenomena in Critical Hyperbolic Equations

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Consider Cauchy problem in $\mathbb{R}^n$ with $\Box = \partial_{tt} - \Delta$,

$$\Box u = N(u), \quad \Box u = N(u, \nabla u) \quad \text{(NLW)}$$

data $(u, \partial_t u)(0) = (u_0, u_1) \in H^s \times H^{s-1}$. Basic question of solvability and uniqueness:

- **Locally well posed (LWP)** in $H^s \times H^{s-1}$?

- If so, solutions global (GWP) or finite time blow-up?

**LWP in $H^s \times H^{s-1}$** if $\forall (u_0, u_1) \in H^s \times H^{s-1}$ there is $T > 0$ and a neighborhood $U$ of $(u_0, u_1)$ so that for all data in $U \exists!$ solution $(u, u_t) \in C((-T, T); H^s) \times C((-T, T); H^{s-1})$, depending continuously on data. Also, higher regularity preserved.

**Easy:** (LWP) for large $s$ via energy methods, (GWP) for small data.
Recast *(NLW)* as

\[ u = S(u_0, u_1) + \Box^{-1}N(u) \]

Find fixed point for small times in Banach spaces *X, Y* with

\[ S : H^s \times H^{s-1} \to X, \quad \Box^{-1} : Y \to X, \quad N : X \to Y \]

Energy method means

\[ X = \{ u \in L^\infty(H^s), \partial u \in L^\infty(H^{s-1}) \}, \quad Y = L^1(H^{s-1}) \]

Boundedness of *S, \Box^{-1}* trivial:

\[ u(t) = \cos(\sqrt{-\Delta}t)u_0 + \frac{\sin(\sqrt{-\Delta}t)}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin(\sqrt{-\Delta}(t-t'))}{\sqrt{-\Delta}}f(t') \, dt' \]

\[ \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_0^t \|f(t')\|_{H^{s-1}} \, dt' \]
Obstruction from $N: s \geq s_0 + 1$ where $s_0$ is the scaling exponent: $s_0 = \frac{n}{2} - \alpha$, (NLW) invariant under $\mathcal{D}_\lambda: u \rightarrow \lambda^\alpha u(\lambda x, \lambda t), \lambda > 0$.

Point: norm of $\dot{H}^{\frac{n}{2} - \alpha}$ unchanged under $\mathcal{D}_\lambda$.

- $s < s_0$ Small data, small time equivalent to large data, large time result. Expect local ill-posedness below scaling.

- $s = s_0$ Small data, small time equivalent to small data, large time result. Same for large data provided time of existence only depends on size of data.

- $s > s_0$ Small data, large time equivalent to large data, small time result. $T_{\text{max}} \gtrsim \| (u_0, u_1) \|_{H^s \times H^{s-1}}^{s_0 - s}$

Other symmetries besides scaling also serve as obstructions to (LWP): Lorentz invariance leads to concentration along light-rays.
(GWP): Suppose energy $E : H^{s_c} \times H^{s_c-1} \rightarrow \mathbb{R}$ preserved under the flow of (NLW). Require that $E(u, u_t) \sim \| (u, u_t) \|^2_{s_c}$.

- subcritical case: $s_c > s_0$. (LWP) at $s \geq s_c$ implies (GWP) at $s \geq s_c$ since norm $\| \cdot \|_{s_c}$ does not grow. Energy penalizes point-singularity formation via scaling.

- critical case: $s_c = s_0$. (LPW) equivalent to (GWP) for small data. Large data: Energy neutral to point-singularity formation via scaling, exclude concentration of energy in the tip of characteristic cone (Morawetz identity).

- supercritical case: $s_c < s_0$. Energy rewards point-singularity formation via scaling. No global large data results known, even for Schwartz data.
Consider (NLW) with \( N(u) = -|u|^{p-1}u \) and energy

\[
E(u) = \int \frac{1}{2} |\partial u|^2 + \frac{1}{p+1} |u|^{p+1} \, dx
\]

Then \( s_c = 1, s_0 = \frac{n}{2} - \frac{2}{p-1} \). If \( n=3, p=5 \) the problem is energy critical, and (GWP) known (Grillakis, 90). **Open problem**: for example, prove that \( p = 7 \) (NLW)

\[
\partial_{tt}u - \Delta u + u^7 = 0
\]

admits *global smooth solutions for all smooth data.*

Expect finite time blow-up in many cases:

\[
\partial_{tt}u - \Delta u - |u|^{p-1}u = 0
\]

Solution \( u(t, x) = c_p \cdot (T - t)^{-\frac{2}{p-1}} \), truncate to light cone.

Merle-Zaag: if \( p \leq 3 \), then blow-up is self-similar with this rate!
The Wave Maps Equation

$\phi : \mathbb{R}^{n+1} \rightarrow M$, $(M, g)$ Riemann manifold with Lagrangian

$$L(\phi) = \frac{1}{2} \int_{\mathbb{R}^{n+1}} \left( -|\partial_t \phi|^2_g + |\nabla \phi|^2_g \right) dx dt$$

and associated Euler-Lagrange equation

$$D^\alpha \partial_\alpha \phi = 0, \quad \square \phi^i = \Gamma^i_{jk} \partial^\alpha \phi^j \partial_\alpha \phi^k, \quad \square \phi \perp T_\phi M$$

E.g. $\phi = \gamma \circ u$, $D\dot{\gamma} = 0$, $\square u = 0$. Conserved energy

$$E(\phi) = \frac{1}{2} \int |\partial_t \phi|^2_g + |\nabla \phi|^2_g \ dx$$

Cauchy Problem $\phi(0) = \phi_0 \in M$, $\partial_t \phi(0) = \phi_1 \in T_{\phi_0} M$. (WM)

scaling $\phi(t, x) \mapsto \phi(\lambda t, \lambda x)$. So $s_0 = \frac{n}{2}$: (LWP) for $s > \frac{n}{2}$

(Machedon, Klainerman, Selberg), (GWP) for $s = \frac{n}{2}$, small data,

"reasonable" $M$ (Tataru, Tao, Krieger), and (LIP) for $s < \frac{n}{2}$

(d’Ancona, Georgiev). Large data (GWP): compare $s_0 = \frac{n}{2}, s_c = 1$
Expect (GWP) for $n = 1$ since $s_c > s_0$ (known, even down to scaling critical), blow-up for $n \geq 3$ (Shatah): there exist self-similar solutions $u(t, x) = v(x/t)$ with $u = v, \partial_t u = -x \cdot \nabla v$ at $t = 1$.

Rewrite the Lagrangian in self-similar coordinates $\tau = \sqrt{t^2 - |x|^2}$, $\xi = \frac{x}{t} = \rho \omega$, $\omega \in S^{n-1}$. Then $v$ solves the elliptic PDE

$$-v_{\rho\rho} - \left( \frac{n-1}{\rho} + \frac{(n-3)\rho}{1-\rho^2} \right) v_\rho + \frac{1}{\rho^2(1-\rho^2)} \Delta_\omega v \perp T_v M$$

which is an harmonic map equation on the unit ball of $\mathbb{R}^n$ with hyperbolic metric

$$\frac{d\rho^2}{(1-\rho^2)^2} + \frac{\rho^2}{1-\rho^2} d\omega^2$$

For $n \geq 3$ non-constant solutions exist (equivariant harmonic maps) which can be smoothly continued beyond $\rho = 1$ with target manifolds given in terms of surfaces of revolution (spheres). Fails for $n = 2$, since the only solutions are $v = \text{const}$.
Critical case: \( n = 2, s_c = s_0 = 1 \). We’ll take target \( M \) a surface of revolution with metric \( ds^2 = d\rho^2 + g^2(\rho)d\theta^2, \theta \in S^1, g \in C^\infty(\mathbb{R}), \)
\( g(0) = 0, g'(0) = 1 \) and also assume this: \( g \) odd and either

\[
g(\rho) > 0 \ \forall \rho > 0, \quad \int_0^\infty |g(\rho)| \, d\rho = \infty \tag{1}
\]

or if \( M \) is compact, that \( g \) has first zero \( \rho_1 > 0, g'(\rho_1) = -1 \) and \( g \) periodic with period \( 2\rho_1 \). Consider equivariant wave maps \( \phi(t, x) : \mathbb{R}^{1+2} \to M \): if \( (r, \phi) \) polar coordinates on \( \mathbb{R}^2 \) then

\[
\rho = u(t, r), \quad \theta = \phi
\]

with equation \( \partial_{tt} u - \partial_{rr} u - \frac{1}{r} \partial_r u + \frac{g(u)g'(u)}{r^2} = 0 \). Smooth data with finite energy have local smooth solutions which cannot be continued beyond \( t = T_\ast \) iff energy concentrates: \( \exists \varepsilon_0 = \varepsilon_0(M) > 0 \) so that

\[
\liminf_{t \to T_\ast} \frac{1}{2} \int_0^{T_\ast-t} |\partial u|^2 + \frac{g^2(u)}{r^2} \, dr > \varepsilon_0
\]
Struwe’s bubbling off theorem: Let $\phi$ be smooth (EQWM) blowing up at $t_0$. Then $\exists r_j \to 0+, t_j \to t_0−$ s.t.

$$
\phi_j(t, x) := \phi(t_j + r_j t, r_j x) \to \Phi_{\infty}(t, x), \quad r_j = o(t_0 − t_j)
$$

strongly in $H^1_{\text{loc}}([-1,1[ \times \mathbb{R}^2)$, where $\Phi_{\infty}$ a nonconstant, time-independent solution generating nonconstant, smooth, (EQHM) $\Psi : S^2 \to M$ ($\Delta_{S^2} \Psi \perp T_{\Psi}M$).

If $M$ is non-compact as in (1), then harmonic spheres are known not to exist: (GWP) for such targets in the equivariant case.
However: for $M = S^2$ harmonic spheres do exist: $z^\ell, \bar{z}^\ell : \mathbb{C} \to \mathbb{C}$, $\ell \geq 1$, via stereographic projection $u(r) = 2 \arctan(r^\ell), \theta = \pm \ell \phi$.

Main result of this talk for wave maps is to show that blow-up does occur for $S^2$! In the non-equivariant case also expect dichotomy between (GWP) for large data and blow-up depending on the geometry of the target manifold ($S^2$ vs. $\mathbb{H}^2$).
Figure 1: Energy concentrates in a cusp
Theorem (K-S-T, 2006): Let $\nu > \frac{1}{2}$, $t_0 > 0$, $\lambda(t) = t^{-1-\nu}$, $N$ large. $\exists$ (WM) $\phi(t, r, \phi) = (u(r, t), \phi) : [0, t_0] \times \mathbb{R}^2 \rightarrow S^2$

$u(r, t) = 2 \arctan(\lambda(t)r) + u^e(r, t) + \varepsilon(r, t), \quad 0 \leq r \leq t$

$\mathcal{E}_{loc}(u^e)(t) \lesssim (t\lambda(t))^{-2} |\log t|^2, \quad \mathcal{E}_{loc}(\varepsilon)(t) \lesssim t^N$ as $t \rightarrow 0$

$u^e \in C^{\nu+1/2-}(\{ t_0 > t > 0, |x| \leq t \}), \quad \varepsilon \in t^N H^{1+\nu-}_{loc}(\mathbb{R}^2)$

Also, $u(0, t) = 0$ for all $0 < t < t_0$. The solution $u(r, t)$ extends as an $H^{1+\nu-}$ solution to all of $\mathbb{R}^2$ and the energy of $u$ concentrates in the cuspidal region $0 \leq r \lesssim \frac{1}{\lambda(t)}$ leading to blow-up at $r = t = 0$.

Remarks: i) Expect $\nu > 0$, improve on nonlinear part of proof; blow-up non-generic, study conditional stability

ii) Rodnianski-Sterbenz 06 independently obtained blow up; bulk term $Q(\lambda(t)r)$ with $Q$ (HM) degree $\ell \geq 4$, obtained $stable$ rate $\lambda(t) \gtrsim t^{-1}\sqrt{-\log t}$ (prior work Bizon-Ovchinnikov-Sigal 04, Cote)
Related result for energy critical (SLWE) in $n = 3$:

$$\partial_{tt} u - \Delta u - u^5 = 0, \quad (t, x) \in \mathbb{R}^{1+3}$$

Stationary solutions $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$.

**Theorem (K-S-T, 2007):** Let $\nu > \frac{1}{2}$, $\delta > 0$, $\lambda(t) = t^{-1-\nu}$. \exists energy solution $u$ of (SLWE) blowing up at $r = t = 0$ and $\forall |x| = r \leq t < t_0$ small,

$$u(x, t) = \lambda^{\frac{1}{2}}(t)W(\lambda(t)r) + \eta(x, t)$$

where $\mathcal{E}_{loc}(\eta(\cdot, t)) \to 0$ as $t \to 0$ and outside the cone

$$\int_{|x| \geq t} \left[ |\nabla u(x, t)|^2 + |u_t(x, t)|^2 + |u(x, t)|^6 \right] dx < \delta$$

for small $t > 0$. In particular, the energy of these blow-up solutions can be chosen arbitrarily close to $\mathcal{E}(W, 0)$, i.e., the energy of the stationary solution.
Background on (SLWE): $s_0 = s_c = 1$, $u(t, x) \rightarrow \lambda^{\frac{1}{2}} u(\lambda t, \lambda x)$ leaves both $\dot{H}^1(\mathbb{R}^3)$ and (SLWE) invariant. (LWP) in energy space $\dot{H}^1 \times L^2(\mathbb{R}^3)$, solution cannot be continued beyond $T_* < \infty$ iff $\|u\|_{L^8([0, T_*) \times \mathbb{R}^3)} = \infty$. Critical points of Lagrangian

$$L(u) = \int_{\mathbb{R}^{1+3}} \frac{1}{2} (-|\partial_t u|^2 + |\nabla u|^2) - \frac{1}{6} |u|^6 \, dt \, dx$$

and conserved energy

$$\mathcal{E}(u) = \int_{\mathbb{R}^3} \frac{1}{2} (|\partial_t u|^2 + |\nabla u|^2) - \frac{1}{6} |u|^6 \, dx$$

The critical Sobolev imbedding $\dot{H}^1 \rightarrow L^6(\mathbb{R}^3)$ has extremizers $W(r) = (1 + r^2/3)^{-\frac{1}{2}}$. Conformal group generates all extremizers. Associated EL-equation is $-\Delta W - W^5 = 0$, hence stationary solutions of (SLWE). Perturb around $W$: Seek $u = W + \psi$ with $\psi$ small. Then $(-\Delta - 5W^4(r))\psi(t, r) = N(W, \psi) = O(|\psi|^2)$ has solutions, with $H = -\Delta - 5W^4(r)$,
\[ \psi(t, x) = \cos(\sqrt{H}t)\psi_0 + \frac{\sin(\sqrt{H}t)}{\sqrt{H}}\psi_1 + \int_0^t \frac{\sin(\sqrt{H}(t-t'))}{\sqrt{H}} N(t') dt' \]

Spectrum of $H$ restricted to $L^2_{\text{rad}}$: since $H(\partial_\lambda [\lambda^{\frac{1}{2}} W(\lambda r)]) = 0$ and the zero-mode has a unique positive zero: zero is second eigenvalue (actually: resonance), bottom eigenvalue negative, $Hg_0 = -k_0^2 g_0$,

\[
\cos(\sqrt{H}t)g_0 = \cosh(k_0 t) g_0, \quad \frac{\sin(\sqrt{H}t)}{\sqrt{H}} g_0 = \frac{\sinh(k_0 t)}{k_0} g_0.
\]

Thus, need to project onto $g_0^\perp$ to have linear stability. Nonlinearly, one has a local center stable mf theorem for radial data (KS 05): In $B_\delta(0) \subset \langle x \rangle^{-\sigma}(H^3_{\text{rad}} \times H^2_{\text{rad}}) \ni$ co-dim one Lipschitz graph $G$ parametrized by tangent plane $\langle k_0 f_1 + f_2, g_0 \rangle = 0$ s. t. data on $(W, 0) + G$ have global solutions $\lambda_\infty^{\frac{1}{3}} W(\lambda_\infty x) + R$ where $R$ disperses and radiates to a free energy wave and $|\lambda_\infty - 1| \lesssim \delta$. 

15
Figure 2: Spectrum of $H = -\Delta - 5W^4$ and $H = -\Delta + \alpha^2 - p\phi^{p-1}$
Figure 3: Manifold of data leading to global solutions
Similar results for nonlinear Schrödinger: cubic NLS in $\mathbb{R}^3$ (S. 2004, Beceanu 2007), all $L^2$-supercritical NLS in $\mathbb{R}^1$ (2005, Krieger-S.), using spectral analysis and dispersive linear theory for the linearized NLS around the standing wave solution. Builds on earlier work on asymptotic stability for standing wave solutions of NLS by Soffer-Weinstein, Buslaev-Perelman, Rodnianski-S-Soffer in the stable regime. The difference here is that one is interested in conditional asymptotic stability in the unstable regime. Our understanding here is still very limited:

- Correct topology: Energy

- Behavior off the manifold, is there a radiation/blow-up dichotomy (as numerics by Bizon et al. suggests)?
**Center manifold theorem:** \( \dot{x} = f(x) \), \( f(0) = 0 \), \( A = Df(0) \), 
\( \mathbb{R}^n = V^s \oplus V^u \oplus V^c \) subspaces corresponding to eigenvalues with negative, positive, and zero, real parts. \( \exists \delta > 0 \) and Lipschitz map \( \phi : B_\delta(0) \cap V^c \to \mathbb{R}^n \) with

\[
\mathcal{M} = \{ \phi(x) : x \in B_\delta(0) \cap V^c \}
\]

tangent to \( V^c \) at the origin and invariant locally at zero.

Bates and Jones 89 (preceded by Clayton Keller, Segal):

\( \dot{u} = Au + f(u) \) on B-space \( X \), \( f(0) = 0 \), \( Df(0) = 0 \), 
\( X = X^s \oplus X^u \oplus X^c \), \( \dim X^{u,s} < \infty \), \( \forall \epsilon > 0, \exists C \) so that 
\( \|S^c(t)u_0\|_X \leq Ce^{|t|}\|u_0\|_X \) for all \( t \geq 0 \). Then there exist (local) Lipschitz graphs \( W^u, W^{cs} \). Applied this to prove instability of stationary solutions of Klein-Gordon in the \( L^2 \) subcritical case (Shatah, Shatah-Strauss, Grillakis-Shatah-Strauss).

**Main issues:** Global existence on \( W^{cs} \), “separatrix” property of \( W^{cs} \) between blow-up and scattering region in \( B_\delta(0) \).
Figure 4: A possible scenario
Expect $G$ to divide $(W, 0) + B_\delta(0)$ into **blow-up/scattering** halves. Karageorgis-Strauss 06 proved that there is blow-up above the tangent plane of $G$ (for $|u|^5$). Graph of energy at $(W, 0)$ is a saddle surface:

$$E(W + f_1, f_2) = E(W, 0) + \langle D\mathcal{E}(W, 0), (f_1, f_2) \rangle +$$

$$+ \frac{1}{2} \langle D^2\mathcal{E}(W, 0)(f_1, f_2), (f_1, f_2) \rangle + \ldots$$

Euler-Lagrange equation for $W$ is equivalent to $D\mathcal{E}(W, 0) = 0$, whereas the second variation is an indefinite quadratic form

$$\langle D^2\mathcal{E}(W, 0)(f_1, f_2), (f_1, f_2) \rangle = -k_0^2 \xi_1^2 + \xi_2^2 + \langle Hf_1^\perp, f_1^\perp \rangle + \langle f_2^\perp, f_2^\perp \rangle$$

with $\xi_1 = \langle f_1, g_0 \rangle, \xi_2 = \langle f_2, g_0 \rangle$.

Compare this to the definition of the tangent plane to $G$. Recent work of Kenig-Merle 06, 07 on the regime of energy below $\mathcal{E}(W, 0)$: for $\mathcal{E}(W + f_1, f_2) < \mathcal{E}(W, 0)$ there is a dichotomy $\|\nabla f_1\|_2 < \|\nabla W\|_2$ scattering, $\|\nabla f_1\|_2 > \|\nabla W\|_2$ blow-up.
Strategy of proof for (WM): We seek a solution of

$$-\partial_{tt} u + u_{rr} + \frac{u_r}{r} - \frac{\sin(2u)}{2r^2} = 0$$

of the form $u(t, r) = Q(\lambda(t)r) + v(t, r)$ inside the light-cone $r \leq t$, with $\mathcal{E}_{loc}(v(t, \cdot)) \to 0$ as $t \to 0^+$. By Struwe's result, $t\lambda(t) \to \infty$.

Applying $\partial_{tt}$ to the bulk-term generates

$$\partial_{tt} Q(\lambda(t)r) = \ddot{\lambda}(t)rQ'(\lambda(t)r) + \dot{\lambda}^2(t)r^2Q''(\lambda(t)r)$$

Note: $rQ'(r) = \frac{r}{1+r^2} \not\in L^2(\mathbb{R}^2)$. To cancel at $r = \infty$ need ODE

$$\lambda(t)\ddot{\lambda}(t) - 2\dot{\lambda}(t)^2 = 0 \Leftrightarrow \lambda(t) = (c_1t + c_2)^{-1}$$

Not allowed by Struwe's theorem!

Way out: Replace $Q(\lambda(t)r)$ by $Q(\lambda(t)r)\chi(t, r/t)$ (a cut-off to light-cone) with $\chi(t, \cdot) = 1 + o_{H^1}(1)$ as $t \to 0^+$. 
Ignoring $\chi_t$, setting $a = \frac{r}{t}$, $\lambda(t) = t^{-1-\nu}$ yields ODE in $a$ with principal Sturm-Liouville term

$$(1 - a)^{\nu + \frac{1}{2}} \partial_a [(1 - a)^{-\nu + \frac{1}{2}} \partial_a] \chi(a) = \ldots$$

Fundamental system $1, (1 - a)^{\nu + \frac{1}{2}}$, gives energy solutions as long as $\nu > 0$.

**Actual renormalization scheme:** $R = \lambda(t)r$, $a = r/t$, $u_0(R) = Q(R)$. Seek $u_k = u_{k-1} + v_k$ approximate solutions of (WME) in light-cone $r \leq t$, improve with $k$. **Iterative scheme** of errors and corrections

$$e_k = (-\partial_{tt} + \partial_{rr} + \frac{1}{r} \partial_r) u_k - \frac{\sin(2u_k)}{2r^2}$$

$$v_{2k+1} = (\partial_{rr} + \frac{1}{r} \partial_r - \frac{\cos(2u_0)}{r^2}) e_{2k}$$

$$v_{2k} = (-\partial_{tt} + \partial_{rr} + \frac{1}{r} \partial_r - \frac{1}{r^2}) e_{2k-1}$$
Then \( e_{2k} = -N_{2k}(v_{2k}) \), \( e_{2k+1} = -\partial_t^2 v_{2k+1} - N_{2k+1}(v_{2k+1}) \) with

\[
N_{2k+1}(v) = \frac{\cos(2u_0) - \cos(2u_{2k})}{r^2} v + \frac{\sin(2u_{2k})}{2r^2} (1 - \cos(2v)) \\
+ \frac{\cos(2u_{2k})}{2r^2} (2v - \sin(2v))
\]

\[
N_{2k}(v) = \frac{1 - \cos(2u_{2k-1})}{r^2} v + \frac{\sin(2u_{2k-1})}{2r^2} (1 - \cos(2v)) \\
+ \frac{\cos(2u_{2k-1})}{2r^2} (2v - \sin(2v))
\]

**Upshot:** Get errors that decay like \( t^m \ \forall \ m \). Indeed,

\[
u_{2k-1}(r, t) = Q(\lambda(t)r) + \frac{c_k}{(t\lambda)^2} R\log(1 + R^2) + O\left(\frac{(\log(1 + R^2))^2}{R(t\lambda)^2}\right)
\]

\[
e_{2k-1} = O\left(\frac{R(\log(2 + R))^{2k-1}}{t^2(t\lambda)^{2k}}\right)
\]

where correction is \( C^\nu + \frac{1}{2} \) at \( r = t \).
Perturbative analysis: Exact (WME) solution \( u = u_{2k-1} + \varepsilon \).

Solve in coordinates \( R = \lambda(t)r, \ d\tau = \lambda(t)dt, \ \tau = \int_t^1 \lambda(s) \, ds + \nu^{-1} \).

Set \( \tilde{\varepsilon} = R^{\frac{1}{2}} \varepsilon(t(\tau), \lambda^{-1}R) \). Main PDE:

\[
\left( - \left( \partial_\tau + \frac{\lambda_\tau}{\lambda} R \partial_R \right)^2 + \frac{1}{4} \left( \frac{\lambda_\tau}{\lambda} \right)^2 + \frac{1}{2} \partial_\tau \left( \frac{\lambda_\tau}{\lambda} \right) \right) \tilde{\varepsilon} - \mathcal{L} \tilde{\varepsilon} \\
= \lambda^{-2} R^{\frac{1}{2}} \left( N_{2k-1} R^{-\frac{1}{2}} \tilde{\varepsilon} + e_{2k-1} \right)
\]

\( \mathcal{L} := -\partial_R^2 + \frac{3}{4R^2} - \frac{8}{(1 + R^2)^2} \)

Note: \( e_{2k-1} \) decays like \( \tau^{-N} \).

Two key issues: Appearance of generator of dilations \( R \partial_R \) and the strongly singular Schrödinger operator \( \mathcal{L} \) on \( L^2(0, \infty) \) as part of the driving linear hyperbolic operator.

- Spectral, scattering theory, Fourier transform \( \mathcal{F} \) of \( \mathcal{L} \)
- \( \mathcal{F} \) hits (PDE), solve transport equation for \( \mathcal{F} \tilde{\varepsilon} \), \( R \partial_R \rightarrow \xi \partial_\xi \)
Spectral theory: Our Sturm-Liouville operator is singular at both $R = 0$ and $R = \infty$. Consider first regular case at $R = 0$:

$$\tilde{\mathcal{L}} = -\frac{d^2}{dR^2} + V(R)$$

with $V(R) \in L^1(0, \infty)$ is self-adjoint subject to Dirichlet BC at zero, say. For $\text{Im}z > 0 \exists$ Weyl-Titchmarsh solution $\psi(R, z) \in L^2(0, \infty)$ - unique up to scalar multiples (limit point case at $R = \infty$). Set $\psi(0, z) = 1$, and write

$$\psi(R, z) = \theta(R, z) + m(z)\phi(R, z)$$

where $\mathcal{L}\theta(\cdot, z) = z\theta(\cdot, z), \mathcal{L}\phi(\cdot, z) = z\phi(\cdot, z)$ and

$$\theta(0, z) = \phi'(0, z) = 1, \quad \theta'(0, z) = \phi(0, z) = 0$$

Then $m(z)$ Herglotz. **Fourier inversion** (where $\chi_{(0, \infty)}(\mathcal{L})f = f$)

$$\hat{f}(\xi) := \int_0^\infty f(R)\phi(R, \xi) \, dR, \quad f(R) = \int_0^\infty \hat{f}(\xi)\phi(R, \xi)\rho(\xi) \, d\xi$$

with spectral measure $\rho(d\xi) := \pi^{-1}\text{Im}m(\xi + i0)\, d\xi$. **Free case:**

$V = 0, \psi = e^{iR\xi^{\frac{1}{2}}}, \theta = \cos(R\xi^{\frac{1}{2}}), \phi = \xi^{-\frac{1}{2}}\sin(R\xi^{\frac{1}{2}}), m(\xi) = i\xi^{\frac{1}{2}}$
Singular case: $\mathcal{L}_0 = -\frac{d^2}{dR^2} + \frac{3}{4R^2}$, $\mathcal{L}_0 R^{\frac{3}{2}} = \mathcal{L}_0 R^{-\frac{1}{2}} = 0$, self-adjoint without BC at $R = 0$, limit point case. For $\mathcal{L} = \mathcal{L}_0 - \frac{8}{(1+R^2)^2}$ have $\mathcal{L}\phi_0 = \mathcal{L}\theta_0 = 0$, $W(\theta_0, \phi_0) = 1$, where

$$\phi_0(R) := \frac{R^{\frac{3}{2}}}{1 + R^2}, \quad \theta_0(R) := \frac{1 - 4R^2 \log R - R^4}{2R^{\frac{1}{2}}(1 + R^2)}$$

$\forall \text{Im}z > 0 \exists$ fund. systems $\phi(R, z), \theta(R, z)$, and $\psi(R, z), \overline{\psi(R, z)}$ of $\mathcal{L}f = zf$. Former entire in $z$, $W(\theta(\cdot, z), \phi(\cdot, a)) = 1$, and

$$\phi(R, z) = \phi_0(R) + R^{-\frac{1}{2}} \sum_{j=1}^{\infty} (R^2 z)^j \phi_j(R)$$

**Fourier inversion:** as above with “Weyl-Titchmarsh” function

$$m(z) = \frac{W(\theta(., z), \psi(., z))}{W(\psi(., z), \phi(., z))}$$

Proved by Gesztesy-Zinchenko 05
For $\mathcal{L}_0 = -\frac{d^2}{dR^2} + \frac{n^2 - 1/4}{R^2}$ this amounts to **Bessel functions**:

$$
\phi(R; z) = \frac{\pi}{2} C^{-1} z^{-n/2} R^{1/2} J_n(z^{1/2} R)
$$

$$
\theta(R; z) = C z^{n/2} R^{1/2}[-Y_n(z^{1/2} R) + \pi^{-1} \log(z) J_n(z^{1/2} R)]
$$

$$
\psi(R; z) = C z^{n/2} R^{1/2}[-Y_n(z^{1/2} R) + i J_n(z^{1/2} R)]
$$

$$
= C z^{n/2} R^{1/2} i H_n^{(1)}(z^{1/2} R)
$$

$$
= \theta(R; z) + m(z) \phi(R; z)
$$

$$
m(z) = C^2 \frac{2}{\pi} z^n [i - \pi^{-1} \log(z)], \quad z \in \mathbb{C} \setminus \mathbb{R}^+
$$

For our $\mathcal{L} = -\frac{d^2}{dR^2} + \frac{3}{4R^2} - \frac{8}{(1+R^2)^2}$ obtain

$$
\rho(\xi) \sim \frac{\chi[\xi \leq 1]}{\xi \log^2 \xi} + \xi \chi[\xi \geq 1]
$$

**Singularity** at $\xi = 0$ due to zero energy resonance.
Transference identity: With $\mathcal{F}$ the Fourier transform of $\mathcal{L}$,

$$\mathcal{F}(\partial_\tau + \frac{\lambda_\tau}{\lambda} R\partial_R) = (\partial_\tau + \frac{\lambda_\tau}{\lambda} (-2\xi \partial_\xi + \mathcal{K}))\mathcal{F}$$

where $\mathcal{K} = -\left(\frac{3}{2} + \frac{\eta'\rho}{\rho}\right)\delta(\xi - \eta) + \frac{\rho(\xi)}{\xi - \eta} F(\xi, \eta)$, and

$$|F(\xi, \eta)| \lesssim \langle \xi + \eta \rangle^{-\frac{3}{2}} (1 + |\xi^{\frac{1}{2}} - \eta^{\frac{1}{2}}|)^{-N}$$

Point: $(R\partial_R - 2\xi \partial_\xi)e^{iR\xi^{\frac{1}{2}}} = 0$ and $\mathcal{K}$ is the error when applied to $\phi(R, \xi)$ rather than $e^{iR\xi^{\frac{1}{2}}}$. Boundedness property $\forall \alpha$

$$\mathcal{K}_0(\xi, \eta) := \frac{\rho(\xi)}{\xi - \eta} F(\xi, \eta) : L^2,\alpha_{\rho} \rightarrow L^2,\alpha+1/2$$

$$[\mathcal{K}_0, \xi \partial_\xi] : L^2,\alpha_{\rho} \rightarrow L^2,\alpha_{\rho}$$

with $\|f\|_{L^2,\alpha}^2 = \int_0^\infty |f(\xi)|^2 \langle \xi \rangle^{2\alpha} \rho(\xi) \, d\xi$. 
Final iteration: Apply Fourier transform $\mathcal{F}$ to main (PDE)

\[-\left(\partial_\tau - 2\beta \xi \partial_\xi\right)^2 x - \xi x = 2\beta \mathcal{K}\left(\partial_\tau - 2\beta \xi \partial_\xi\right)x + \beta^2(\mathcal{K}^2 + 2[\xi \partial_\xi, \mathcal{K}])x\]

\[-\left(\frac{\beta^2}{4} + \frac{\dot{\beta}}{2}\right)x + \lambda^{-2} FR^{\frac{1}{2}}(N_{2k-1}(R^{-\frac{1}{2}} F^{-1} x) + e_{2k-1})\]

where $\beta = \frac{\lambda_\tau}{\lambda}$. We solve this with zero terminal conditions. I.e., let $H(\tau, \sigma)$ be the backward fundamental solution for the operator

\[
\left(\partial_\tau - 2\frac{\lambda_\tau}{\lambda} \xi \partial_\xi\right)^2 + \xi
\]

and by $H(\tau, \sigma)$ its kernel, $x(\tau) = \int_\tau^\infty H(\tau, \sigma)f(\sigma) d\sigma$. Then

\[\forall \alpha \geq 0 \ \exists C = C(\alpha) \text{ large so that uniformly in } \sigma \geq \tau\]

\[
\|H(\tau, \sigma)\|_{L^2,\alpha} \to L^2,\alpha+1/2 \lesssim \tau \left(\frac{\sigma}{\tau}\right)^C
\]

\[
\left\|\left(\partial_\tau - \frac{\lambda_\tau}{\lambda} 2\xi \partial_\xi\right) H(\tau, \sigma)\right\|_{L^2,\alpha} \to L^2,\alpha \lesssim \left(\frac{\sigma}{\tau}\right)^C
\]
**Main linear estimate, small Lipschitz constant:** Define

$$\|f\|_{L^\infty,N L^2_\rho,\alpha} := \sup_{\tau \geq 1} \tau^N \|f(\tau)\|_{L^2_\rho,\alpha}$$

*Given* $\alpha \geq 0$ *let* $N$ *be large enough. Then*

$$\|Hb\|_{L^\infty,N-2 L^2_\rho,\alpha+1/2} + \left\| \left( \partial_\tau - 2\beta \xi \partial_{\xi} \right) Hb \right\|_{L^\infty,N-1 L^2_\rho,\alpha} \lesssim \frac{1}{N} \|b\|_{L^\infty,N L^2_\rho,\alpha}$$

To close the loop, it remains to show this:

**Assume that* $N$ *is large enough and* $\frac{\nu}{2} + \frac{3}{4} > \alpha > \frac{1}{4}$. Then the map

$$x \mapsto \lambda^{-2} FR^{1/2} (N_{2k-1} (R^{-1/2} F^{-1} x))$$

*is locally Lipschitz from* $L^\infty,N-2 L^2_\rho,\alpha+1/2$ *to* $L^\infty,N L^2_\rho,\alpha$.*

In order to lower $\nu > \frac{1}{2}$ to $\nu > 0$, say, would need to improve on this nonlinear estimate. Need $\frac{\nu}{2} > \alpha > \frac{1}{4}$ due to regularity of $e_{2k-1}$ from renormalization step.
The semilinear case Similar scheme, but some major differences:

- No ”cuspidal result” as Struwe’s theorem expected (SLWE); in fact, blow-up could occur on complicated set. Currently little intuition why \( \lambda(t) = t^{-1-\nu} \) rates appear. Role of criticality? Compare work of Merle-Zaag for (SLWE) and Merle-Raphael, Perelman for \( L^2 \) critical (NLS).

- **Spectrum** of linearized operator \(-\Delta - 5W^4\) (restricted to \( L^2_{\text{rad}} \)) is \( \{-k_0^2\} \cup [0, \infty) \) with a zero energy resonance. Need to kill the exponentially growing mode; heuristically (!?), tie eval \(-k_0^2\) to generic, self-similar blow-up rate \( \lambda(t) = t^{-1} \) (with corrections), whereas the resonance (by itself) leads to all the slow rates that we observed. Since (WM) has no negative spectrum, perhaps no generic rate for (WM) relative to the ground state (HM). **Note:** If \( \text{deg}(Q) > 1 \), then \( L_Q \) has zero eigenvalue!! In fact, Rodnianski-Sterbenz 06 obtain stable rate \( \lambda(t) \gtrsim t^{-1} \sqrt{-\log t} \).
\begin{itemize}
  \item In (WM) the scaling is $Q(\lambda r)$ whereas for (SLWE) it is $\lambda^{1/2} W(\lambda r)$. Difference in the nonlinearity: $\frac{\sin(2u)}{2r^2}$ versus $u^5$. The former comes from $\Box \phi = \phi(|\nabla \phi|^2 - |\phi_t|^2)$, the latter has no gradient; $\lambda^{1/2}$ in front of $W$ produces same scaling as $r^{-2}$.
  \item The role of the zero energy resonance in the formation of blow-up not understood. It appears in both (WM) and (SLWE). In 4 + 1-dim. energy critical Yang-Mills, however, the linearized operator has a zero energy eigenvalue which appears to destroy the renormalization procedure:
    \begin{align*}
      \partial_{tt} u - \Delta_{\mathbb{R}^2} u - \frac{2}{r^2} u(1 - u^2) &= 0 \\
      Q(r) &= \frac{1 - r^2}{1 + r^2}, \quad rQ'(r) = -4 \frac{r^2}{(1 + r^2)^2} \\
      \mathcal{L} &= -\frac{d^2}{dR^2} + \frac{15}{4R^2} - \frac{24}{(1 + R^2)^2}, \quad \mathcal{L}R^{-\frac{3}{2}} = \mathcal{L}R^{\frac{5}{2}} = 0
    \end{align*}
    \end{itemize}

\textit{Slow blow-up} does occur here, but at a very different rate!
Yang-Mills: $G$ compact Lie group, Lie algebra $\mathfrak{g}$ with an invariant inner product $\langle \cdot, \cdot \rangle$, one forms $A = A_\alpha \, dx^\alpha$, $A_\alpha \in \mathfrak{g}$. Co-variant derivatives

$$D_\beta B = \partial_\beta B + [A_\beta, B]$$

curvature

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]$$

invariant under gauge transformations $A_\alpha \mapsto OA_\alpha O^{-1} - \partial_\alpha OO^{-1}, O \in G$. Yang-Mills connection is a critical point for the functional

$$I(A) = \int_{\mathbb{R}^n \times \mathbb{R}} \langle F_{\alpha\beta}, F^{\alpha\beta} \rangle \, dx \, dt = \int_{\mathbb{R}^n \times \mathbb{R}} \eta^{\alpha\beta} \langle F_{\alpha\beta}, F_{\alpha\beta} \rangle \, dx \, dt$$

Yang-Mills equations: $D^\beta F_{\alpha\beta} = 0$, conserved energy:

$$E(A) = \int \langle F_{\alpha\beta}, F_{\alpha\beta} \rangle \, dx.$$
Take $G = SO(d)$, $A_{\alpha} = \{A^{ij}_{\alpha}\}_{i,j=1}^{d}$ skew-symmetric, spherically symmetric ansatz

$$A^{ij}_{\alpha} = (\delta^{i}_{\alpha} x^{j} - \delta^{j}_{\alpha} x^{i}) \frac{1-u(t,r)}{r^{2}}$$

leads to Yang-Mills equations

$$\Box_{d-2} u = \frac{d-2}{r^{2}} u(1-u^{2})$$

Invariant under the scaling $u \mapsto u(t/\lambda, r/\lambda)$ and energy

$$E = \int_{0}^{\infty} \left[ u_{t}^{2} + u_{r}^{2} + \frac{d-2}{2r^{2}}(1-u^{2})^{2} \right] r^{d-3} dr$$

invariant under this scaling iff $d = 4$. Instanton $Q(r) = \frac{1-r^{2}}{1+r^{2}}$ is a stationary solution in that case.
Theorem 1. (Krieger-S-Tataru 2008) Let \( \lambda(t) = t^{-1} |\log t|^\beta \), \( \beta > 1 \). There exists an energy solution inside the cone \( \{ r < t, t < t_0 \} \) of the form

\[
 u(x, t) = Q(\lambda(t)r) + v(r, t)
\]

where

\[
 \|\nabla v(t)\|_2 + \sup_{|x| < t} |v(x, t)| \lesssim |\log t|^{-1}
\]