ON SURFACES WITH ONLY SPHERICAL POINTS

BENJAMIN SCHMIDT

Abstract. Götz and Rybarski asked whether the round spheres are the only convex surfaces with the property that every pair of points joined by more than one minimizing geodesic are joined by infinitely many minimizing geodesics. Zamfirescu answered this question affirmitavely. We similarly characterize round spheres amongst smooth Riemannian surfaces.

Götz and Rybarski asked whether round spheres are the only convex surfaces with the property that every pair of points is joined either by a unique minimizing geodesic or by infinitely many minimizing geodesics ([GoRy51] or [CrFaGu91, pg. 44]). In [Za95], Zamfirescu answered their question affirmatively. In this short note, we similarly characterize round spheres amongst all smooth Riemannian surfaces.

Definition. *Let (M, g) denote a closed Riemannian manifold. A point $p \in M$ is defined to be spherical if for each $q \in M$ the set of minimizing geodesics joining p to q has either one element or at least three elements.*

Every point in a constant curvature sphere S^n is spherical. More generally, every point in a simply connected compact rank one symmetric space is spherical. It is natural to consider the following question.

Question: Assume that g is a Riemannian metric on an n-dimensional closed manifold M^n all of whose points are spherical. Must (M^n, g) be a simply connected compact rank one symmetric space?

The theorem below implies that when $n = 2$ the answer to this question is yes. Before stating the theorem, we fix some notation. For more details concerning the basics of Riemannian manifolds, we refer the reader to [ChEb75].

Let (M, g) be a closed Riemannian manifold. For a point $p \in M$, we let $C(p) \subset M$ denote the cut locus of p in M. The injectivity radius of M at p is defined by $\text{inj}_p(M, g) = d(p, C(p))$. The injectivity radius of (M, g) is defined by $\text{inj}(M, g) = \inf_{p \in M}\{\text{inj}_p(M, g)\}$ and is realized by at least one point of M.

Theorem. *Assume that* (M^2, g) *is a closed Riemannian surface which has a spherical point realizing the injectivity radius. Then* g *is a constant curvature metirc on the sphere* S^2.

Proof. Let $p \in M^2$ be a spherical point realizing the injectivity radius of (M^2, g). We first argue that since p is spherical, its cut locus $C(p)$ reduces to a point.

Recall that a point $q \in C(p)$ is called a *cleave cut point* if there are precisely two minimizing geodesic segments joining p to q such that q is not conjugate to p along either of these segments. According to [He87, Proposition 1.2], the 1-dimensional Hausdorff measure of non-cleave cut points in $C(p)$ is zero. The hypothesis that p is spherical implies that no point in $C(p)$ is a cleave cut point and hence $C(p)$ has zero 1-dimensional Hausdorff measure. On the other hand, according to [My36], $C(p)$ is a *local tree* (see [He94, Theorem 2.3] for detailed definitions). These results together easily imply that $C(p) = q$ for some point $q \in M^2$. The proof of the theorem now follows from the next lemma. □

Lemma. *Assume that* M^n *is a closed Riemannian* n*-manifold and that* $p \in M^n$ *realizes the injectivity radius. If* $C(p)$ *is a single point, then* M^n *is isometric to a constant curvature metric on* S^n.

Proof. Note that since $C(p)$ is a single point, M^n is homeomorphic to the n-sphere S^n. We now argue that the metric on M is Blaschke, i.e. that $\text{inj}(M) = \text{diam}(M)$.

As $\text{inj}(M) \leq \text{diam}(M)$ always holds, we must argue that $\text{diam}(M) \leq \text{inj}(M)$. Let $q = C(p)$ so that every unit speed geodesic leaving p minimizes up to time $\text{inj}(M)$ at which point it reaches q. Choose points $x, y \in M$ such that $d(x, y) = \text{diam}(M)$. As $\exp_p$ is surjective there exists vectors $v_x, v_y \in T_pM$ such that $\exp_p(v_x) = x$ and $\exp_p(v_y) = y$. Let $\gamma_x, \gamma_y : [0, \text{inj}(M)] \to M$ be the unit speed minimizing geodesics defined by $\gamma_x(t) = \exp_p(t\frac{v_x}{||v_x||})$ and $\gamma_y(t) = \exp_p(t\frac{v_y}{||v_y||})$.

If γ_x and γ_y are the same minimizing geodesic, then since x and y realize the diameter, $\{x, y\} = \{p, q\}$, concluding the proof in this case. Otherwise, these geodesics are distinct and $\gamma_x \cup \gamma_y \subset M$ forms a subset homeomorphic to the circle containing the points x and y and having total length $2\,\text{inj}(M)$. It follows easily that $d(x, y) \leq \text{inj}(M)$, concluding the proof that $\text{inj}(M) = \text{diam}(M)$.

By the Bott-Samelson theorem (see e.g. [Be78]), the integral cohomology ring of a Blaschke manifold coincides with the cohomology ring of one of the compact rank one symmetric spaces. Hence, a Blaschke manifold has a model symmetric space. By the above two paragraphs,

M^n is a Blaschke manifold modeled on S^n. Finally, Berger (see e.g. [Be78]) proved that the Blaschke manifolds modeled on a sphere are actually isometric with the model, concluding the proof of the lemma. □

References

[Be78] A. L. Besse, Manifolds all of whose geodesics are closed, Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[ChEb75] J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland Mathematical Library, Volume 9, 1975.

[CrFaGu91] H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved problems in geometry,* Springer-Verlag, New York, 1991.

[GoRy51] A. Götz and A. Rybarski, *Problem 102*, Colloquium Mathematicum, No. 2, pg. 301-302, 1951.

[He87] J. Hebda, Parallel translation of curvature along geodesics, Transactions of the American Mathematical Society, Vol. 299, pg. 559-572, 1987.

[He94] J. Hebda, Metric structure of cut loci in surfaces and Ambrose's problem, J. Differential Geometry, 40, pg. 621-642, 1994.

[My36] S. B. Myers, Connections between differential geometry and topology. II: Closed surfaces, Duke Math. J., 2, pg. 95-102, 1936.

[Za95] T. Zamfirescu, On some questions about convex surfaces, Math. Nachr., 172, pg. 313-324, 1995.

Benjamin Schmidt, University of Chicago, schmidt@math.uchicago.edu