ON THE VOLUME OF TUBES.*

By HERMANN WEYL.

1. The problem. In a lecture before the Mathematics Club at Princeton last year Professor Hotelling stated the following geometric problem [1] as one of primary importance for certain statistical investigations:

Let there be given in the n-dimensional Euclidean space E_n or spherical space S_n a closed ν-dimensional manifold C_ν. The solid spheres of given radius a around all the points of C_ν cover a certain part $C_\nu(a)$ of the embedding space E_n or S_n, the volume $V(a)$ of which is to be determined. We call $C_\nu(a)$ an (n,ν)-tube (of radius a around C_ν).

For small values of a one will have in the first approximation

$$V(a) = \Omega_m a^m \cdot k_0,$$

where $\Omega_m a^m$ is the volume of the solid m-dimensional sphere

$$\sigma_m(a): \quad t_1^2 + \cdots + t_m^2 \leqq a^2 \tag{1}$$

$(m = n - \nu)$, and k_0 the area of the "surface" C_ν. Professor Hotelling showed that this formula is exact in E_n and a similar formula prevails in S_n, *for* $\nu = 1$. I shall here treat the problem for higher dimensionalities ν. The result in E_n is a formula consisting of $1 + [\frac{1}{2}\nu]$ terms, of the following type (§ 3):

$$V(a) = \Omega_m \cdot \sum_e \frac{a^{m+e}}{(m+2)(m+4)\cdots(m+e)} k_e \tag{2}$$
$$(e \text{ even}, \ 0 \leqq e \leqq \nu),$$

where k_e is a certain integral invariant of the surface C_ν determined by the intrinsic metric nature of C_ν only, and thus independent of its embedding in E_n. I shall express these invariants (§ 4) in terms of the Riemannian tensor of C_ν. An analogous result is obtained for S_n.

2. The fundamental formulas for the volume of tubes. If an n-dimensional manifold M_n consisting of points u and locally referred to

* Received October 14, 1938.

[1] See his paper "Tubes and spheres in n-spaces, and a class of statistical problems" which precedes this article in this Journal, pp. 440-460.

parameters $u^1, \cdots, u^n$ is mapped upon the Euclidean space E_n with the coördinates $(x_1, \cdots, x_n) = \mathfrak{r}$,

$$(3) \qquad \mathfrak{r} = \mathfrak{r}(u) = \mathfrak{r}(u^1, \cdots, u^n),$$

then the volume V of the image of M_n in E_n may be computed by means of the formula

$$(4) \qquad V = \int [\mathfrak{r}_1 \cdots \mathfrak{r}_n] du^1 \cdots du^n,$$

where $[\mathfrak{r}_1 \cdots \mathfrak{r}_n]$ designates the determinant of the n columns $\mathfrak{r}_i$, each consisting of the components of the vector

$$\mathfrak{r}_i = \partial \mathfrak{r} / \partial u^i.$$

This formula takes account of the $\pm$ orientation and multiplicity with which the mapping $u \rightarrow \mathfrak{r}$ covers the several parts of E_n. The covering will be locally a one-to-one mapping without folds and ramifications wherever $[\mathfrak{r}_1 \cdots \mathfrak{r}_n] > 0$. But even if this condition is satisfied everywhere, multiple covering might occur. This question is essentially one of topological rather than differential geometric nature. It is with this reservation in mind that in the following we apply formula (4).

When dealing with the spherical space S_n we employ homogeneous coördinates $(x_0, x_1, \cdots, x_n) = \mathfrak{r}$, the set ρx_i meaning the same point as x_i, whatever the factor $\rho \neq 0$. Sometimes we use the normalization

$$\mathfrak{r}^2 = x_0{}^2 + x_1{}^2 + \cdots + x_n{}^2 = 1.$$

S_n then appears as the unit sphere in the Euclidean E_{n+1}. (4) must be replaced by the formula

$$(5) \qquad V = \int \frac{[\mathfrak{r}\mathfrak{r}_1 \cdots \mathfrak{r}_n]}{(\mathfrak{r}^2)^{(n+1)/2}} du^1 \cdots du^n$$

as one easily verifies by observing the following facts: (1) the integrand is orthogonally invariant; (2) it is not affected by the gauge factor $\rho = \rho(u)$ because

$$(\rho\mathfrak{r})_i = \rho \cdot \mathfrak{r}_i + \frac{\partial \rho}{\partial u^i} \cdot \mathfrak{r};$$

(3) at the point $\mathfrak{r} = (1, 0, \cdots, 0)$ the integrand reduces to the "Euclidean" value

$$\begin{vmatrix} \dfrac{\partial x_1}{\partial u^1}, \cdots, \dfrac{\partial x_n}{\partial u^1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{\partial x_1}{\partial u^n}, \cdots, \dfrac{\partial x_n}{\partial u^n} \end{vmatrix}.$$

After these preliminary remarks I now turn to our problem in E_n. Let a piece of the ν-dimensional manifold C_ν be given in the Gaussian representation

$$\mathfrak{r} = \mathfrak{r}(u^1 \cdots u^\nu). \tag{6}$$

At each point we can determine $m = n - \nu$ normal vectors $\mathfrak{n} = \mathfrak{n}(1), \cdots, \mathfrak{n}(m)$ satisfying the equations

$$\mathfrak{r}_\alpha \cdot \mathfrak{n} = 0 \qquad (\alpha = 1, \cdots, \nu)$$

which are mutually normalized by

$$\mathfrak{n}(p) \cdot \mathfrak{n}(q) = \delta_{pq} \qquad (p, q = 1, \cdots, m).$$

$\mathfrak{r}_\alpha$ is the derivative $\partial\mathfrak{r}/\partial u^\alpha$. In using the radius vector (6) and these normals, the part $C_\nu(a)$ of the space covered by the spheres of radius a around the points of C_ν allows the representation

$$\mathfrak{x} = \mathfrak{r} + t_1\mathfrak{n}(1) + \cdots + t_m\mathfrak{n}(m), \qquad (t_1^2 + \cdots + t_m^2 \leqq a^2), \tag{7}$$

in terms of the parameters $u^1, \cdots, u^\nu, t_1, \cdots, t_m$. Hence its volume $V(a)$ is the integral

$$\int [\mathfrak{x}_1, \cdots, \mathfrak{x}_\nu, \mathfrak{n}(1), \cdots, \mathfrak{n}(m)]\, dt_1 \cdots dt_m du^1 \cdots du^\nu. \tag{8}$$

Following Gauss we describe the surface C_ν embedded in E_n by its metric ground form

$$(d\mathfrak{r})^2 = \sum_{\alpha,\beta} g_{\alpha\beta} du^\alpha du^\beta, \qquad (g_{\alpha\beta} = \mathfrak{r}_\alpha \cdot \mathfrak{r}_\beta)$$

together with the linear pencil of the second fundamental forms

$$-\sum_{p=1}^{m} \{t_p \sum_{\alpha,\beta} G_{\alpha\beta}(p)\, du^\alpha du^\beta\},$$

which is the scalar product of

$$d^2\mathfrak{r} = \sum_{\alpha,\beta} \mathfrak{r}_{\alpha\beta} du^\alpha du^\beta$$

with an arbitrary normal $\mathfrak{n} = t_1\mathfrak{n}(1) + \cdots + t_m\mathfrak{n}(m)$.

$$G_{\alpha\beta}(p) = G_{\beta\alpha}(p) = -\mathfrak{r}_{\alpha\beta} \cdot \mathfrak{n}(p) = \mathfrak{r}_\alpha \cdot \mathfrak{n}_\beta(p).$$

A Greek subscript α attached to the vectors $\mathfrak{r}$, $\mathfrak{n}$ and $\mathfrak{x}$ always denotes differentiation by u^α.

Each vector at the point u of C_ν is a linear combination of the basic vectors $\mathfrak{r}_\alpha$, $\mathfrak{n}(p)$. On applying this remark to $\mathfrak{n}_\alpha(p)$ we set

$$\mathfrak{n}_\alpha(p) = \sum_\beta G_\alpha{}^\beta(p) \cdot \mathfrak{r}_\beta + \cdots,$$

where . . . indicates a linear combination of the normal vectors $\mathfrak{n}(p)$. By scalar multiplication with $\mathfrak{r}_\beta$ one finds

$$G_{\alpha\beta}(p) = \sum_\lambda g_{\alpha\lambda} G_\beta{}^\lambda(p).$$

From (7) one infers that

$$\mathfrak{x}_\alpha = \sum_\beta \{\delta_\alpha{}^\beta + \sum_{p=1}^m t_p G_\alpha{}^\beta(p)\}\mathfrak{r}_\beta + \cdots.$$

Therefore the integrand in (8)

$$= \det \{\delta_\alpha{}^\beta + \sum_p t_p G_\alpha{}^\beta(p)\} \cdot [\mathfrak{r}_1 \cdots \mathfrak{r}_\nu \mathfrak{n}(1) \cdots \mathfrak{n}(m)].$$

Because of the general identity

$$[\mathfrak{a}_1 \cdots \mathfrak{a}_n]^2 = \det (\mathfrak{a}_i \mathfrak{a}_k),$$

$$[\mathfrak{r}_1 \cdots \mathfrak{r}_\nu \mathfrak{n}(1) \cdots \mathfrak{n}(m)]^2 = |\, g_{\alpha\beta} \,|,$$

and considering that

$$ds = |\, g_{\alpha\beta} \,|^{\frac{1}{2}} du^1 \cdots du^\nu$$

is the area element of C_ν, one arrives at the fundamental formula

$$V(a) = \int_{C_\nu} \left\{ \int \cdots \int_{(t_1^2 + \cdots + t_m^2 \leqq a^2)} |\, \delta_\alpha{}^\beta + \sum_p t_p G_\alpha{}^\beta(p) \,|\, dt_1 \cdots dt_m \right\} ds$$

in the Euclidean case. The integrand is independent of the choice of the parameters u^α on C_ν.

In the spherical case, let the manifold C_ν be given by the parametric representation (6) with the normalization $\mathfrak{r}^2 = 1$. Therefore $\mathfrak{r} \cdot \mathfrak{r}_\alpha = 0$. The mutually orthogonal normal vectors $\mathfrak{n} = \mathfrak{n}(1), \cdots, \mathfrak{n}(m)$ satisfy the equations

$$\mathfrak{r} \cdot \mathfrak{n} = 0, \qquad \mathfrak{r}_\alpha \cdot \mathfrak{n} = 0.$$

From both equations there follows

$$\mathfrak{r} \cdot \mathfrak{n}_\alpha = 0.$$

The part $C_\nu(\alpha)$ of the space S_n covered by the $m = (n - \nu)$-dimensional solid spheres of spherical radius α is represented by

$$\mathfrak{x} = \mathfrak{r} + t_1\mathfrak{n}(1) + \cdots + t_m\mathfrak{n}(m),$$

where the argument u in $\mathfrak{r}, \mathfrak{n}(1), \cdots, \mathfrak{n}(m)$ ranges over the whole C_ν, while the parameters $t_1, \cdots, t_m$ are bound by

$$t_1^2 + \cdots + t_m^2 \leqq a^2, \qquad (a = \tan \alpha).$$

According to equation (5) the volume $V(\alpha)$ of $C_\nu(\alpha)$ is given by the integral of

$$(9) \qquad \frac{[\mathfrak{x}\mathfrak{x}_1 \cdots \mathfrak{x}_\nu \mathfrak{n}(1) \cdots \mathfrak{n}(m)]}{(\mathfrak{x}^2)^{(n+1)/2}} du^1 \cdots du^\nu dt_1 \cdots dt_m$$

extended with respect to $u^1, \cdots, u^\nu$ over the whole of C_ν, with respect to $t_1, \cdots, t_m$ over the sphere $\sigma_m(a)$. Application of the same procedure as before results in the formula

$$(10) \quad V(\alpha) = \int_{C_\nu} \left\{ \int \cdots \int_{(t_1^2 + \cdots + t_m^2 \leqq a^2)} | \delta_\alpha^\beta + \sum_p t_p G_\alpha^\beta(p) | \times \frac{dt_1 \cdots dt_m}{(1 + t_1^2 + \cdots + t_m^2)^{(n+1)/2}} \right\} ds.$$

3. Evaluation. For any function $\phi(t) = \phi(t_1, \cdots, t_m)$, let $\langle \phi(t) \rangle_t$ designate its mean value over the sphere

$$(11) \qquad t_1^2 + \cdots + t_m^2 = 1.$$

The mean value $\langle t_1^{e_1} \cdots t_m^{e_m} \rangle_t$ of a monomial is obviously zero unless all exponents e_p are even. In the latter case one has the well-known formula

$$(12) \qquad \langle t_1^{e_1} \cdots t_m^{e_m} \rangle_t = \frac{e_1) \cdots e_m)}{m(m+2) \cdots (m+e-2)} \qquad (e_p \text{ even, } e = e_1 + \cdots + e_m),$$

where

$$0) = 1, \quad e) = 1 \cdot 3 \cdots (e-1) \qquad [\text{for } e = 2, 4, \cdots].$$

[(12) is most easily proved by multiplying the monomial by

$$e^{-t_1^2} \cdots e^{-t_m^2} = e^{-(t_1^2 + \cdots + t_m^2)}$$

and then integrating over

$$-\infty < t_p < \infty \qquad (p = 1, \cdots, m).$$

One thus obtains

$$\int t_1^{e_1}\cdots t_m^{e_m}d\omega_t\cdot\int_0^\infty e^{-r^2}r^{e+m-1}dr=\prod_p\left(\int_{-\infty}^{+\infty}e^{-t^2}t^{e_p}dt\right),$$
$$(e=e_1+\cdots+e_m),$$

with $\int\cdots d\omega_t$ indicating the "solid angle" integration over the sphere (11), and hence

$$\frac{1}{2}\int t_1^{e_1}\cdots t_m^{e_m}d\omega_t=\frac{\Gamma\left(\frac{1+e_1}{2}\right)\cdots\Gamma\left(\frac{1+e_m}{2}\right)}{\Gamma\left(\frac{m+e}{2}\right)}.$$

In particular, for the surface $\omega_m=\int d\omega_t$ of the sphere,

$$\frac{1}{2}\omega_m=\left[\Gamma\left(\frac{1}{2}\right)\right]^m\Big/\Gamma\left(\frac{m}{2}\right).\tag{13}$$

Division results in the desired equation

$$\langle t_1^{e_1}\cdots t_m^{e_m}\rangle_t=\frac{\Gamma\left(\frac{1}{2}+\frac{e_1}{2}\right)\cdots\Gamma\left(\frac{1}{2}+\frac{e_m}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\cdots\Gamma\left(\frac{1}{2}\right)}\Bigg/\frac{\Gamma\left(\frac{m}{2}+\frac{e}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}$$
$$=\frac{e_1)\cdots e_m)}{m(m+2)\cdots(m+e-2)}.$$

The volume of the solid sphere $\sigma_m(a)$ amounts to

$$\omega_m\int_0^a r^{m-1}dr=\frac{\omega_m}{m}\cdot a^m;$$

hence $\Omega_m=\omega_m/m$. Specialization of (13) for $m=2$ yields $[\Gamma(\frac{1}{2})]^2=\pi$. The numbers ω_m, Ω_m are best defined by the recursive formulas readily derived from (13):

$$\omega_{m+2}=\frac{2\pi}{m}\cdot\omega_m\quad(m\geqq 1);\qquad(\omega_1=2,\quad\omega_2=2\pi).$$
$$\Omega_{m+2}=\frac{2\pi}{m+2}\cdot\Omega_m\quad(m\geqq 0);\qquad(\Omega_0=1,\ \Omega_1=2).]$$

We expand the determinant

$$\psi(t_1\cdots t_m)=|\,\delta_\alpha^\beta+\sum_p t_pG_\alpha^\beta(p)\,|=\psi_0+\psi_1+\cdots+\psi_\nu$$

according to degrees in the variables $t_1,\cdots,t_m$:

$$\psi_e(t_1\cdots t_m)=\sum\phi_{e_1\ldots e_m}t_1^{e_1}\cdots t_m^{e_m}\qquad(e_1+\cdots+e_m=e)$$

is homogeneous of degree e. $\psi_0 = 1$. This decomposition is conveniently described by introducing an artificial parameter λ:

$$|\delta_\alpha{}^\beta + \lambda \sum_p t_p G_\alpha{}^\beta(p)| = 1 + \lambda\psi_1 + \lambda^2\psi_2 + \cdots.$$

We set

$$\langle\psi_e(t_1 \cdots t_m)\rangle_t = \frac{H_e}{m(m+2)\cdots(m+e-2)}.$$

By its definition, H_e is a point invariant of C_ν. H_e is zero for odd e, while for even e one derives from (12) the explicit expression

$$H_e = \sum e_1) \cdots e_m) \cdot \phi_{e_1 \ldots e_m}, \qquad (e_p \text{ even, } e_1 + \cdots + e_m = e).$$

The integral over the solid sphere $\sigma_m(a)$,

$$\int \cdots \int_{\sigma_m(a)} \psi_e(t_1 \cdots t_m)\, dt_1 \cdots dt_m$$

then will turn out to be

$$\frac{\omega_m H_e}{m(m+2)\cdots(m+e-2)} \int_0^a r^{e+m-1} dr = \omega_m H_e \cdot \frac{a^{m+e}}{m(m+2)\cdots(m+e)}.$$

Thus we find in the Euclidean case

$$V(a) = \Omega_m \sum_e k_e \frac{a^{m+e}}{(m+2)(m+4)\cdots(m+e)}, \qquad (e \text{ even, } 0 \leqq e \leqq \nu), \tag{14}$$

with the coefficients

$$k_e = \int_{C_\nu} H_e ds. \tag{15}$$

In the spherical case one gets

$$\int \cdots \int_{\sigma_m(a)} \psi_e(t) \frac{dt_1 \cdots dt_m}{(1 + t_1{}^2 + \cdots + t_m{}^2)^{(n+1/)2}}$$
$$= \frac{\omega_m H_e}{m(m+2)\cdots(m+e-2)} \int_0^a \frac{r^{e+m-1} dr}{(1+r^2)^{(n+1/)2}}.$$

On putting $r = \tan\rho$ the integral at the right side becomes

$$\int_0^a (\sin\rho)^{m+e-1} (\cos\rho)^{\nu-e} d\rho,$$

and instead of (14) one obtains

$$V(\alpha) = \omega_m \cdot \sum_e k_e J_e(\alpha), \qquad (e \text{ even, } 0 \leqq e \leqq \nu), \tag{16}$$

where

$$(17)\quad m(m+2)\cdots(m+e-2)J_e(\alpha)=\int_0^\alpha(\sin\rho)^{m+e-1}(\cos\rho)^{\nu-e}d\rho.$$

One may notice the recurrent equation

$$\frac{(\sin\alpha)^{e+m}(\cos\alpha)^{\nu-e-1}}{m(m+2)\cdots(m+e)}=J_e(\alpha)-(\nu-e-1)J_{e+2}(\alpha).$$

THEOREM. *The volumes of (n,ν)-tubes in Euclidean and in spherical space are given by the formulas* (14), (16) *respectively, $J_e(\alpha)$ being defined by* (17). *k_e,* (15), *are certain integral invariants of C_ν, in particular k_0 is its surface.*

4. Intrinsic nature of the invariants k_e. So far we have hardly done more than what could have been accomplished by any student in a course of calculus. However, some less obvious argument is needed for ascertaining that more explicit form of the point invariant H_e which enables one to replace the curvature $G_\alpha{}^\beta(p)$ by the Riemannian tensor $R^\lambda{}_{\mu\alpha\beta}$ of C_ν. I repeat the definition of this tensor in terms of the metric ground tensor:

$$\sum_\lambda g_{\kappa\lambda}\Gamma^\lambda{}_{\alpha\beta}=\tfrac{1}{2}\left(\frac{\partial g_{\alpha\kappa}}{\partial u^\beta}+\frac{\partial g_{\beta\kappa}}{\partial u^\alpha}-\frac{\partial g_{\alpha\beta}}{\partial u^\kappa}\right)$$

[definition of the affine connection $\Gamma^\kappa{}_{\alpha\beta}$],

$$R^\kappa{}_{\lambda\alpha\beta}=\left(\frac{\partial\Gamma^\kappa{}_{\lambda\beta}}{\partial u^\alpha}-\frac{\partial\Gamma^\kappa{}_{\lambda\alpha}}{\partial u^\beta}\right)+\sum_\rho(\Gamma^\kappa{}_{\rho\alpha}\Gamma^\rho{}_{\lambda\beta}-\Gamma^\kappa{}_{\rho\beta}\Gamma^\rho{}_{\lambda\alpha}).$$

After raising the index λ according to

$$R^\kappa{}_{\lambda\alpha\beta}=\sum_\mu g_{\lambda\mu}R^{\kappa\mu}_{\alpha\beta},$$

$R^{\kappa\lambda}_{\alpha\beta}$ is not only skew-symmetric in $\alpha\beta$, but also in $\kappa\lambda$. As a part of the integrability conditions expressing the *Euclidean* nature of the embedding space E_n, one has the relations [2]

$$(18)\qquad R^{\kappa\lambda}_{\alpha\beta}=\sum_{p=1}^m\{G_\alpha{}^\kappa(p)G_\beta{}^\lambda(p)-G_\beta{}^\kappa(p)G_\alpha{}^\lambda(p)\}.$$

In the *spherical* case we look upon C_ν as a surface in E_{n+1}. To the set of m normals $\mathfrak{n}(p)$, $(p=1,\cdots,m)$ one has simply to add $\mathfrak{n}(0)=\mathfrak{r}$. Since

[2] See H. Weyl, *Mathematische Zeitschrift*, vol. 12 (1922), p. 154.

$$\mathfrak{n}_\alpha(0) = \mathfrak{r}_\alpha \text{ or } G_\alpha{}^\beta(0) = \delta_\alpha{}^\beta,$$

(18) changes into the equation

$$(19) \quad R^{\kappa\lambda}_{\alpha\beta} - (\delta_\alpha{}^\kappa\delta_\beta{}^\lambda - \delta_\beta{}^\kappa\delta_\alpha{}^\lambda) = \sum_{p=1}^{m} \{G_\alpha{}^\kappa(p)G_\beta{}^\lambda(p) - G_\beta{}^\kappa(p)G_\alpha{}^\lambda(p)\}.$$

[It is a pity that the inadequate name "curvature," which ought to be reserved for $G_\alpha{}^\beta(p)$, has been attached to the Riemann tensor. In the paper just quoted I proposed the more descriptive term "vector vortex." The left side of (19), and also of (18), is the excess of the vortex of C_ν over that of the embedding space. In this form the relation would hold with an arbitrary embedding Riemann space.]

We must try then to express the spherical average

$$\langle \det (\delta_\alpha{}^\beta + \lambda \sum_p t_p G_\alpha{}^\beta(p))\rangle_t$$

in terms of the quantities

$$(20) \qquad H\binom{\kappa\lambda}{\alpha\beta} = \sum_p G_\alpha{}^\kappa(p)G_\beta{}^\lambda(p) - \sum_p G_\alpha{}^\lambda(p)G_\beta{}^\kappa(p).$$

In this investigation the

$$G_\alpha{}^\beta = (G_\alpha{}^\beta(1), \cdots, G_\alpha{}^\beta(m)),$$

just as

$$t = (t_1, \cdots, t_m),$$

may be looked upon as arbitrary vectors in an m-dimensional Euclidean space E_m. Using for a moment the abbreviation

$$z_\alpha{}^\beta = (t \cdot G_\alpha{}^\beta) = \sum_p t_p G_\alpha{}^\beta(p),$$

one has

$$\psi_e = \sum_{a_1 < \cdots < a_e} \begin{vmatrix} z_{a_1}{}^{a_1}, & \cdots, & z_{a_1}{}^{a_e} \\ \cdot & \cdot & \cdot \\ z_{a_e}{}^{a_1}, & \cdots, & z_{a_e}{}^{a_e} \end{vmatrix}.$$

Hence we try to determine

$$\langle \det (t \cdot G_\alpha{}^\beta)\rangle_t,$$

where $G_\alpha{}^\beta$, $(\alpha, \beta = 1, \cdots, e)$ are any e^2 given vectors in E_m.

LEMMA.

$$(21)\quad \underset{(\alpha,\beta=1,\cdots,e)}{\langle \det\,(t\cdot G_\alpha{}^\beta)\rangle_t} = \frac{1}{m(m+2)\cdots(m+e-2)} \sum_{[\alpha,\beta]} \delta\begin{pmatrix}\beta\\ \alpha\end{pmatrix} H\begin{pmatrix}\beta_1\beta_2\\ \alpha_1\alpha_2\end{pmatrix}\cdots H\begin{pmatrix}\beta_{e-1}\beta_e\\ \alpha_{e-1}\alpha_e\end{pmatrix}.$$

$\alpha_1\cdots\alpha_e$, $\beta_1\cdots\beta_e$ are the numbers $1,\cdots,e$ in any two arrangements, $\delta\begin{pmatrix}\beta\\ \alpha\end{pmatrix}=\pm 1$ according as the permutation carrying the α- into the β-arrangement is even or odd. The sum extends over all couplings of pairs

$$(22)\qquad \begin{array}{c|c|c} (\alpha_1\alpha_2) & (\alpha_3\alpha_4) & \\ (\beta_1\beta_2) & (\beta_3\beta_4) & \cdots. \end{array}$$

By a "pair" $(\alpha_1\alpha_2)$ we mean here two distinct numbers α_1, α_2, *irrespective* of their order. Indeed the term $T\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$ under the sum $\sum_{[\alpha\beta]}$ on the right side of (21) does not change under reversal of an α-pair, $(\alpha_1\alpha_2)\rightarrow(\alpha_2\alpha_1)$, or of a β-pair. Nor does it change under permutation of its $e/2$ factors H; therefore only the coupling of the α-pairs with the β-pairs, but not the order of the $e/2$ blocks of the scheme (22) matters. Of the $2^e\cdot(\frac{1}{2}e)!$ equal terms arising from $T\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$ by inverting any of the e pairs of indices and by permuting the $e/2$ factors H, only one is retained in the sum.

Taking the lemma for granted, we find at once

$$(23)\qquad H_e = \sum_{[\alpha,\beta]} \delta\begin{pmatrix}\beta\\ \alpha\end{pmatrix} H\begin{pmatrix}\beta_1\beta_2\\ \alpha_1\alpha_2\end{pmatrix}\cdots H\begin{pmatrix}\beta_{e-1}\beta_e\\ \alpha_{e-1}\alpha_e\end{pmatrix},$$

where the sum now extends to all couplings of pairs (22) from the larger range $1,2,\cdots,\nu$ for which the β-sequence consists of the same e distinct figures as the α-sequence. The invariant nature of the sum to the right is evidenced when we first write it as

$$\frac{1}{2^e(e/2)!}\sum_{\alpha_1,\ldots,\alpha_e}\ \sum_{1',\ldots,e'} \pm H\begin{pmatrix}\alpha_{1'}\ \alpha_{2'}\\ \alpha_1\ \ \alpha_2\end{pmatrix} H\begin{pmatrix}\alpha_{3'}\ \alpha_{4'}\\ \alpha_3\ \ \alpha_4\end{pmatrix}\cdots, \qquad (e/2 \text{ factors}),$$

the inner sum alternatingly running over the permutations $1',\cdots,e'$ of $1,\cdots,e$. The limitation of distinctness imposed upon $\alpha_1,\cdots,\alpha_e$ can be canceled, as the inner sum vanishes if two of the α's coincide. Hence

$$(24)\qquad H_e = \frac{1}{2^e(e/2)!}\sum_{1',\ldots,e'} \left\{ \pm \sum_{\alpha_1,\ldots,\alpha_e} H\begin{pmatrix}\alpha_{1'}\ \alpha_{2'}\\ \alpha_1\ \ \alpha_2\end{pmatrix} H\begin{pmatrix}\alpha_{3'}\ \alpha_{4'}\\ \alpha_3\ \ \alpha_4\end{pmatrix}\cdots \right\}.$$

The inner sum in which each α runs independently from 1 to ν is a scalar. We have thus arrived at the decisive

THEOREM. *The scalar H_e on C_ν is determined by the formulas* (23), (24) *where* $H\binom{\lambda\mu}{\alpha\beta}$ *is the Riemann tensor or vortex* $R^{\lambda\mu}_{\alpha\beta}$ *in the Euclidean case, and the vortex excess* (19) *in the spherical case.*

These metric scalars H_e deserve attention on their own merits: they are probably the simplest and most fundamental scalars built up by the Riemann tensor.

As a very special case of our theorem we find that *the one term formulas*

$$V(a) = \Omega_m a^m \cdot k_0, \qquad V(\alpha) = \omega_m J_0(\alpha) \cdot k_0$$

prevail if C_ν is applicable on E_ν or S_ν respectively. k_0 denotes the surface of C_ν. Professor Hotelling's result concerning the tubes around a *curve*, $\nu = 1$, is fully contained in this special case.

The lemma is proved by an invariant-theoretic argument as follows. We consider the e^2 vectors $G_\alpha{}^\beta$ as independent variables.

$$\Phi = \langle \det (t \cdot G_\alpha{}^\beta) \rangle_t$$

is an orthogonal invariant of these variables and therefore, according to the theory of orthogonal vector invariants,[3] expressible as a polynomial in terms of the scalar products $(G_\alpha{}^\lambda \cdot G_\beta{}^\mu)$. Observing that Φ is linear and homogeneous in the components of the vectors of each row and each column of the scheme

$$\begin{Vmatrix} G_1{}^1, \cdots, G_1{}^e \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ G_e{}^1, \cdots, G_e{}^e \end{Vmatrix}$$

we realize that it must be a linear combination of terms

$$(G_{\alpha_1}{}^{\beta_1} \cdot G_{\alpha_2}{}^{\beta_2}) \cdots (G_{\alpha_{e-1}}^{\beta_{e-1}} \cdot G_{\alpha_e}{}^{\beta_e}),$$

where the α and β are any two arrangements of $1, \cdots, e$. Moreover Φ is skew-symmetric with respect to the columns. Hence, by summing alternatingly over the $e!$ permutations of the superscripts β we find that Φ is a linear combination of the following functions

[3] E. Study, *Ber. Sächs. Akad. Wissensch.* 1897, p. 442. H. Weyl, *Mathematische Zeitschrift*, vol. 20 (1924), p. 136.

$$\sum_{(\beta)} \delta\binom{\beta}{\alpha} (G_{\alpha_1}{}^{\beta_1} \cdot G_{\alpha_2}{}^{\beta_2}) (G_{\alpha_3}{}^{\beta_3} \cdot G_{\alpha_4}{}^{\beta_4}) \cdots = \sum_{[\beta]} \delta\binom{\beta}{\alpha} H\binom{\beta_1\beta_2}{\alpha_1\alpha_2} H\binom{\beta_3\beta_4}{\alpha_3\alpha_4} \cdots.$$

The first sum runs over all $e!$ permutations $\beta_1 \cdots \beta_e$ of $1, \cdots, e$, the second over all their $e!/2^{e/2}$ arrangements in "pairs"

$$(\beta_1\beta_2), \quad (\beta_3\beta_4), \cdots.$$

By applying the same argument to the subscripts α one concludes that Φ is a constant multiple c of H_e, (23).

The constant c is determined by the specialization

$$G_\alpha{}^\beta = (\delta_\alpha{}^\beta, 0, \cdots, 0)$$

for which

$$\Phi = \langle t_1{}^e \rangle_t = \frac{e)}{m(m+2)\cdots(m+e-2)}$$

and

$$H\binom{\lambda\mu}{\alpha\beta} = \delta_\alpha{}^\lambda \delta_\beta{}^\mu - \delta_\alpha{}^\mu \delta_\beta{}^\lambda, \quad H_e = e!/2^{e/2}(\tfrac{1}{2}e)! = e).$$

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