CAN YOU T THE PLANE ?

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In this short note, I will introduce a question that is very simple to state. I would love to hear from anybody that has a solution.

Let $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ be the standard inclusion given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$ and $\mathbb{R}^{n+1}_+ = \{(x_1, \ldots, x_{n+1}) \,|\, x_{n+1} \geq 0\}$.

A *letter* T *on the point* $p \in \mathbb{R}^n$ consists of the union of two straight line segments: the *base* of the T is a segment beginning at p that meets $\mathbb{R}^n$ orthogonally and is contained in $\mathbb{R}^{n+1}_+$, and the *top* of the T is a segment bisected by the endpoint of the base which is contained in the hyperplane parallel to $\mathbb{R}^n$ and containing the endpoint. We only require that the base and top of a T both have positive length so that Tees could be short and wide or tall and narrow.

We say that $\mathbb{R}^n$ can be Teed if it is possible to put a T on every point of $\mathbb{R}^n$ with no two intersecting.

Question: For which $n \in \mathbb{N}$ can $\mathbb{R}^n$ be Teed?

As an undergraduate, I was asked this question in the special case of $n = 1$. I do not know who originally posed this problem, but believe it comes from the Hungarian school of problem solving. Anyhow, it is not difficult to argue that a set of pairwise non-intersecting Tees on $\mathbb{R}^1$ is countable. For $n \geq 2$, I suspect that $\mathbb{R}^n$ cannot be Teed.

If a Ting does exist it cannot be too regular. Assume that $\mathbb{R}^2$ has been Teed so that at each point $p \in \mathbb{R}^2$ we have a T which we denote by T_p. The Ting induces functions

$$h, w : \mathbb{R}^2 \to \mathbb{R}_+$$

$$\theta : \mathbb{R}^2 \to \mathbb{RP}^1$$

that record the height, width, and directions of Tees respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{RP}^1$ be the function given by $T(p) = (h(T_p), w(T_p), \theta(T_p))$.

Claim: The set of continuity points of the function T has empty interior.

Proof. We argue by contradiction. Assume then that T is continuous on an open ball $B \subset \mathbb{R}^2$. By applying a translation, we may assume that B is centered at the origin. Orient the top of the Tees in B in a consistent fashion so that the restriction of T to B is a continuous function with image in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{S}^1$. By possibly replacing B with a ball of smaller radius, there is an $h > 0$ such that $h(x) \geq h$ for every $x \in B$.

Let

$$f : B \times (-h, h) \to \mathbb{R}^3$$

be the function which for each $x \in B$ maps $\{x\} \times (-h, h)$ linearly and preserving orientation into the top of T_x, taking $(x, 0)$ to the top of the base of T_x, and having image filling out $\frac{h}{h(x)}$ worth of the top of T_x. As T is continuous, so is f. As no two Tees in B intersect, f is injective. By Brouwer's theorem on invariance of domain [Br12], f maps onto a neighborhood of the top of the base of T_0. In particular, points in the base of T_0 slightly below the top of this base are in the image of f. As points in the image of f are all contained in the tops of Tees in B, there exist Tees that intersect T_0, a contradiction.

□

We could just as well ask about Ting a side of a smooth embeddings of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$. Here, one requires that Tees at a point in the image are Tees for the tangent space of the embedded $\mathbb{R}^n$ at that point. An argument analogous to the one above proves that the set of continuity points of a Ting of an embedding cannot have interior.

REFERENCES

[Br12] L. Brouwer. Zur Invarianz des n-dimensionalalen Gebiets, *Mathematische Annalen*, **72** (1912), p. 55-56.

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