

Energy growth for Schrödinger equations with Markovian forcing

Burak Erdogan, Rowan Killip, W. S.

Consider the Schrödinger equation on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$

$$(1) \quad i\partial_t\psi(t, \theta) = -\partial_\theta^2\psi(t, \theta) + V(t, \theta)\psi(t, \theta)$$

where V is a real time-dependent potential. One has $\|\psi(t, \cdot)\|_2 = \text{const.}$, but $\|\psi(t, \cdot)\|_{H^s}$ can grow for $s > 0$. For example:

$$V(t, \theta) = \sum_{n \in \mathbb{Z}} \delta_n(t) [r_n \cos(\theta) + r'_n \sin(\theta)]$$

where r_n, r'_n are i.i.d. with $\mathbb{P}[r_n = \pm 1] = \frac{1}{2}$. Then

$$\begin{aligned} & \|\partial_\theta\psi(n+1, \cdot)\|_2^2 \\ &= \|\partial_\theta e^{-iH_0} e^{-i[r_n \cos(\theta) + r'_n \sin(\theta)]} \psi(n, \cdot)\|_2^2 \\ (2) &= \|\partial_\theta\psi(n, \cdot) - i[-r_n \sin(\theta) + r'_n \cos(\theta)]\psi(n, \cdot)\|_2^2. \end{aligned}$$

Let $\mathcal{F}_n = \text{span}\{r_j, r'_j \mid 0 \leq j \leq n\}$. Then (2) implies

$$\begin{aligned} \mathbb{E}[\|\partial_\theta\psi(n+1, \cdot)\|_2^2 \mid \mathcal{F}_n] &= \|\partial_\theta\psi(n, \cdot)\|_2^2 + \|\psi(n, \cdot)\|_2^2 \\ \mathbb{E}\|\partial_\theta\psi(n+1, \cdot)\|_2^2 &= \|\partial_\theta\psi(0, \cdot)\|_2^2 \\ &\quad + (n+1)\|\psi(0, \cdot)\|_2^2. \end{aligned}$$

So expected “energy”: $\mathbb{E}\|\psi(t, \cdot)\|_{H^1}^2 \sim t$.

Bourgain 1998: Let

$$\sup_{\theta, t} \left| \partial_{\theta}^{\alpha} \partial_t^{\beta} V(\theta, t) \right| < \infty \quad \text{for all } \alpha, \beta = 0, 1, \dots,$$

then for $s, \epsilon > 0$ one has $\|\psi\|_{H^s} = O(t^{\epsilon})$ as $t \rightarrow \infty$. He also showed that some slight growth of higher H^s norms is possible. It is natural to consider potentials that are intermediate between the smooth and the totally disordered cases. Suppose

$$V(t, \theta) = \lambda X(t) \cos(\theta)$$

where $X(t)$ a real-valued stationary random process. Conjecture of Tom Spencer and Zhakarov: Suppose

$$\kappa(t) := \mathbb{E}[X(t)X(0)] \quad \text{satisfies} \quad \hat{\kappa}(\tau) \sim |\tau|^{-p}$$

as $|\tau| \rightarrow \infty$. Then

$$\frac{1}{T} \int_0^{\infty} e^{-\frac{t}{T}} \mathbb{E} \|\partial_{\theta} \psi(t, \cdot)\|_2^2 dt \lesssim T^{\frac{2}{2+p}}.$$

We prove this for $p = 2$ with Markovian $X(t)$, and also that the power $\frac{1}{2}$ is correct. Possible motivation: Power spectrum of the forcing process,

$$(3) \quad \hat{\kappa}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{4T} \left| \int_{-T}^T e^{i\omega t} X(t) dt \right|^2 \quad \omega \in \mathbb{R}.$$

Idea: Current charact. frequency $\omega_c \approx \dot{\theta} \approx \sqrt{E}$

$$\dot{E} \approx \hat{\kappa}(\sqrt{E}) \sim E^{-\frac{p}{2}} \implies E(t) \approx t^{\frac{2}{2+p}}.$$

Markov processes: Conditional expectations satisfy

$$\begin{aligned} & \mathbb{P}[X(t_1) \in A_1 \mid X(t_2) \in A_2, \dots, X(t_k) \in A_k] \\ &= \mathbb{P}[X(t_1) \in A_1 \mid X(t_2) \in A_2] \end{aligned}$$

for any $t_1 > t_2 > t_3 > \dots > t_k$. Therefore, process determined by transition probabilities $P[dy, t \mid x, s]$ on the state space E for $t > s$ satisfying the Chapman-Kolmogoroff equation

$$(4) \quad P[dy, t_1 \mid x, t_3] = \int_E P[dy, t_1 \mid z, t_2] P[dz, t_2 \mid x, t_3]$$

for any $t_1 > t_2 > t_3$. We always assume $X(t)$ stationary with unique stationary measure μ_0 . Let

$$\begin{aligned} P[dy, t \mid x, s] &= p(y, t \mid x, s) \mu_0(dy) \\ (5) \quad (T_t f)(y) &= \int_E p(y, t \mid x, 0) f(x) \mu_0(dx). \end{aligned}$$

Then (4) implies $T_{t+s} = T_t \circ T_s$. With $T_t = e^{-tB}$,

$$\begin{aligned} & \langle \chi_{A_2}, e^{-Bt} \chi_{A_1} \rangle \\ &= \int_{A_2} \int_{A_1} p(y, t \mid x, 0) \chi_{A_1}(x) \mu_0(dx) \mu_0(dy) \\ (6) \quad &= \mathbb{P}[X(t) \in A_2, X(0) \in A_1]. \end{aligned}$$

Thus B is self-adjoint in $L^2(S, \mu_0)$ provided the process is reversible. On the state space $S = \mathbb{R}^n$ and under suitable conditions, B is a second order elliptic operator ($X(t)$ is a diffusion process). See for example Gardiner.

We shall assume that B is positive semi-definite with eigenvalues $0 = \kappa_0 < \kappa_1 \leq \kappa_2 \leq \dots$, $u_0 = 1$ being the eigenfunction for the simple eigenvalue $\kappa_0 = 0$. Also, let $\mathbb{E} X(0) = 0$, $\mathbb{E} X(0)^2 = 1$. For example, the Ornstein-Uhlenbeck process has $S = \mathbb{R}$,

$$B = -\partial_x^2 + x\partial_x, \quad \mu_0(dx) = (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} dx,$$

eigenvalues $\kappa_N = N$ and eigenfunctions $u_N = (N!)^{-\frac{1}{2}} He_N(x)$, the L^2 -normalized Hermite polynomials. For the purposes of this talk, we take $X(t)$ to be this process.

Formalism of density matrices: System in state $|\psi\rangle$ described by projection $|\psi\rangle\langle\psi|$. System in states $|\psi_j\rangle$ with probabilities p_i described by $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. Expected value of an observable A is $\sum_j p_j \langle\psi_j | A \psi_j\rangle = \text{tr}(A\rho)$. Also,

$$i\partial_t \psi_j = H\psi_j \implies i\partial_t \rho = [H, \rho].$$

More generally, density matrix ρ is positive trace-class operator, $\text{tr}(\rho) = 1$. Expected value of an observable A is $\text{tr}(A\rho)$. Inner product $\langle\rho|\sigma\rangle = \text{tr}(\rho^\dagger\sigma)$, same as in $\mathfrak{J}_2 = \mathfrak{J}_2(L^2(\mathbb{R}/2\pi\mathbb{Z}))$.

The combined state of Markov process and solution ψ of PDE described by element of Hilbert space $\mathcal{H} := L^2(S; \mu_0) \otimes \mathfrak{I}_2$, denoted by $P(t)$. Initial condition $P(t = 0) = u_0 \otimes \rho_0$, $\rho_0 = |\psi_0\rangle\langle\psi_0|$. At later times P is no longer a tensor product. In fact, evolution equation for P (see Pillet, Tcheremchantsev):

$$\begin{aligned} \frac{d}{dt}P &= -L_\lambda P \quad \text{where} \\ (7) \quad L_\lambda &= B \otimes I + iI \otimes [H_0, \cdot] + i\lambda x \otimes [\cos, \cdot] \end{aligned}$$

Indeed, since general P limit of convex combinations of uncorrelated states $u \otimes \rho$, suffices to prove (7) for uncorrelated states. But

$$\begin{aligned} \frac{d}{dt}u \otimes \rho &= -(Bu) \otimes \rho + u \otimes \frac{1}{i}[H, \rho] \\ &= -(Bu) \otimes \rho - iu \otimes [H_0, \rho] - i\lambda x u \otimes [\cos, \rho]. \end{aligned}$$

Relative to uncorrelated $P = u \otimes \rho$ expectation of “combined observable” $f(X)A$ is

$$\begin{aligned} \mathbb{E}(f(X)A) &= \int f(x)u(x) \mu_0(dx) \text{tr}(A\rho) \\ (8) \quad &= \langle \bar{f} \otimes A^\dagger | P \rangle_{L^2(\mu_0) \otimes \mathfrak{I}_2}, \end{aligned}$$

so by linearity same relative to general P .

Combine (7) with (8) (with $f = 1 = u_0$ and any Hilbert-Schmidt A) to conclude:

$$\begin{aligned}
& \frac{1}{T} \int_0^\infty e^{-t/T} \mathbb{E} \langle \psi(t) | A\psi(t) \rangle dt \\
&= \frac{1}{T} \int_0^\infty e^{-t/T} \langle u_0 \otimes A^\dagger | P(t) \rangle dt \\
&= \frac{1}{T} \int_0^\infty e^{-t/T} \langle u_0 \otimes A^\dagger | e^{-tL_\lambda} P(0) \rangle dt \\
(9) \quad &= \left\langle u_0 \otimes A^\dagger \left| \frac{\beta}{L_\lambda + \beta} P(0) \right. \right\rangle
\end{aligned}$$

where $\beta = \frac{1}{T}$. Inverse of $L_\lambda + \beta$ exists since L_λ is sum of nonnegative and skew-adjoint operator. We need energy, i.e., $A = -\Delta$: restrict to finite modes and pass to limit by monotone convergence theorem. Use this to prove our

Main Theorem: Let $X(t)$ be an Ornstein-Uhlenbeck process. If λ is sufficiently small, one has

$$(10) \quad \frac{1}{T} \int_0^\infty e^{-t/T} \mathbb{E} \|\partial_\theta \psi(t, \cdot)\|_2^2 dt \asymp T^{\frac{1}{2}}$$

as $T \rightarrow \infty$, where ψ denotes a (random) solution of

$$\begin{aligned}
i\partial_t \psi(\theta, t) &= -\partial_\theta^2 \psi(\theta, t) + \lambda X(t) \cos(\theta) \psi(\theta, t) \\
\psi(0, \cdot) &= 1.
\end{aligned}$$

Comments:

- Can treat more general initial conditions, for example any trigonometric polynomial.
- Argument works for any H^s , $s \geq 0$, not just $s = 1$. In fact,

$$\frac{1}{T} \int_0^\infty e^{-t/T} \mathbb{E} \|\psi(t, \cdot)\|_{H^s}^2 dt \asymp T^{\frac{s}{2}}.$$

- Ornstein-Uhlenbeck process only for convenience. Any process with generator B as above (see page 4; in particular, we need a spectral gap) can be treated by the same method. Note that such processes have exponentially decaying correlations. More precisely,

$$h(t) := \mathbb{E}[X(t)X(0)] = e^{-|t|\kappa_1} \implies \hat{h}(\tau) \sim |\tau|^{-2}.$$

This is the case $p = 2$ of the conjecture by Spencer, Zakharov, see page 2.

- Can treat potentials $V(t, \theta) = \lambda V_0(\theta)X(t)$ provided V_0 analytic and even.

By (9) the proof of our main theorem reduces to bounds for the resolvent $(L_\lambda + \beta)$. The subspace of odd and even functions are both invariant under the Schrödinger flow (even potential). Introduce basis $|N, n, m\rangle$, $N, n, m = 0, 1, 2, \dots$ where

$$|n\rangle = \pi^{-\frac{1}{2}} \cos(n\theta) \quad \text{for } n \geq 1, \quad |0\rangle = (2\pi)^{-\frac{1}{2}}$$

$$|N, n, m\rangle = u_N \otimes |n\rangle \langle m|.$$

Then

$$\mathcal{H} = L^2(\mathbb{R}; \mu_0) \otimes \mathfrak{J}_2(L_{\text{even}}^2) \simeq \ell^2(\mathbb{Z}_0^+) \otimes \ell^2(\mathbb{Z}_0^+ \times \mathbb{Z}_0^+).$$

One has

$$(11) \quad L_0 |N, n, m\rangle = (N + i(n^2 - m^2)) |N, n, m\rangle$$

$$(12) \quad x |N\rangle = \sqrt{N+1} |N+1\rangle + \sqrt{N} |N-1\rangle$$

$$(13) \quad \cos(\theta) |n\rangle = \begin{cases} \frac{1}{\sqrt{2}} |1\rangle & n = 0 \\ \frac{1}{2} |2\rangle + \frac{1}{\sqrt{2}} |0\rangle & n = 1 \\ \frac{1}{2} |n+1\rangle + \frac{1}{2} |n-1\rangle & n \geq 2. \end{cases}$$

Thus $PL_\lambda P = 0$, $PL_0 P^\perp = 0$, and $P^\perp L_0 P = 0$, where P is the orthogonal projection onto

$$\mathcal{H}_0 := \text{span}\{|0, n, n\rangle \mid n \in \mathbb{Z}_0^+\}.$$

Hence (with $U = L_\lambda - L_0$)

$$(14) \quad L_\lambda = \begin{bmatrix} 0 & P(L_\lambda - L_0)P^\perp \\ P^\perp(L_\lambda - L_0)P & P^\perp L_\lambda P^\perp \end{bmatrix}$$

$$P(L_\lambda + \beta)^{-1}P = P\left(\beta - P U R_\lambda(\beta) U P\right)^{-1}P$$

$$R_\lambda(\beta) = P^\perp(P^\perp L_\lambda P^\perp + \beta)^{-1}P^\perp$$

The inverse in the second line suffices, since by (9)

$$(15) \quad \begin{aligned} & \frac{1}{T} \int_0^\infty e^{-t/T} \mathbb{E} \|\partial_\theta \psi(t, \cdot)\|_2^2 dt \\ & = \beta \sum_{n=0}^\infty n^2 \langle 0, n, n | P(L_\lambda + \beta)^{-1}P | 0, 0, 0 \rangle. \end{aligned}$$

Invert the operator in the second line of (14) perturbatively, i.e., let

$$(16) \quad H = -P(L_\lambda - L_0) R_0(\beta) (L_\lambda - L_0)P$$

as operator on $\mathcal{H}_0 \simeq \ell^2(\mathbb{Z}_0^+)$. Using (11)-(13) one checks that

$$H = \nabla^\dagger a \nabla \quad \text{with } a_n = \frac{\lambda^2}{2} \begin{cases} \frac{1}{(1+\beta)^2 + (2n+1)^2} & n \geq 1 \\ \frac{2}{(1+\beta)^2 + 1} & n = 0 \end{cases}$$

and with $\nabla = 1 - S$, $(S\psi)_n = \psi_{n+1}$. Hence H is positive, compact, and $\inf(\text{spec}(H)) = 0$. We need to understand the Green's function $(H + \beta)^{-1}$ as $\beta \rightarrow 0$.

Written out, H has the form

$$(H\psi)_n = \begin{cases} (a_n + a_{n-1})\psi_n - a_{n-1}\psi_{n-1} - a_n\psi_{n+1} & n \geq 1 \\ a_0\psi_0 - a_0\psi_1 & n = 0 \end{cases}$$

or in matrix notation

$$H = \begin{bmatrix} a_0 & -a_0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ -a_0 & a_0 + a_1 & -a_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & -a_1 & a_1 + a_2 & -a_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

To guess the Green's function $(H + \beta)^{-1}$, recall that (on the whole line \mathbb{Z})

(17)

$$(\nabla^\dagger \nabla + \beta)^{-1}(0, n) \sim \begin{cases} \frac{1}{\sqrt{\beta}} e^{-\sqrt{\beta}|n|} & 0 < \beta < 1 \\ \frac{\beta-1}{\beta(\beta+1)} \beta^{-|n|} & \beta \geq 1. \end{cases}$$

Let $\beta^{-\frac{1}{4}} \ll N < \beta^{-\frac{1}{2}}$ and assume that a_n basically constant on $[N, 3N]$. Then one gets formally

$$\begin{aligned} (H + \beta)^{-1}(2N, 3N) &\approx (N^{-2} \nabla^\dagger \nabla + \beta)^{-1}(2N, 3N) \\ &\approx N^2 \exp(-\sqrt{N^2 \beta} N) \approx \exp(-\sqrt{\beta} N^2) \end{aligned}$$

from the first line of (17).

This suggests (e.g., resolvent identity) that for some constant $k = k(\beta)$

$$(18) \quad (H + \beta)^{-1}(0, n) \approx k e^{-\sqrt{\beta} n^2} \quad \text{for } 0 \leq n \leq \beta^{-\frac{1}{2}}.$$

If $n > \beta^{-\frac{1}{2}}$, then the second case of (17) suggests that

$$(19) \quad (H + \beta)^{-1}(0, n) \approx k' (\beta n^2)^{-n}.$$

Since

$$\sum_{n=0}^{\infty} (H + \beta)^{-1}(0, n) = \beta^{-1},$$

one is lead to the choice $k = \beta^{-\frac{3}{4}}$ above. This would indeed give the desired bound

$$\beta \sum_{n=0}^{\infty} n^2 \langle n | (H + \beta)^{-1} | 0 \rangle \lesssim \beta^{-\frac{1}{2}} = T^{\frac{1}{2}}.$$

Recall, however, that we need this for $P(L_\lambda + \beta)^{-1}P$, and not just for $(H + \beta)^{-1}$, cf. (14) and (16). Strategy: Use aspect of Combes-Thomas and Agmon method, namely bound $(H + \beta)^{-1}$ in weighted ℓ^2 spaces (also for all terms in the Neumann series of $P(L_\lambda + \beta)^{-1}P$). **Notice, however**: Combes-Thomas gives the wrong prefactor β^{-1} , whereas we need $\beta^{-\frac{3}{4}}$. To pass to the correct point-wise bounds need to consider also $\sqrt{a} \nabla (H + \beta)^{-1} \nabla^\dagger \sqrt{a}$.

Recall Combes-Thomas argument: Let $H = -\Delta + V$ in \mathbb{R}^n . Then, with $\nu \in \mathbb{R}^n$,

$$\begin{aligned} H(\nu) &:= e^{\nu \cdot x} (H - E) e^{-\nu \cdot x} \\ &= H - E + 2\nu \cdot \nabla - |\nu|^2 = A_1 + iA_2, \end{aligned}$$

where $A_1 = H - E - |\nu|^2$, $A_2 = -2i\nu \cdot \nabla$. If $E \leq \inf(\text{spec}(H)) - \delta$, then

$$\langle A_1 f | f \rangle \geq \frac{\delta}{2} \|f\|_2^2 \quad \text{provided} \quad |\nu|^2 \leq \frac{1}{2} \delta.$$

Now

$$\begin{aligned} \|(A_1 \pm iA_2)f\| \|f\| &\geq |\langle (A_1 \pm iA_2)f | f \rangle| \\ &\geq \text{Re} \langle (A_1 \pm iA_2)f | f \rangle \geq \frac{\delta}{2} \|f\|^2 \end{aligned}$$

so that for $|\nu| \leq \sqrt{\frac{\delta}{2}}$

$$\left\| e^{-\nu \cdot x} (H - E)^{-1} e^{\nu \cdot x} \right\| = \left\| (A_1 + iA_2)^{-1} \right\| \leq \frac{2}{\delta}.$$

But this implies that

$$\begin{aligned} \|\chi_1 (H - E)^{-1} \chi_0\| &\lesssim \frac{1}{\delta} \inf_{|\nu| = \sqrt{\frac{\delta}{2}}} e^{-\nu \cdot (x_0 - x_1)} \\ &= \frac{1}{\delta} \exp\left(-\sqrt{\delta/2} |x_0 - x_1|\right) \end{aligned}$$

where χ_1 and χ_0 are supported in balls of size one around x_1 and x_0 , respectively.

Do this calculation with $H = \nabla^\dagger a \nabla$ on $\ell^2(\mathbb{Z}_0^+)$:

$$H_\rho := e^\rho \nabla^\dagger a \nabla e^{-\rho} = A_1 + iA_2$$

where A_1, A_2 are self-adjoint, $A_1 = H + kS + S^\dagger k$ with

$$(20) \quad k_n = 2a_n \sinh^2\left[(\rho(n) - \rho(n+1))/2\right].$$

Thus

$$\|(H_\rho + \beta)^{-1}\| \lesssim \beta^{-1} \quad \text{provided} \quad \|k\|_\infty \leq \beta/4.$$

A possible choice of ρ in (20) is therefore (c small)

$$(21) \quad \boxed{\rho(n) = cn \min(1, n\sqrt{\beta})},$$

which agrees with (18). One can also choose $\rho(n) \sim n \log(\sqrt{\beta} n)$ if $n \geq \beta^{-\frac{1}{2}}$, see (20) and (19).

Basic intuition: Main difficulty with Combes-Thomas bound is the “wrong” prefactor β^{-1} . Of course,

$$\|(H + \beta)^{-1}\| \sim \beta^{-1},$$

but functions $\psi \in \ell^2(\mathbb{Z}_0^+)$ that attain this bound are spread out. Indeed, our intuition from above tells us that, roughly speaking,

$$(H + \beta)^{-1}(x, y) \approx \beta^{-\frac{3}{4}} \chi_{[0, \beta^{-\frac{1}{4}}]}(x) \chi_{[0, \beta^{-\frac{1}{4}}]}(y).$$

Hence the function ψ that maximizes the norm is

$$\psi = \chi_{[0, \beta^{-\frac{1}{4}}]}.$$

So cannot hope to get the correct bound on $(H + \beta)^{-1}(x, y)$ by plugging in δ functions. On the other hand, also suggests that we should look at ∇G , or $G\nabla$ or $\nabla G\nabla$ to capture the fact that G is spread out. This is indeed what we will do! The main tool for this turns out to be the commutator relation, see (26) below. Method robust!

In order to use this relation, could say that $H = \nabla^\dagger \sqrt{a} \sqrt{a} \nabla$. It will be convenient to set things up slightly differently, though. In fact, we write

$$\begin{aligned} H &= -P(L_\lambda - L_0)R_0(\beta)(L_\lambda - L_0)P \\ &= -P(L_\lambda - L_0) \frac{1}{2} \left(R_0(\beta) + \overline{R_0(\beta)} \right) (L_\lambda - L_0)P \\ &= (1 + \beta)D^\dagger D \quad \text{where} \end{aligned}$$

$$D = -iR_0(\beta)(L_\lambda - L_0)P : \quad \mathcal{H}_0 \rightarrow \mathcal{H}_1$$

$$\begin{aligned} \mathcal{H}_1 &:= \text{span}\{|1, n+1, n\rangle, |1, n, n+1\rangle \mid n \geq 0\} \\ (22) \quad &\simeq \ell^2(\mathbb{Z}_0^+; \mathbb{C}^2). \end{aligned}$$

In coordinates,

$$D = \begin{bmatrix} \alpha \nabla \\ -\bar{\alpha} \nabla \end{bmatrix} \quad \text{with} \quad \alpha_n = \frac{\lambda}{2} \begin{cases} \frac{1}{1+\beta+i(2n+1)} & n \geq 1 \\ \frac{\sqrt{2}}{1+\beta+i} & n = 0 \end{cases}$$

and

$$D^\dagger \begin{bmatrix} \psi \\ \phi \end{bmatrix} = \nabla^\dagger(\bar{\alpha} \psi) - \nabla^\dagger(\alpha \phi).$$

A calculation shows that

$$(23) \quad \begin{aligned} D_\rho &:= e^\rho D e^{-\rho} = D + \frac{1}{2}(\xi - \eta) + \frac{1}{2}(\xi + \eta) \\ (D^\dagger)_\rho &:= e^\rho D^\dagger e^{-\rho} = D^\dagger + \frac{1}{2}(\xi - \eta)^\dagger - \frac{1}{2}(\xi + \eta)^\dagger \end{aligned}$$

where

$$\xi = \begin{bmatrix} \alpha k S \\ -\bar{\alpha} k S \end{bmatrix} \quad \text{and} \quad \eta = - \begin{bmatrix} \alpha S \ell \\ -\bar{\alpha} S \ell \end{bmatrix}$$

and $k_n = 1 - e^{\rho(n) - \rho(n+1)}$, $\ell_n = 1 - e^{\rho(n) - \rho(n-1)}$.

Thus one checks that

$$\xi + \eta = -2 \sinh(\nabla \rho) \begin{bmatrix} \alpha S \\ -\bar{\alpha} S \end{bmatrix}.$$

From (23) with $\zeta = \frac{1}{2}(\xi + \eta)$, and $A = D + \frac{1}{2}(\xi - \eta)$,

$$e^\rho D^\dagger D e^{-\rho} = A^\dagger A - \zeta^\dagger \zeta - \zeta^\dagger A + A^\dagger \zeta,$$

the sum of the last two operators being skew-adjoint.

The “eikonal equation” (cf. (20))

$$(24) \quad \begin{aligned} 4\|\zeta^\dagger \zeta\| &= \|(\xi + \eta)^\dagger(\xi + \eta)\| = \|\xi + \eta\|^2 \\ &= \boxed{\|\sinh^2(\nabla\rho) a\|_\infty \leq \beta} \end{aligned}$$

therefore ensures that $\|((D^\dagger)_\rho D_\rho + \beta)^{-1}\| \lesssim \beta^{-1}$. Same works for $D_\rho(D^\dagger)_\rho$. Let ρ be as in (21) (which satisfies (24)), set $w(n) = e^{\rho(n)}$, $\langle \phi | \psi \rangle_w := \sum_{n=0}^\infty \overline{\phi_n} \psi_n w(n)$. Then we have shown

$$\|(D D^\dagger + \beta)^{-1}\|_{\ell_w^2} \lesssim \beta^{-1}, \quad \|(D^\dagger D + \beta)^{-1}\|_{\ell_w^2} \lesssim \beta^{-1}$$

Now use the commutator relation

$$(26) \quad D(D^\dagger D + \beta)^{-1} D^\dagger = 1 - \beta(DD^\dagger + \beta)^{-1}$$

to conclude that

$$(27) \quad \boxed{\|(H + \beta)^{-1}\|_{\ell_w^2} \lesssim \beta^{-1}}, \quad \|D(H + \beta)^{-1} D^\dagger\|_{\ell_w^2} \lesssim 1.$$

The latter can also be written as

$$(28) \quad \boxed{\|\sqrt{a} \nabla (H + \beta)^{-1} \nabla^\dagger \sqrt{a}\|_{\ell_w^2} \lesssim 1.}$$

These bounds imply that

$$(29) \quad \boxed{\|\sqrt{a} \nabla (H + \beta)^{-1}\|_{\ell_w^2} \lesssim \beta^{-\frac{1}{2}}.}$$

These weighted inequalities (27)-(29) (which hold uniformly in $0 < \lambda < 1$) will yield the point-wise ones.

Proof of (29): Let $T = \sqrt{w} \sqrt{a} \nabla (H + \beta)^{-1} w^{-\frac{1}{2}}$.
Then, using $\nabla(fg) = f\nabla g + \nabla(f)Sg$,

$$\begin{aligned} T^\dagger T &= w^{-\frac{1}{2}} (H + \beta)^{-1} \nabla^\dagger a w \nabla (H + \beta)^{-1} w^{-\frac{1}{2}} \\ &= w^{-\frac{1}{2}} (H + \beta)^{-1} \nabla^\dagger a \nabla w (H + \beta)^{-1} w^{-\frac{1}{2}} \\ &\quad - w^{-\frac{1}{2}} (H + \beta)^{-1} \nabla^\dagger a \nabla(w) S (H + \beta)^{-1} w^{-\frac{1}{2}}. \end{aligned}$$

Since $\nabla^\dagger a \nabla = H + \beta - \beta$,

$$\begin{aligned} \|T^\dagger T\| &\leq \left\| \sqrt{w} (H + \beta)^{-1} w^{-\frac{1}{2}} \right\| \\ &\quad + \beta \left\| w^{-\frac{1}{2}} (H + \beta)^{-1} \sqrt{w} \right\| \left\| \sqrt{w} (H + \beta)^{-1} w^{-\frac{1}{2}} \right\| \\ &\quad + \left\| w^{-\frac{1}{2}} (H + \beta)^{-1} \nabla^\dagger \sqrt{a} \sqrt{w} \right\| \left\| \sqrt{a} \frac{\nabla w}{\sqrt{w}} S (H + \beta)^{-1} w^{-\frac{1}{2}} \right\| \\ &\lesssim \frac{1}{\beta} + \|T^\dagger\| \left\| \sqrt{a} \frac{\nabla w}{\sqrt{w}} S (w^{-\frac{1}{2}}) \right\| \frac{1}{\beta} \lesssim \frac{1}{\beta} + \|T^\dagger\| \beta^{-\frac{1}{2}}. \end{aligned}$$

To pass to the final expression use (21). In fact, $w = e^\rho$ and $|\nabla w| \sqrt{a} \lesssim w \sqrt{\beta}$. Conclude that

$$\|T\|^2 = \left\| \sqrt{a} \nabla (H + \beta)^{-1} \right\|_{\ell_w^2}^2 \lesssim \beta^{-1},$$

as desired. This also proves that

$$(30) \quad \left\| D(H + \beta)^{-1} \right\|_{\ell_w^2} \lesssim \beta^{-\frac{1}{2}}.$$

Recall that $(H + \beta)^{-1}$ is an approximation to $P(L_\lambda + \beta)^{-1}P$. In fact, let

$$C = P(L_\lambda - L_0)(R_\lambda(\beta) - R_0(\beta))(L_\lambda - L_0)P.$$

Then, see (14)

$$\begin{aligned} P(L_\lambda + \beta)^{-1}P &= (H + \beta - C)^{-1} \\ (31) \quad &= \sum_{j=0}^{\infty} (H + \beta)^{-1}(-C(H + \beta)^{-1})^j. \end{aligned}$$

We claim that each term in this series (and not just $j = 0$) satisfy the bounds (27)-(29), and for λ small also $P(L_\lambda + \beta)^{-1}P$ itself. More precisely, one has

$$\begin{aligned} (32) \quad &\| (H + \beta)^{-1} (C(H + \beta)^{-1})^j \|_{\ell_w^2} \lesssim \lambda^{2j} \beta^{-1} \\ &\| \sqrt{a} \nabla (H + \beta)^{-1} (C(H + \beta)^{-1})^j \nabla^\dagger \sqrt{a} \|_{\ell_w^2} \lesssim \lambda^{2j} \\ &\| \sqrt{a} \nabla (H + \beta)^{-1} (C(H + \beta)^{-1})^j \|_{\ell_w^2} \lesssim \lambda^{2j} \beta^{-\frac{1}{2}}. \end{aligned}$$

Firstly, apply the resolvent identity twice to C , keeping in mind that it takes four hops to go from $\mathcal{H}_0 = \text{Ran}(P)$ back to itself, see (22):

$$C = -D^\dagger (L_\lambda - L_0) R_\lambda(\beta) (L_\lambda - L_0) D.$$

Expanding the j^{th} powers above thus leads to products of terms each of which are controlled by (27), (30), or the following:

Let P_1 be the projection onto \mathcal{H}_1 , see (22). Then

$$(33) \quad \left\| P_1 (L_\lambda - L_0) R_\lambda(\beta) (L_\lambda - L_0) P_1 \right\|_{\ell_w^2} \lesssim \lambda^2.$$

In view of the preceding, all bounds in (32) hold once (33) is established. To prove (33), expand

$$(34) \quad \begin{aligned} R_\lambda(\beta) &= \left(P^\perp (L_0 + \beta + L_\lambda - L_0) P^\perp \right)^{-1} \\ &= \sum_{k=0}^{\infty} R_0(\beta) \left(-(L_\lambda - L_0) R_0(\beta) \right)^k. \end{aligned}$$

Let

$$\|\{a_{N,n,m}\}\|_{\ell_w^2} := \left(\sum_{N,n,m} |a_{N,n,m}|^2 \sqrt{w(n)w(m)} \right)^{\frac{1}{2}}$$

be a weighted norm on \mathcal{H}_0^\perp . Since $R_0(\beta)$ is diagonal, $\|R_0(\beta)\|_{\ell_w^2} \lesssim 1$. Also,

$$\|P^\perp (L_\lambda - L_0) R_0(\beta)\|_{\ell_w^2} \lesssim \lambda.$$

Since $\langle N', n', m' | (L_\lambda - L_0) R_0(\beta) | N, n, m \rangle = 0$ if $|N - N'| + |n - n'| + |m - m'| > 2$, it suffices to check this on basis vectors. But this is easy from (11)-(13). Hence (34) converges in the operator norm on ℓ_w^2 , and (33) holds. The conclusion from all this is that the boxed estimates (27)-(29) hold also for $P(L_\lambda + \beta)^{-1}P$ and not just for $(H + \beta)^{-1}$. It remains to estimate $\langle 0, n, n | (L_\lambda + \beta)^{-1} | 0, 0, 0 \rangle$.

Point-wise lemma: Let $0 < \lambda < 1$ and $w = e^\rho$ with ρ as in (21). Suppose G is an operator on $\ell^2(\mathbb{Z}_0^+)$ that satisfies the weighted estimates (27)-(29). Then the matrix coefficients $G(n, 0)$ with $0 \leq n < \beta^{-\frac{1}{2}}$ satisfy

$$(35) \quad |G(n, 0)| \lesssim \lambda^{-\frac{1}{2}} \beta^{-\frac{3}{4}} \left(1 + \lambda^{-\frac{1}{4}} \beta^{\frac{1}{8}} \sqrt{n}\right) e^{-c\sqrt{\beta}n^2}.$$

One can also obtain bounds on $G(n, m)$ for general n, m . Ignoring the tails $n > \beta^{-\frac{1}{2}}$, this finishes the proof of the upper bound in (10), see (15):

$$(36) \quad \sum_{n=0}^{\infty} n^2 \left| \langle 0, n, n | (L_\lambda + \beta)^{-1} | 0, 0, 0 \rangle \right| \lesssim \lambda^{-\frac{3}{4}} \beta^{-\frac{3}{2}}.$$

A possible proof of (35) proceeds as follows: Let

$$\Lambda_{n,N}(k) = \chi_{(n, n+N]}(k) \left(1 - \frac{k - n - 1}{N}\right).$$

Then one checks that

$$(37) \quad \begin{aligned} G(n, m) &= \left\langle \frac{1}{N} \chi_{(n, n+N]} \middle| G \frac{1}{M} \chi_{(m, m+M]} \right\rangle \\ &\quad - \left\langle \frac{1}{N} \chi_{(n, n+N]} \middle| G \nabla^\dagger \Lambda_{m, M} \right\rangle \\ &\quad - \left\langle \Lambda_{n, N} \middle| \nabla G \frac{1}{M} \chi_{(m, m+M]} \right\rangle \\ &\quad + \left\langle \Lambda_{n, N} \middle| \nabla G \nabla^\dagger \Lambda_{m, M} \right\rangle. \end{aligned}$$

Let

$$\Omega_{n,N}(k) = \frac{1}{N} \chi_{(n,n+N]}(k), \quad \tilde{\Lambda}_{n,N}(k) = \frac{1}{\sqrt{a(k)}} \Lambda_{n,N}(k).$$

Invoking our assumptions and (37) leads to

$$(38) \quad |G(n, m)| \lesssim \left(\left\| \beta^{-\frac{1}{2}} \Omega_{n,N} \right\|_{\ell_{1/w}^2} + \left\| \tilde{\Lambda}_{n,N} \right\|_{\ell_{1/w}^2} \right) \cdot \left(\left\| \beta^{-\frac{1}{2}} \Omega_{m,M} \right\|_{\ell_w^2} + \left\| \tilde{\Lambda}_{m,M} \right\|_{\ell_w^2} \right).$$

The point-wise bound follows by making suitable choices of M, N . For example, if $m = 0$, then one checks that $M = \sqrt{\lambda} \beta^{-\frac{1}{4}}$ gives $\lambda^{-\frac{1}{4}} \beta^{-\frac{3}{8}}$ for the terms involving m, M in (38). For the other terms involving n, N , one needs to take $N = \sqrt{\lambda} \beta^{-\frac{1}{4}}$ if $0 \leq n \leq \sqrt{\lambda} \beta^{-\frac{1}{4}}$, and $N = \lambda \beta^{-\frac{1}{2}} / n$ if $\beta^{-\frac{1}{4}} \leq n \leq \beta^{-\frac{1}{2}}$. In view of the definition (21) of ρ , it is easy to see that this finally proves (35).

To prove the lower bound in (10), we first reduce ourselves to $G := (H + \beta)^{-1}$. In fact, we claim that

$$(39) \quad \sum_{n=0}^{\infty} n^2 G(n, 0) \gtrsim \lambda \beta^{-\frac{3}{2}}.$$

By (31), (32), and (36), the contribution of

$$P(L_\lambda + \beta)^{-1}P - G$$

to the energy is at most $\lambda^2 \lambda^{-\frac{3}{4}} \beta^{-\frac{3}{2}}$. Hence (39) dominates for λ small. To show (39), we use arguments as in (37), (38) together with the fact that $G := (H + \beta)^{-1}$ solves $(H + \beta)G(\cdot, 0) = \delta_0$. The latter implies that $G(n, 0)$ is nonincreasing in n (and therefore positive), and also that

$$(40) \quad \sum_{n=0}^{\infty} G(n, 0) = \beta^{-1}.$$

We will argue below that $G(0, 0) \lesssim \lambda^{-\frac{1}{2}} \beta^{-\frac{3}{4}}$. If so, then by (40) and monotonicity of $G(\cdot, 0)$ one has

$$\begin{aligned} & \sum_{n=0}^{\infty} n^2 G(n, 0) \geq n_0^2 \sum_{n=n_0}^{\infty} G(n, 0) \\ & \geq n_0^2 \left(\beta^{-1} - \sum_{n=0}^{n_0-1} G(n, 0) \right) \geq n_0^2 \beta^{-1} - n_0^3 G(0, 0) \\ (41) \quad & \geq \frac{1}{2} n_0^2 \beta^{-1} \gtrsim \lambda \beta^{-\frac{3}{2}} \end{aligned}$$

as desired. To obtain (41), set $n_0 = (2\beta G(0, 0))^{-1}$ and use the upper bound on $G(0, 0)$. To prove the latter, invoke a representation similar to (37).

In fact, for any $n, M \leq \beta^{-\frac{1}{4}}$,

$$\begin{aligned}
G(0, 0) - G(n, 0) &= \left\langle \chi_{[0,n)} \left| \nabla G \nabla^\dagger \frac{1}{M} \sum_{k=1}^M \chi_{[0,k)} \right. \right\rangle \\
&\quad + \left\langle \chi_{[0,n)} \left| \nabla G \frac{1}{M} \chi_{[1,M]} \right. \right\rangle \\
&\lesssim \left\| \chi_{[0,n)} \frac{1}{\sqrt{a}} \right\|_{\ell_{1/w}^2} \left\| \sqrt{a} \nabla G \nabla^\dagger \sqrt{a} \right\|_{\ell_w^2} \left\| \frac{1}{\sqrt{a}} \chi_{[0,M]} \right\|_{\ell_w^2} \\
&\quad + \left\| \chi_{[0,n)} \frac{1}{\sqrt{a}} \right\|_{\ell_{1/w}^2} \left\| \sqrt{a} \nabla G \right\|_{\ell_w^2} \left\| \frac{1}{M} \chi_{[1,M]} \right\|_{\ell_w^2} \\
&\lesssim \lambda^{-2} n^{\frac{3}{2}} M^{\frac{3}{2}} + n^{\frac{3}{2}} \beta^{-\frac{1}{2}} \lambda^{-1} M^{-\frac{1}{2}} \\
&\lesssim \lambda^{-\frac{5}{4}} \beta^{-\frac{3}{8}} n^{\frac{3}{2}}.
\end{aligned}$$

To pass to the final inequality, set $M = \lambda^{\frac{1}{2}} \beta^{-\frac{1}{4}}$. Now average this inequality over $0 \leq n \leq N$, $N \leq \beta^{-\frac{1}{4}}$ to be determined. In view of (40) this yields

$$\begin{aligned}
G(0, 0) &\lesssim N^{-1} \left(\beta^{-1} + \lambda^{-\frac{5}{4}} \beta^{-\frac{3}{8}} N^{\frac{5}{2}} \right) \\
&\lesssim \lambda^{-\frac{1}{2}} \beta^{-\frac{3}{4}},
\end{aligned}$$

where we have set $N = \lambda^{\frac{1}{2}} \beta^{-\frac{1}{4}}$ in the last line. This is what we claimed, and we are done.

J. Bourgain, *Growth of Sobolev norms in linear Schrödinger equations with quasi-periodic potential*, Comm. Math. Phys. **204** (1999), no. 1, 207–247.

J. Bourgain, *On growth of Sobolev norms in linear Schrödinger equations with smooth time dependent potential*, J. Anal. Math. **77** (1999), 315–348.

J. Bourgain, *Estimates on Green's functions, Localization, and the quantum kicked rotor model*. to appear in Annals of Math.

C. W. Gardiner, *Handbook of stochastic methods*, Second edition, Springer, Berlin, 1985

C.-A. Pillet, *Some results on the quantum dynamics of a particle in a Markovian potential*, Comm. Math. Phys. **102** (1985), no. 2, 237–254.

S. Tcheremchantsev, *Markovian Anderson model: bounds for the rate of propagation*. Comm. Math. Phys. **187** (1997), no. 2, 441–469.

S. Tcheremchantsev, *Transport properties of Markovian Anderson model*. Comm. Math. Phys. **196** (1998), no. 1, 105–131.