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Notes on Gromov's systolic estimate

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Abstract These notes cover some of the main results of Gromov's paper Filling Riemannian manifolds. The goal of these notes is to make the results and proofs accessible to more people. The main result is that if (M, g) is a Riemannian manifold of dimension n, then there is a non-contractible curve in (M, g) of length at most C_n $\mathrm{Vol}(M, g)^{1/n}$.

Keywords Systole · Isoperimetric inequality · Filling radius

These notes are an exposition of Gromov's systolic estimate, which first appeared in the long paper Filling Riemannian manifolds [4]. Gromov's paper is one of the most important in metric geometry—that part of geometry in which distance, length, area, and volume are the main characters.

The systole of a Riemannian manifold (M, g) is defined to be the smallest length of a non-contractible curve in M. We recall that a manifold M is aspherical if $\pi_i(M) = 0$ for all $i \geq 2$. For example, the torus T^n is aspherical.

Theorem *For any n-dimensional closed aspherical Riemannian manifold (M, g), the systole is bounded in terms of the volume by the following formula.*

$$\mathrm{Systole}(M, g) \leq C(n)\mathrm{Volume}(M, g)^{1/n}.$$

In order to prove the theorem, Gromov invented a new geometric invariant, called the filling radius. Roughly speaking, the filling radius measures how "thick" a Riemannian manifold is. For example, the filling radius of the cylinder $S^1 \times \mathbb{R}$ (with the standard product metric) is $\pi/3$, but the filling radius of $\mathbb{R}^2$ (with the Euclidean metric) is infinite. We will define the filling radius below.

Using the filling radius, the proof of the theorem breaks into two pieces.

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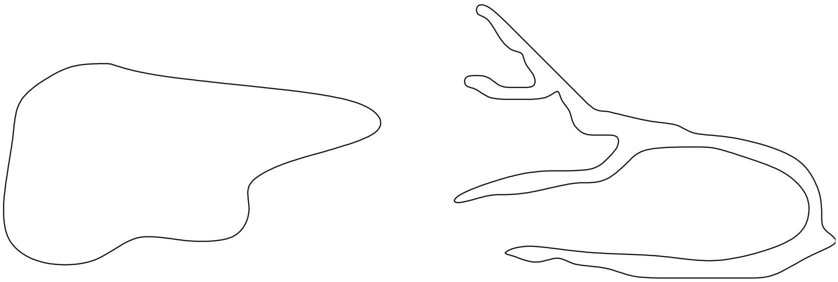


Fig. 1 Surfaces with different filling radii

Theorem A *If M is a closed aspherical manifold, then the systole of* (M, g) *is bounded in terms of the filling radius by the formula* Systole(M,g)$\leq$ 6 Fill Rad(M,g).

Theorem B *For any closed manifold M, the filling radius is bounded in terms of the volume by the formula* Fill Rad*(M,g)*$\leq C(n)\text{Volume}(M, g)^{1/n}$.

In Sect. 1, we define the filling radius. In Sect. 2, we prove Theorem A. In Sect. 3, we prove Theorem B. The sections are rather unequal though. The third section contains most of the work. In an appendix afterwards, we extend the systolic estimate to a few more manifolds, such as $\mathbb{RP}^n$.

1 The definition of filling radius

The filling radius of a Riemannian manifold is defined by analogy with an invariant defined for submanifolds of Euclidean space. Suppose that $M^n \subset \mathbb{R}^N$ is a closed submanifold. By a filling of M, we mean an (n+1)-chain C with $\partial C = M$. The filling radius of M is defined to be the smallest number R so that there exists a filling of M inside the R-neighborhood of M.

(Technical detail: we need to specify what coefficients to allow for the chain C. Gromov makes the convention to use integral coefficients for oriented manifolds and mod 2 coefficients for non-oriented manifolds.)

We give a few examples. The filling radius of an ellipse is equal to its smallest principal axis. The filling radius of the cylinder $S^1 \times \mathbb{R} \subset \mathbb{R}^3$ is equal to 1. (I have in mind the cylinder defined by the equation $x^2 + y^2 = 1$.) In Fig. 1, there are some examples of surfaces with different filling radii.

Gromov gave an analogous definition for a closed Riemannian manifold (M, g). At first sight, a Riemannian metric g on M is a completely different thing from an embedding of M into Euclidean space. At this point, Gromov used a construction of Kuratowski. Using the metric g on M, Kuratowski constructed a canonical embedding from M into the Banach space $L^\infty(M)$. Kuratowski's embedding sends a point $x \in M$ to the function dist_x. (The function dist_x is a bounded function on M because M is closed.) We call this embedding K, for Kuratowski, and so we can write $K(x) = \text{dist}_x$. This embedding is an exact isometry according to the following lemma.

Lemma 1 $\|\text{dist}_x - \text{dist}_y\|_{L^\infty} = \text{dist}(x, y)$.

Proof For any point $z \in M$, the triangle inequality gives $|\text{dist}_x(z) - \text{dist}_y(z)| = |\text{dist}(x, z) - \text{dist}(y, z)| \leq \text{dist}(x, y)$. Therefore, $\|\text{dist}_x - \text{dist}_y\|_{L^\infty} \leq \text{dist}(x, y)$. Moreoever, if we take $z = y$, then $|\text{dist}_x(z) - \text{dist}_y(z)| = |\text{dist}(x, y) - 0| = \text{dist}(x, y)$. Therefore, $\|\text{dist}_x - \text{dist}_y\|_{L^\infty} = \text{dist}(x, y)$. □

For each metric g, we get a canonical embedding K: $M \to L^\infty(M)$. Gromov then defined the filling radius of (M, g) to be the infimal R so that the embedded submanifold $K(M)$ bounds a chain inside its R-neighborhood.

When I first saw it, this definition seemed very abstract to me. One thing that I found intimidating was the presence of an infinite-dimensional space $L^\infty(M)$. We can also formulate the definition of filling radius in terms of finite-dimensional spaces with very large dimensions. The resulting construction is not as canonical but it is easier to work with.

Suppose that S is a set of points, $x_1, \ldots, x_N \in M$. We can then define a map K_S from M to the N-dimensional Banach space l^N_∞. (This is the space of vectors $v = (v_1, \ldots, v_\mathrm{N})$ with the norm $|v| = \sup |v_i|$.) The map K_S sends a point x to a vector with ith component $K_S(x)_i = \text{dist}_{x_i}(x)$. By taking x_i sufficiently dense, we can arrange that this map is an embedding and that it is almost an isometry.

Lemma 2 *Given a closed manifold (M, g), for each $\epsilon > 0$, we can choose a finite set of points S sufficiently densely so that the following inequality holds for any $x, y \in M$.*

$$(1 - \epsilon)\text{dist}(x, y) \leq |K_S(x) - K_S(y)|_\infty \leq \text{dist}(x, y). \qquad (*)$$

Proof (sketch) First note that the upper bound $|K_S(x) - K_S(y)|_\infty \leq \text{dist}(x, y)$ holds for any set S.

We will pick a small number $\delta > 0$, and take S to be any δ-net in (M, g). If $\text{dist}(x, y) >> \delta$, then the lower bound follows by choosing a point $x_i \in S$ with $\text{dist}(x, x_i) < \delta$.

Since the problem is scale-invariant, we may assume that we first scaled (M, g) so that its injectivity radius is at least 10 and its curvature is at most δ^2. Suppose for a moment that every unit ball in (M, g) was Euclidean. Then we could prove the lower bound as follows. Look at the line segment from x to y, and then extend it past y a distance (1/10). Let z denote the end of the extension. Then $\text{dist}(x, z) = \text{dist}(x, y) + \text{dist}(y, z)$. If the point z were in our set S, this equation would imply the lower bound $|K_S(x) - K_S(y)|_\infty \geq \text{dist}(x, y)$. Most likely, z is not in S, but we can choose $x_i \in S$ with $\text{dist}(x_i, z) < \delta$. Then it follows from trigonometry that $\text{dist}(x_i, x) - \text{dist}(x_i, y) > (1 - \epsilon)\text{dist}(x, y)$.

Finally, we can generalize the trigonometry argument to almost flat manifolds using the Toponogov comparison theorem. □

We now fix a set S so that $(1/2)\text{dist}(x, y) \leq |K_S(x) - K_S(y)| \leq \text{dist}(x, y)$. We will use this finite set for the rest of the paper. The map K_S is as useful for our purposes as the full Kuratowski embedding, and the filling radius of $K_S(M)$ is as useful as the official filling radius of M.

For context, we include one more fact about the Banach spaces l^N_∞ and L^∞. This fact allows us to show that if K_S is nearly isometric, then the filling radius of $K_S(M)$ is nearly equal to the filling radius of M. The rest of the results in this section are not needed to prove the systolic estimate.

Lemma 3 (Extension property) *Let B denote the Banach space l_∞^N or L^∞. Given a metric space X with and a subset $Y \subset X$, and a function $f\colon Y \to B$ with Lipschitz constant 1, there exists an extension F of f to X with the same Lipschitz constant.*

Proof We work up through a sequence of cases. First we take $B = l_\infty^1 = \mathbb{R}$. We define $F(x)$ to be $\sup_{y\in Y}\big[f(y) - \mathrm{dist}(x,y)\big]$. Any extension G of f with Lipschitz constant 1 must have $G(x) \geq f(y) - \mathrm{dist}(x,y)$ for every y in Y. Hence the formula for F was motivated by putting down the minimal plausible value for $F(x)$. The graph of F somewhat resembles a sheet or string that has been pinned in place over Y and hangs down from the pins over the rest of X.

If $x \in Y$, then we can take $y = x$, and we get $F(x) \geq f(x) - \mathrm{dist}(x,x) = f(x)$. Also $F(x) = \sup_y f(y) - \mathrm{dist}(x,y) \leq f(x)$ because f has Lipschitz constant 1. Therefore, F restricted to Y is f. If x_1, x_2 are two points in X, then the difference of functions $\big[f(y) - \mathrm{dist}(x_1,y)\big] - \big[f(y) - \mathrm{dist}(x_2,y)\big] = \mathrm{dist}(x_2,y) - \mathrm{dist}(x_1,y)$ has norm at most $\mathrm{dist}(x_1,x_2)$ at every point. Therefore, the difference between the suprema of the two functions is bounded by $\mathrm{dist}(x_1,x_2)$. In other words, $|F(x_1) - F(x_2)| \leq \mathrm{dist}(x_1,x_2)$, and F has Lipschitz constant 1.

Next we take $B = l_\infty^N$ for a finite N. In this case, we use the last argument on each component of the vector-valued function f. More precisely, we define $F_i(x) = \sup_{y\in Y}[f_i(y) - \mathrm{dist}(x,y)]$. The argument above shows that $F|_Y = f$ and that $|F(x_1) - F(x_2)|_{l_\infty^N} = \sup|F_i(x_1) - F_i(x_2)| \leq \mathrm{dist}(x_1,x_2)$.

Finally, we take $B = L^\infty(M)$. In this case, we still use essentially the same formula, $F(x) = \sup_{y\in Y}\big[f(y) - \mathrm{dist}(x,y)\big]$, but interpreted in a slightly more complicated way. In this case, $f(y)$ is a function in L^∞, and $\mathrm{dist}(x,y)$ is interpreted as a constant function in L^∞. The sup denotes a (pointwise) supremum of functions. This case is morally the same as the vector-valued case that we just did, except that instead of indexing $F(x)$ by a number $i \in 1,\ldots,N$, we index it by a point p in the space M where our measurable functions are defined. The proof is the same as for vectors, except that we need to check that $F(x)$ is a bounded measurable function. Since each function $f(y) - \mathrm{dist}(x,y)$ is measurable, the supremum is also measurable. Next we check that $F(x)$ is bounded. Fix a point $y_0 \in Y$. The function $F(x)$ is greater than or equal to the function $f(y_0) - \mathrm{dist}(x,y_0)$, which is bounded below. Next we check that $F(x)$ is bounded above. We note that $\sup f(y) \leq \sup(f(y_0)) + |f(y) - f(y_0)|_\infty \leq \sup(f(y_0)) + \mathrm{dist}(y,y_0) \leq \sup(f(y_0)) + \mathrm{dist}(y_0,x) + \mathrm{dist}(x,y)$. Hence $f(y) - \mathrm{dist}(x,y)$ is bounded above by $\sup(f(y_0)) + \mathrm{dist}(x,y_0)$, and so our function $F(x)$ is bounded above. □

As a corollary to this lemma, we see that if K_S obeys the quasi-isometric estimate $(*)$, then the filling radius of $K_S(M)$ is pinched between $(1-\epsilon)\mathrm{FillRad}(M,g)$ and $\mathrm{FillRad}(M,g)$. As another corollary, we see that if (M,g) and (N,h) are closed Riemannian manifolds of dimension n, and if ϕ is a degree 1 map from (M,g) to (N,h) with Lipschitz constant 1, then $\mathrm{FillRad}(M,g) \geq \mathrm{FillRad}(N,h)$.

2 Systoles and fillings

In this Section, we establish a relationship between the systole of M and the filling radius of $K_S(M)$.

Lemma 4 *Suppose that M is a closed aspherical manifold, and that g is a Riemannian metric on M. Let K_S be an embedding from (M,g) to l_∞^N that obeys the following quasi-isometric estimate for any pair of points $x,y \in M$.*

$$(1/2)\mathrm{dist}(x,y) \le |K_S(x) - K_S(y)|_\infty \le \mathrm{dist}(x,y).$$

Then the filling radius of $K_S(M)$ *is at least* Systole$(M,g)/12$.

Proof The proof is by contradiction. We assume that $K_S(M)$ bounds a chain C inside its R-neighborhood, for $R <$ Systole$(M)/12$. We will get our contradiction by constructing a map ϕ: $C \to M$ with $\phi|_{\partial C}$ equal to the identity.

We pick a fine triangulation of the chain C so that each edge has length at most $\delta > 0$, a small number we can choose later. We have already defined $\phi|_{\partial C}$. We will define ϕ on the rest of C one skeleton at a time. We begin by defining ϕ on the vertices of the triangulation. Let v be a vertex. Since C lies inside the R-neighborhood of $K_S(M)$, we can pick a point $w \in K_S(M)$ within a distance R of v. We define $\phi(v) = w$.

Next we define ϕ on the edges of the triangulation. Suppose that E is an edge of the triangulation and that E has boundary vertices v_1 and v_2. The map ϕ moved each boundary vertex at most R. Since the triangulation was δ-fine, the distance from $\phi(v_1)$ to $\phi(v_2)$ in l^N_∞ is at most $2R + \delta$. Now $\phi(v_1)$ and $\phi(v_2)$ are each points in $K_S(M)$. Because of the quasi-isometric estimate for K_S, we know that the distance between the points in (M,g) is at most $4R + 2\delta$. We map E onto a minimal path in (M,g) between the two points.

Next we define ϕ on the 2-simplices of the triangulation. Suppose that Δ is a 2-simplex. We have already defined ϕ on the boundary of the 2-simplex, mapping the boundary to a circle in (M,g) of length at most $12R+6\delta$. By assumption, $R <$ Systole$(M,g)/12$. We can choose $\delta > 0$ sufficiently small so that $12R + 6\delta < Systole(M,g)$. Therefore, the map of the boundary of Δ into M is contractible. We pick a contraction and use it to define ϕ on Δ.

Finally, we define ϕ on the higher-dimensional simplices of the triangulation, working one skeleton at a time. Suppose we have extended ϕ to the $(k-1)$-skeleton, and consider a k-simplex Δ. The restriction of ϕ to the boundary of Δ is null-homotopic, because M is aspherical. Therefore, ϕ can be extended to Δ. We pick any extension.

This finishes the construction of a retraction ϕ: $C \to K_S(M)$. Since the fundamental class $[M]$ is non-trivial in M, this retraction is a contradiction. We may conclude that $R \ge$ Systole$(M,g)/12$. □

Exercise: Using the same argument in the infinite-dimensional space L^∞, check that Fill Rad$(M,g) \ge$ Systole$(M,g)/6$. This argument proves Theorem A.

3 Filling radius inequality and isoperimetric inequality

We now turn to the main estimate in the proof of our theorem.

Theorem B *Let C be any n-cycle in l^N_∞. Then the filling radius of C is controlled in terms of its volume by the following formula.*

$$\mathrm{FillRad}(C) \le C_n \mathrm{Vol}(C)^{1/n}.$$

This estimate is slightly more general than the filling radius inequality in the introduction. At the end of the section, we explain how it implies the original version of Theorem B. Here, we explain how this version of Theorem B implies our main theorem.

Let M be a closed aspherical manifold of dimension n with a Riemannian metric g. Using Lemma 2, we pick a finite set of points $S \subset M$ so that $K_S: M \to l^N_\infty$ is quasi-isometric with the bound $(1/2)\mathrm{dist}(x,y) \le |K_S(x) - K_S(y)|_{l^N_\infty} \le \mathrm{dist}(x,y)$. The image $K_S(M)$ is a cycle C with volume at most $Vol(M,g)$. Applying Theorem B above, we see that the filling radius of $K_S(M)$ is at most $C_n Vol(M,g)^{\frac{1}{n}}$. According to Lemma 4, the systole of (M,g) is at most 12 times the filling radius of $K_S(M)$. Therefore, $\mathrm{Systole}(M,g) \le C_n Vol(M,g)^{\frac{1}{n}}$.

The proof of Theorem B is fairly substantial. To get some perspective, we recall some facts about the analogous question for cycles in Euclidean space. The first work on this problem was done by Federer and Fleming [3]. They gave an elegant construction which shows the following bound.

Theorem (Federer and Fleming) *For any n-cycle C in Euclidean space $\mathbb{R}^N$, the following inequality holds.*

$$\mathrm{FillRad}(C) \le C_N \mathrm{Vol}(C)^{1/n}.$$

We will give the proof of this estimate below. The proof generalizes easily to the Banach space l^N_∞. But the result is weaker than the lemma we want to prove, because the constant C_N depends on the ambient dimension N, whereas the constant in our lemma depends only on n. This point is crucial to the systolic estimate, because we have no control over N, the number of points we need to get a good approximation of the Kuratowski embedding.

For cycles in Euclidean space, the sharp constant was found by Simon and Bombieri [2]. It occurs when the cycle C is a round n-sphere, and it does not depend on N. The proof is based on examining the minimal-volume filling of C and using minimal surface theory. Their argument does not generalize to Banach spaces (at least not easily).

The filling radius estimate is in the same spirit as a more famous result: the isoperimetric inequality. We recall the isoperimetric inequality for cycles in Euclidean space.

Theorem (Federer–Fleming, Michael–Simon, Almgren) *Suppose that C is an n-cycle in Euclidean space $\mathbb{R}^N$. Then C bounds a chain F obeying the following estimate.*

$$\mathrm{Vol}(F) \le C_n \mathrm{Vol}(C)^{\frac{n+1}{n}}.$$

In [3], Federer and Fleming proved this inequality with a constant C_N depending on the ambient dimension. Then in [5], Michael and Simon proved the inequality with a non-sharp constant C_n depending only on the dimension of C. Finally, in [1], Almgren proved the inequality with a sharp constant. (The sharp constant occurs when C is a round sphere.) As with the filling radius, the Federer–Fleming argument adapts to Banach spaces, but the other arguments don't (at least not easily).

In analogy with the term filling radius, we will call the smallest volume of any chain F with $\partial F = C$ the filling volume of C.

We now begin to discuss filling estimates in l^N_∞. First of all, we have to deal with a technical point. What do we mean by the volume of a cycle or a chain in l^N_∞? There are several different definitions that seem reasonable. Fortunately, all the definitions that I've seen agree up to a factor C_n. In this paper, we define the volume to be the n-dimensional Hausdorff measure, because this definition is familiar to most people. The main fact about volume that we will use is the coarea inequality. We first recall the version in Euclidean space.

Proposition (The coarea inequality) *Let C be an n-dimensional chain in Euclidean space $\mathbb{R}^N$. Let f be a function on $\mathbb{R}^N$ with Lipschitz constant 1. Then the following inequality holds.*

$$\mathrm{Vol}_n\,(C \cap \{x|a \le f(x) \le b\}) \ge \int_a^b \mathrm{Vol}_{n-1}\,(C \cap \{x|f(x) = t\})\,\mathrm{d}t.$$

With our definition of volume, I don't see why the coarea formula should hold for chains in l_∞^N, but it does hold with a fudge-factor. The proof is that the chain C is bilipschitz to a piecewise Riemannian chain with bilipschitz constant at most C_n. Therefore, we get the following coarea inequality in l_∞^N.

Proposition (The coarea inequality in Banach space) *There is a constant $C_n > 0$ so that the following estimate holds. If C is any n-dimensional chain in the Banach space l_∞^N, and f is any function on $\mathbb{R}^N$ with Lipschitz constant 1, then*

$$\mathrm{Vol}_n\,(C \cap \{x|a \le f(x) \le b\}) \ge C_n \int_a^b \mathrm{Vol}_{n-1}\,(C \cap \{x|f(x) = t\})\,\mathrm{d}t.$$

This constant C_n in the coarea formula is a penalty that we pay for using the most well-known definition of volume. In Filling Riemannian manifolds, Gromov surveys several definitions of volume, including one where the coarea formula holds on the nose. The constants in Theorem B are not sharp anyway, so this penalty is not too important.

Throughout this section, it is crucial to distinguish between constants that depend only on the dimension n of the chain C and constants that depend on the dimension N of the ambient space. We use C to denote an absolute constant, C_n to denote a constant that depends only on n, and C_N to denote a constant that depends on N. All constants are positive, but their exact value may change from line to line.

We now give Federer and Fleming's construction, leading to bounds for the filling radius and the filling volume. Although the bounds of Federer and Fleming are weaker than the ones in Theorem B, their estimate is a necessary part of the proof of the theorem.

Theorem (Federer and Fleming) *Suppose that z is an n-cycle in $\mathbb{R}^N$. Then there is a chain A with $\partial A = z$, obeying the following estimates.*

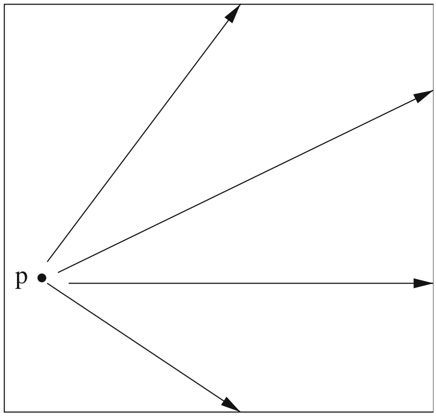
1. *The volume of A is at most $C_N \mathrm{Vol}(z)^{\frac{n+1}{n}}$.*
2. *The distance from any point $x \in A$ to z is at most $C_N \mathrm{Vol}(z)^{1/n}$.*

Proof By a scaling argument, it suffices to prove the theorem when $\mathrm{Vol}(z) = 1$.

We consider the cubical lattice with side length S, for a constant S that we will choose later. We will construct a sequence of cycles $z = z_N \sim z_{N-1} \sim \cdots \sim z_n$ with z_k contained in the k-skeleton of the lattice. To get from z_k to z_{k-1} we use the following lemma.

Lemma 5 (Pushing a cycle to the boundary of a cube) *Suppose that z is a relative n-cycle inside a k-dimensional cube Q with side length S, $k > n$. Then there is an n-chain $z' \subset \partial Q$ with $\partial z' = \partial z$ and $\mathrm{vol}(z') \le C_k \mathrm{vol}(z)$. Moreover, $z - z' = \partial A$, for an $(n{+}1)$-chain A in Q with volume at most $C_k S \mathrm{vol}(z)$.*

Fig. 2 The retraction Ψ_p



A retraction to the boundary of the cube

Proof By a scaling argument, it suffices to prove the result when Q is a unit cube.

Let p be a point in Q which does not lie in z. There is a radial retraction Ψ_p from $Q-p$ to ∂Q, which maps each ray r leaving p to the point on $r \cap \partial Q$. See Fig. 2 for an illustration of this map.

The image $\Psi_p(z)$ is a chain z' in ∂Q with $\partial z' = \partial z$, since $\Psi_p|_{\partial z}$ is the identity. The volume of $\Psi_p(z)$ could possibly be very large, because any piece of z that lies close to p will be stretched by the map $\Psi_p(z)$.

The next step in the proof is to estimate the average value of the volume of $\Psi_p(z)$, averaging as we consider various points $p \in Q$. First we upper bound the volume of $\Psi_p(z)$.

$$\mathrm{Vol}[\Psi_p(z)] \leq C \int_z [\mathrm{dist}(p,x)]^{-n} \mathrm{d}x.$$

This formula follows because the derivative of the map Ψ_p at the point x is at most $C\mathrm{dist}(p,x)^{-1}$.

Therefore, the average volume of $\Psi_p(z)$ is controlled by the following expression.

$$C \int_Q \int_z [\mathrm{dist}(p,x)]^{-n} \mathrm{d}x \, \mathrm{d}p.$$

We estimate this integral by changing the order of integration.

$$= C \int_z \left(\int_Q [\mathrm{dist}(p,x)]^{-n} \mathrm{d}p \right) \mathrm{d}x.$$

The interior integral converges because Q has dimension $k > n$. Its value depends on x, but it can be bounded uniformly by $\int_{B(0,10\sqrt{N})} \mathrm{dist}(0,p)^{-n} \mathrm{d}p$.

$$\leq \int_z C_k \mathrm{d}x \leq C_k \mathrm{Vol}(z).$$

We choose a point p so that the volume of $\Psi_p(z)$ is at most average and define $z' = \Psi_p(z)$.

We take A to be the difference $\mathrm{Cone}_p(z') - \mathrm{Cone}_p(z)$, where $\mathrm{Cone}(p,z)$ is defined to be the cone over z with apex p. The chain A has volume at most $C_k \mathrm{vol}(z') + C_k \mathrm{vol}(z) \leq C_k \mathrm{vol}(z)$. □

Using this lemma repeatedly, we construct cobordisms from z_k to z_{k-1}. Pick an open k-face F of our lattice and consider $z_k \cap F$, which is a relative n-cycle in F. Using the last lemma, we push $z_k \cap F$ to the boundary of F, and doing this on each k-face F we arrive at a cycle z_{k-1} in the $(k-1)$-skeleton of our lattice. The volume of z_{k-1} is at most C_k times the volume of z_k. Also, by adding together the chains A from each face F, we get a chain A_k with boundary $z_k - z_{k-1}$ and with volume at most $C_k S\mathrm{vol}(z_k)$. Finally we end up with a chain z_n in the n-skeleton of the cubical lattice with volume at most $C_N\mathrm{vol}(z) = C_N$.

The estimate for the volume of z_n did not depend on the choice of lattice scale S. We now choose S so that S^N is greater than the upper bound for $\mathrm{vol}(z_n)$. Therefore, for each n-face F, we can choose a point $x \in F$ disjoint from z_n. Using the point x to push out from, it follows that z_n is homologous to a cycle z_{n-1} in the $(n-1)$-skeleton of our lattice, via a cobordism with zero $(n+1)$-dimensional volume. In other words, we can construct an $(n{+}1)$-chain A_n in the n-skeleton of our lattice with $\partial A_n = z_n$ and with zero $(n{+}1)$-dimensional volume.

We define A to be the sum $\sum_{k=n}^{N} A_n$. The boundary of A is our original cycle $z = z_N$. The volume of A is at most C_N. Therefore, the filling obeys estimate 1. Also, the chain A intersects a given closed N-cube of our lattice only if z itself intersects that N-cube. Therefore, the filling A lies within $S\sqrt{N}$ of the cycle z. The choice of S depended only on N, and so the filling obeys estimate 2. □

This finishes the proof of the Federer–Fleming isoperimetric inequality. I believe that it was the first isoperimetric inequality for surfaces with codimension greater than 1. I think it is one of the most fundamental results in metric geometry. Later proofs have improved the constant in the inequality, but I think that they are all more complicated than this one.

The Banach norm l_∞^N given by $|(v_1, \dots, v_N)|_\infty = \max |v_i|$ and the Euclidean norm given by $|(v_1, \dots, v_N)|_2 = (\sum v_i^2)^{1/2}$ agree up to a factor of $\sqrt{N}$. Therefore, the above theorem applies equally well to cycles in l_∞^N after modifying the constants C_N. We state this as a corollary.

Corollary *Suppose that z is an n-cycle in l_∞^N. Then there is a chain A with $\partial A = z$, obeying the following estimates.*

1. *The volume of A is at most $C_N \mathrm{Vol}(z)^{\frac{n+1}{n}}$.*
2. *The distance from any point $x \in A$ to z is at most $C_N \mathrm{Vol}(z)^{1/n}$.*

We are not satisfied with estimates 1 and 2 because the constants in the estimates depend on N. Ideally we would like to know the sharp constants. At this point, we give the sharp constants names.

1. We use I_N to denote the supremal filling volume of any n-cycle in l_∞^N of volume 1.
2. We use J_N to denote the supremal filling radius of any n-cycle in l_∞^N of volume 1.

Our goal is to prove that J_N is bounded above independent of N. According to the following lemma, it is enough to show that I_N is bounded above independent of N.

Lemma 6 *The filling radius constant J_N is controlled in terms of the isoperimetric constant I_N by the formula $J_N \le C_n I_N$.*

Proof Fix an n-dimensional cycle C in our Banach space. For any $\epsilon > 0$, we need to find a filling D lying in the $C_n I_N \mathrm{Vol}(C)^{1/n}$ neighborhood of C.

Suppose that there is a chain D of minimal volume filling C. In Euclidean space, it is an important result of geometric measure theory that such a chain exists. If the chain D exists, we show that it obeys the estimate we want. We will not assume that a minimizing chain actually exists. Instead, we will then modify the proof so that it uses an almost-minimizing chain.

For now, suppose that D is a minimal chain filling C. Because D is minimal, it obeys the following isoperimetric inequality.

Isoperimetric inequality: let $U \subset D$ be any open set that does not border on C.

$$\mathrm{Vol}(U) \leq I_N \mathrm{Vol}(\partial U)^{\frac{n+1}{n}}. \tag{$*$}$$

This isoperimetric inequality holds because otherwise we could reduce the volume of D by removing U and replacing it with a new chain filling ∂U with smaller volume.

Next, let $V(R)$ denote the volume of the open set $\{x \in D | \mathrm{dist}(x, C) > R\}$. Let $A(R)$ denote the area of $\{x \in D | \mathrm{dist}(x, C) = R\}$. As R increases, $V(R)$ decreases. By the coarea inequality, it follows that $V'(R) \leq -C_n A(R)$. But the area $A(R)$ is bounded below in terms of $V(R)$ by the isoperimetric inequality $(*)$, giving the following inequality.

$$V'(R) \leq -C_n A(R) \leq -C_n I_N^{-\frac{n}{n+1}} V(R)^{\frac{n}{n+1}}.$$

As long as $V(R) > 0$, we can rearrange this to give the following.

$$\frac{\mathrm{d}}{\mathrm{d}R}[V(R)^{\frac{1}{n+1}}] = \frac{1}{n+1} V(R)^{-\frac{n}{n+1}} V'(R) \leq -C_n I_N^{-\frac{n}{n+1}}.$$

Hence $V(R) = 0$ for some R at most $C_n \mathrm{Vol}(D)^{\frac{1}{n+1}} I_N^{\frac{n}{n+1}}$. Since the volume of D is in turn bounded by $I_N \mathrm{Vol}(C)^{\frac{n+1}{n}}$, we can conclude that D lies in the R-neighborhood of C for $R = C_n I_N \mathrm{Vol}(C)^{1/n}$. This is the estimate we want to prove.

We are not finished because we do not know that there is any chain D of minimal volume. For each $\delta > 0$, we do know that there is a chain with volume within δ of the infimal volume. In other words, we can find a chain D so that $\mathrm{vol}(D) \leq \mathrm{vol}(D') + \delta$ for any chain D' with $\partial D' = C$. This chain D obeys a slightly weaker version of the isoperimetric inequality $(*)$.

Isoperimetric inequality: let $U \subset D$ be any open set that does not border on C.

$$\mathrm{Vol}(U) \leq I_N \mathrm{Vol}(\partial U)^{\frac{n+1}{n}} + \delta. \tag{$*'$}$$

This isoperimetric inequality holds because otherwise we could reduce the volume of D by more than δ by removing U and replacing it with a new chain filling ∂U.

Next, let $V(R)$ denote the volume of the open set $\{x \in D | \mathrm{dist}(x, C) > R\}$. Let $A(R)$ denote the area of $\{x \in D | \mathrm{dist}(x, C) = R\}$. As R increases, $V(R)$ decreases, and $V'(R) \leq -C_n A(R)$. But the area $A(R)$ is bounded below in terms of $V(R)$ by the isoperimetric inequality $(*')$, giving the following inequality.

$$V'(R) \leq -C_n A(R) \leq -C_n I_N^{-\frac{n}{n+1}} [V(R) - \delta]^{\frac{n}{n+1}}.$$

We rewrite this equation as

$$\frac{\mathrm{d}}{\mathrm{d}R}[V(R) - \delta] \leq -C_n I^{\frac{n}{n+1}} [V(R) - \delta]^{\frac{n}{n+1}}.$$

As long as $V(R)-\delta>0$, we can rearrange this to give the following.

$$\frac{\mathrm{d}}{\mathrm{d}R}\left([V(R)-\delta]^{\frac{1}{n+1}}\right)\leq -C_n I_N^{-\frac{n}{n+1}}.$$

Hence $V(R)\leq\delta$ for some R at most $C_n I_N^{\frac{n}{n+1}}\mathrm{Vol}(D)^{\frac{1}{n+1}}$. Since the volume of D is in turn bounded by $I_N\mathrm{Vol}(C)^{\frac{n+1}{n}}$, $V(R)\leq\delta$ for some R at most $C_n I_N \mathrm{Vol}(C)^{1/n}$.

In other words, if D is δ-almost minimal, then it lies in the R-neighborhood of C except for a piece of volume at most δ. This piece resembles an appendix of small volume that may stretch arbitrarily far from C. Our next step is to cut it off.

Since $V'(R)\leq -C_n A(R)$, we can find a radius $R_0\leq C_n I_N\mathrm{Vol}(C)^{1/n}+\sqrt{\delta}$ where $A(R)\leq C_n\sqrt{\delta}$. Next, we cut D along the hypersurface $\mathrm{dist}(x,C)=R_0$, creating a new boundary of area at most $C_n\sqrt{\delta}$. Now we fill in this new boundary using the Federer-Fleming construction. The filling of the this boundary has filling radius at most $C_N(\sqrt{\delta})^{1/n}$. Therefore, the entire chain that we have constructed lies inside of the $[C_n I_N\mathrm{Vol}(C)^{1/n}+\sqrt{\delta}+C_N(\sqrt{\delta})^{1/n}]$-neighborhood of C. Since this estimate holds for every $\delta>0$, we are done.

Our last task is to prove that the isoperimetric constant I_N is bounded by a constant C_n depending only on n. The proof has many similarities to the one we have just used, but it is one level deeper.

Lemma 7 *Any n-cycle C in l_∞^N has a filling D with volume at most $C_n\mathrm{Vol}(C)^{\frac{n+1}{n}}$.*

Proof The proof is by induction on the dimension n. The base case for the induction is $n=1$. In this case, C is a union of circles. We can assume that there are k circles with lengths $L_1,\ldots,L_k$. Now to fill each circle, we pick a point x in that circle, and use the cone over the circle with vertex x. Since the diameter of a circle is bounded by its length, the cones have total area at most $C(L_1^2+\cdots+L_k^2)\leq C(L_1+\cdots+L_k)^2$.

Now we come to the inductive step. This step is probably the heart of the whole proof. By a scaling argument, it suffices to consider the case $\mathrm{vol}(C)=1$. Among all of the possible cycles with volume 1, we want to pick the one with the *largest* filling volume. Suppose for a moment that such a cycle exists. Somewhat paradoxically, this cycle would have special properties that make its filling volume easier to bound above than the filling volume of a general cycle. Using those properties, we could prove that its filling volume is at most C_n. Since we assumed that it had the largest filling volume of any cycle with volume 1, it would follow that every cycle of volume 1 had filling volume at most C_n. There is a technical problem, though, because it looks hard to prove that a cycle with maximal filling volume exists. We do not prove that such a cycle exists. As in the proof of the last lemma, we choose a cycle with almost maximal filling volume, and modify our argument to work for it.

Suppose for now that C is an n-cycle in l_∞^N with volume 1, and with maximal filling volume I_N among all n-cycles of volume 1 in l_∞^N. Using the inductive hypothesis, we will show that C obeys an isoperimetric inequality with a constant depending only on n. Algebraically, the proof is a little tedious, but geometrically it is quite straightforward.

Suppose that $A\subset C$ is a subset with boundary ∂A. By induction, we can assume that any $(n-1)$-cycle z in l_∞^N bounds an n-chain y with volume at most $C_n|z|^{\frac{n}{n-1}}$. We apply this assumption to ∂A, and we conclude that it bounds a chain B with volume at most $C_n|\partial A|^{\frac{n}{n-1}}$. Now we view C as a sum of two cycles $C_1=A+B$, and $C_2=-B+A^c$.

(Here A^c denotes the complement of A in C.) The volumes of C_1 and C_2 are controlled as follows.

$$|C_1| \leq |A| + C_n|\partial A|^{\frac{n}{n-1}},$$

$$|C_2| \leq (1 - |A|) + C_n|\partial A|^{\frac{n}{n-1}}.$$

Now by assumption we can find a filling D_1 of C_1 with volume at most $I_N|C_1|^{\frac{n+1}{n}}$ and we can find a filling D_2 of C_2 with volume at most $I_N|C_2|^{\frac{n+1}{n}}$. The sum $D_1 + D_2$ constitutes a filling of C, and therefore must have volume at least I_N. Therefore, $|C_1|^{\frac{n+1}{n}} + |C_2|^{\frac{n+1}{n}} \geq 1$. Plugging in our upper bounds on $|C_1|$ and $|C_2|$, we get the following inequality.

$$\left[|A| + C_n|\partial A|^{\frac{n}{n-1}}\right]^{\frac{n+1}{n}} + \left[(1 - |A|) + C_n|\partial A|^{\frac{n}{n-1}}\right]^{\frac{n+1}{n}} \geq 1. \tag{1}$$

Now we claim that this inequality implies an isoperimetric inequality of the form $\min(|A|, 1 - |A|) \leq C_n|\partial A|^{\frac{n}{n-1}}$. This claim is only a matter of elementary algebra, but it is a nasty mess. I think the computation of this step is one of the harder parts to read in [4]. Here we give a geometric explanation of this step.

Consider the quadrant $x \geq 0, y \geq 0$ in $\mathbb{R}^2$. In this quadrant, we look at the graph G defined by the equation $x^{\frac{n+1}{n}} + y^{\frac{n+1}{n}} = 1$. The graph meets the coordinate axes at right angles at the two points $(1, 0)$ and $(0, 1)$. The line segment L given by $x + y = 1$ joins the two points. The situation is illustrated below (Fig. 3).

Lemma 8 *Suppose that* $x, y > 0$ *and* $x + y = 1$, *and that* $(x + d)^{\frac{n+1}{n}} + (y + d)^{\frac{n+1}{n}} \geq 1$. *Then* $\min(x, y) \leq C_n d$.

Proof If $(x + d)^{\frac{n+1}{n}} + (y + d)^{\frac{n+1}{n}} > 1$, then we can decrease d until $(x + d)^{\frac{n+1}{n}} + (y + d)^{\frac{n+1}{n}} = 1$. It suffices to check that in this case $\min(x, y) \leq C_n d$. By a compactness argument, it suffices to check this inequality as (x, y) approaches either of the endpoints of L. Since G is perpendicular to the coordinate axes at each endpoint, it follows that when (x, y) is very close to one of the endpoints, then $\min(x, y) \leq (1 + \epsilon)d$. □

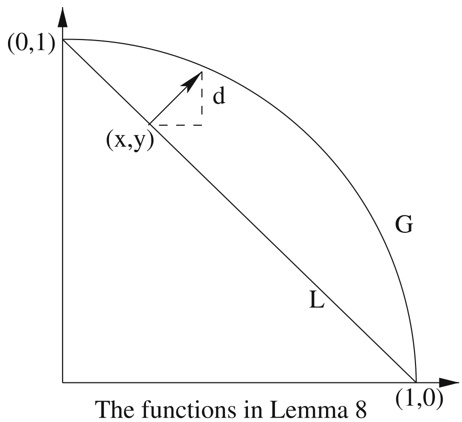


Fig. 3 The functions in Lemma 8

We apply this lemma to Eq. 1. (To do this, we make the substitutions $x = |A|$, $y = 1 - |A|$, and $d = C_n|\partial A|^{\frac{n}{n-1}}$.)

$$\min[|A|, 1 - |A|] \le C_n[C_n|\partial A|^{\frac{n}{n-1}}].$$

This is the inequality we were trying to prove.

Armed with this isoperimetric inequality, we next prove an upper bound on the diameter of the cycle C. We pick a coordinate function x_i on l_∞^N. We translate C so that the plane $x_i = 0$ bisects the volume of C. Then we let $V(h)$ denote the volume of $C \cap \{x|x_i \ge h\}$ and we let $A(h)$ denote the area of $C \cap \{x|x_i = h\}$. For now, we consider the range $h \ge 0$. Since the plane $x_i = 0$ bisects C, we know that $V(0) = 1/2$. By the coarea inequality, we know that $V'(h) \le -C_nA(h)$. On the other hand, the isoperimetric inequality tells us that $A(h) \ge C_nV(h)^{\frac{n-1}{n}}$. Assembling these formulas, we get the following.

$$V'(h) \le -C_nV(h)^{\frac{n-1}{n}}.$$

As long as $V(h) > 0$, we can divide by a power of $V(h)$ to get the following inequality.

$$[V(h)^{1/n}]' = (1/n)V'(h)V(h)^{-\frac{n-1}{n}} \le -C_n.$$

Therefore, we must have $V(h) = 0$ before C_n. In other words, C lies below the plane $x_i = C_n$. By a symmetric argument, it lies above the plane $x_i = -C_n$. Repeating this argument for all coordinate functions, we see that C lies in the box $[-C_n, C_n]^N$. Since we are in the Banach space l_∞^N, this box has bounded diameter C_n.

Finally, we construct a filling of C by taking the cone through C with apex at the origin. Since the distance from any point of C to the origin is at most C_n, the volume of this cone is at most C_n. On the other hand, the filling volume of C is equal to I_N. Therefore, I_N is bounded by a dimensional constant C_n.

This argument would prove our lemma except for one snag. We assumed that we could find a cycle of volume 1 with filling volume equal to I_N—the supremal filling volume of any cycle with volume 1. It's not at all clear that this supremum is realized. What is clear is that for any $\delta > 0$, we can find a cycle C with volume 1 and filling volume at least $(1 - \delta)I_N$. Our last task is to adapt the above argument to an almost maximizing cycle instead of an exactly maximizing cycle.

Let $\delta > 0$ be a small number that we will choose later. Suppose that C is a cycle in l_∞^N with volume 1 and filling volume at least $(1 - \delta)I_N$. By imitating the argument above, we will prove that C can be cut into two pieces, one of them having diameter at most C_n and the other having small volume.

Suppose that $A \subset C$ is a subset with boundary ∂A. By induction, we can assume that any $(n-1)$-cycle z in l_∞^N bounds an n-chain y with volume at most $C_n|z|^{\frac{n}{n-1}}$. We apply this assumption to ∂A, and we conclude that it bounds a chain B with volume at most $C_n|\partial A|^{\frac{n}{n-1}}$. Now we view C as a sum of two cycles $C_1 = A + B$, and $C_2 = -B + A^c$. (Here A^c denotes the complement of A in C.) The volumes of C_1 and C_2 are controlled as follows.

$$|C_1| \le |A| + C_n|\partial A|^{\frac{n}{n-1}}.$$

$$|C_2| \le (1 - |A|) + C_n|\partial A|^{\frac{n}{n-1}}.$$

Now by assumption we can find a filling D_1 of C_1 with volume at most $I_N|C_1|^{\frac{n+1}{n}}$ and we can find a filling D_2 of C_2 with volume at most $I_N|C_2|^{\frac{n+1}{n}}$. The sum $D_1 + D_2$ constitutes a filling of C, and therefore must have volume at least $(1-\delta)I_N$. Therefore, $|C_1|^{\frac{n+1}{n}} + |C_2|^{\frac{n+1}{n}} \geq (1-\delta)$. Plugging in our upper bounds on $|C_1|$ and $|C_2|$, we get the following inequality.

$$\left[|A| + C_n|\partial A|^{\frac{n}{n-1}}\right]^{\frac{n+1}{n}} + \left[(1-|A|) + C_n|\partial A|^{\frac{n}{n-1}}\right]^{\frac{n+1}{n}} \geq (1-\delta) \geq (1-\delta)^{\frac{n+1}{n}}. \quad (2)$$

Now we claim that this inequality implies a conditional isoperimetric inequality of the following form.

Conditional Isoperimetric Inequality: there is a constant $C_n > 0$ so that the following estimate holds. If $A \subset C$ with $|A| > C_n\delta$ and $1 - |A| > C_n\delta$, then the following inequality holds.

$$\min(|A|, 1-|A|) \leq C_n|\partial A|^{\frac{n}{n-1}}.$$

Lemma 9 *Suppose that* $x, y > 0$ *and* $x + y = 1 - \delta$, *and that* $(x+d)^{\frac{n+1}{n}} + (y+d)^{\frac{n+1}{n}} \geq (1-\delta)^{\frac{n+1}{n}}$. *Then* $\min(x, y) \leq C_n d$.

Proof This result follows from the last lemma by scaling. More precisely, $x' = (1-\delta)^{-1}x$, $y' = (1-\delta)^{-1}y$, and $d' = (1-\delta)^{-1}d$. Then x', y', and d' obey the hypotheses of the last lemma, and hence $\min(x', y') \leq C_n d'$, which implies that $\min(x, y) \leq C_n d$. □

We apply this lemma to Eq. 2. To do the application, we take $x = |A| - \delta/2$, $y = 1 - |A| - \delta/2$, and $d = \delta/2 + C_n|\partial A|^{\frac{n}{n-1}}$.

$$\min[|A| - \delta/2, 1 - |A| - \delta/2] \leq C_n[\delta/2 + C_n|\partial A|^{\frac{n}{n-1}}].$$

Therefore, if $|A| > C_n\delta$ and if $1-|A| > C_n\delta$, then we have the isoperimetric inequality $|A| < C_n|\partial A|^{\frac{n}{n-1}}$.

Armed with this conditional isoperimetric inequality, we next prove that most of the cycle C lies in a box of controlled diameter. We pick a coordinate function x_i on l^N_∞. We translate C so that the plane $x_i = 0$ bisects the volume of C. Then we let $V(h)$ denote the volume of $C \cap \{x|x_i \geq h\}$ and we let $A(h)$ denote the area of $C \cap \{x|x_i = h\}$. For the time being, we consider the range $h \geq 0$. Since the plane $x_i = 0$ bisects C, we know that $V(0) = 1/2$. By the coarea formula, we know that $V'(h) \leq -C_n A(h)$. On the other hand, the isoperimetric inequality tells us that $A(h) \geq C_n V(h)^{\frac{n-1}{n}}$, as long as $V(h) > C_n\delta$. Assembling these formulas, we get the following inequality.

$$V'(h) \leq -C_n V(h)^{\frac{n-1}{n}} \quad \text{if } V(h) > C_n\delta.$$

We can divide by a power of $V(h)$ to get the following inequality.

$$[V(h)^{1/n}]' = (1/n)V'(h)V(h)^{-\frac{n-1}{n}} \leq -C_n \quad \text{if } V(h) > C_n\delta.$$

Therefore, we must have $V(h) = C_n\delta$ before h reaches C_n. In other words, the part of C lying above the plane $x_i = C_n$ has volume at most $C_n\delta$. Similarly, the part of C lying below the plane $x_i = -C_n$ has volume at most $C_n\delta$. Repeating the same argument for each coordinate function, we find that the part of C outside of the box $[-C_n, C_n]^N$ has volume at most $NC_n\delta$. By choosing δ, we can make this volume as small as we like.

Next, we let $A(R)$ denote the volume of C intersected with the edge of the box $[-C_n - R, C_n + R]^N$. By the coarea formula, we can choose $0 \leq R_0 \leq 1$ so that $A(R_0) \leq$

$NC_n\delta$. We let A denote the intersection of C with the box $[-C_n - R_0, C_n + R_0]^N$. By the Federer–Fleming lemma, we can fill ∂A by a chain B with volume at most $C_N[NC_n\delta]^{\frac{n}{n-1}}$, lying inside a box with sides of length $C_n + C_N\delta^{\frac{1}{n}}$. Now we view C as a sum of two cycles, $C_1 = A+B$, and $C_2 = -B+A^c$, where A^c denotes the complement of A in C. By choosing $\delta > 0$ sufficiently small, we can arrange that C_1 has volume at most $(1+\epsilon)$ and that C_2 has volume at most ϵ, for any $\epsilon > 0$, and that C_1 lies in a cube of diameter at most C_n. Since C_1 lies in a cube of diameter C_n, it can be filled by a cone with of volume at most $(1+\epsilon)C_n$. Using the Federer–Fleming construction, it follows that C_2 can be filled by a chain with volume at most $C_N\epsilon^{\frac{n+1}{n}}$. By choosing ϵ sufficiently small, it follows that C can be filled by a cycle with volume at most C_n. Now, the filling volume of C is at least $(1-\delta)I_N$. Therefore, we can conclude that I_N is bounded by a constant C_n, independent of the ambient dimension N. □

This finishes the proof of the filling radius estimate in l^N_∞ and also the proof of the systolic estimate.

To conclude this section, we prove the filling radius estimate given in the introduction.

Theorem B *For any closed manifold M, the filling radius is bounded in terms of the volume by the formula* $\mathrm{FillRad}(M,g) \leq C(n)\mathrm{Volume}(M,g)^{1/n}$.

Proof Let (M,g) be a closed Riemannian manifold of dimension n. By Lemma 2, we can find a finite set of points $S \subset M$ so that K_S: $M \to l^N_\infty$ is an embedding obeying the following quasi-isometric estimate for any pair of points x, y in M.

$$(1/2)\mathrm{dist}(x,y) \leq |K_S(x) - K_S(y)| \leq \mathrm{dist}(x,y).$$

The image $K_S(M)$ is a subset of l^N_∞. We can map this subset to $L^\infty(M)$ by sending $K_S(x)$ to $K(x)$. Because of the quasi-isometric bound for K_S, this map has Lipschitz constant 2. By Lemma 3, we can extend this map to a map F from $K_S(M)$ to all of l^N_∞ with the same Lipschitz constant 2. According to the main theorem of this section, $K_S(M)$ bounds some chain A in its $C_n\mathrm{Vol}(M,g)^{1/n}$—neighborhood in l^N_∞. Now $F(A)$ is a chain in $L^\infty(M)$, with boundary $K(M)$. Because the map F stretches distances by at most a factor of 2, the image $F(A)$ lies in the $2C_n\mathrm{Vol}(M,g)^{1/n}$ neighborhood of $K(M)$. Therefore, $\mathrm{Fill\ Rad}(M,g) \leq C_n\mathrm{Vol}(M,g)^{1/n}$. □

Appendix: essential manifolds

The systolic estimate $\mathrm{Systole}(M,g) \leq C_n\mathrm{Vol}(M,g)^{1/n}$ requires a topological assumption about the manifold M. The simplest topological assumption is that M is a closed aspherical manifold, but the theorem remains true for a broader class of manifolds called essential manifolds. For example, the real projective space $\mathbb{RP}^n$ is essential but not aspherical.

A closed n-dimensional manifold M is called essential if there is an aspherical space X and a map f from M to X so that $f_*([M]) \neq 0$ in $H_n(X)$. (If M is oriented, we use the integral fundamental class and integral homology groups, and if M is not orientable, we use the mod 2 fundamental class and mod 2 homology groups.) The manifold $\mathbb{RP}^n$ is essential, because the linear inclusion $\mathbb{RP}^n \subset \mathbb{RP}^\infty$ sends $[\mathbb{RP}^n]$ to the non-zero class in $H_n(\mathbb{RP}^\infty, \mathbb{Z}_2)$, and because $\mathbb{RP}^\infty$ is aspherical.

Lemma 4 can be extended to essential manifolds by slightly modifying the proof as follows.

Lemma 10 *Suppose that M is a closed essential manifold, and that g is a Riemannian metric on M. Let K_S be an embedding from (M, g) to l_∞^N that obeys the following quasi-isometric estimate for any pair of points $x, y \in M$.*

$$(1/2)\mathrm{dist}(x, y) \leq |K_S(x) - K_S(y)|_\infty \leq \mathrm{dist}(x, y).$$

Then the filling radius of $K_S(M)$ is at least Systole$(M, g)/12$.

Proof Since M is essential, we know that there is an aspherical space X and a map f from M to X that does not kill the fundamental homology class $[M]$.

The proof is by contradiction. We assume that $K_S(M)$ bounds a chain C inside its R-neighborhood, for $R <$ Systole$(M)/12$. We will get our contradiction by constructing a map ϕ: $C \to X$ with $\phi|_{\partial C}$ equal to the map f.

We pick a fine triangulation of the chain C so that each edge has length at most $\delta > 0$, a small number we can choose later. We have already defined $\phi|_{\partial C}$. We will define ϕ on the rest of C one skeleton at a time. We begin by defining ϕ on the vertices of the triangulation. Let v be a vertex. Since C lies inside the R-neighborhood of $K_S(M)$, we can pick a point $w \in K_S(M)$ within a distance R of v. We define $\phi(v) = f(w)$.

Next we define ϕ on the edges of the triangulation. Suppose that E is an edge of the triangulation and that E has boundary vertices v_1 and v_2. We know that there is a point w_1 within R of v_1 and that $\phi(v_1) = f(w_1)$, and similarly for v_2. Since the triangulation was δ-fine, the distance from w_1 to w_2 in l_∞^N is at most $2R + \delta$. Now w_1 and w_2 are each points in $K_S(M)$. Because of the quasi-isometric estimate for K_S, we know that the distance between the corresponding points in (M, g) is at most $4R + 2\delta$. Let P be a path from w_1 to w_2 in (M, g) of length at most $4R + 2\delta$. We have already defined ϕ on the endpoints of E, mapping them to the endpoints of $f(P)$. Now we extend ϕ so that it maps E onto $f(P)$.

Next we define ϕ on the 2-simplices of the triangulation. Suppose that Δ is a 2-simplex. We have already defined ϕ on the boundary of the 2-simplex. Let E_1, E_2, and E_3 be the three edges of Δ. For each edge E_i of Δ, we chose a path P_i in (M, g) so that $\phi(E_i)$ is equal to $f(P_i)$. The total length of the three paths P_i is at most $12R + 6\delta$. By assumption, $R <$ Systole$(M, g)/12$. We can choose $\delta > 0$ sufficiently small so that $12R + 6\delta <$ Systole(M, g). Therefore, the union of the paths P_i is a contractible loop in (M, g), and so ϕ maps the boundary of Δ to a contractible loop in X. We pick a contraction and use it to define ϕ on Δ.

Finally, we define ϕ on the higher-dimensional simplices of the triangulation, working one skeleton at a time. Suppose we have extended ϕ to the $(k-1)$-skeleton, and consider a k-simplex Δ. The restriction of ϕ to the boundary of Δ is null-homotopic, because X is aspherical. Therefore, ϕ can be extended to Δ. We pick any extension.

This finishes the construction of a map ϕ: $C \to X$, with $\phi|_{\partial C} = f$. Since the homology class $f_*([M])$ is non-trivial in X, this map is a contradiction. We may conclude that $R \geq$ Systole$(M, g)/12$. □

Using this lemma in place of Lemma 4, we get a systolic estimate for essential manifolds.

Theorem *For any n-dimensional closed essential Riemannian manifold (M, g), the systole is bounded in terms of the volume by the following formula.*

$$\mathrm{Systole}(M, g) \leq C(n)\mathrm{Volume}(M, g)^{1/n}.$$

References

1. Almgren, F.: Optimal isoperimetric inequalities. Indiana Univ. Math. J. **35**(3), 451–547 (1986)
2. Bombieri, E., Simon, L.: On the Gehring link problem. In *Seminar on minimal submanifolds*, Ann. of Math. Stud. 103, pp. 271–274. Princeton University Press, Princeton NJ (1983)
3. Federer, H., Fleming, W.: Normal and integral currents. Ann. Math. **72**(2), 458–520 (1960)
4. Gromov, M.: Filling Riemannian manifolds. J. Diff. Geom. **18**(1), 1–147 (1983)
5. Michael, J., Simon, L.: Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$. Comm. Pure Appl. Math. **26**, 361–379 (1973)