Geometriae Dedicata **57**: 195–206, 1995.

Counterexamples to Isosystolic Inequalities

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(Received: 5 September 1994)

Abstract. We explore M. Gromov's counterexamples to systolic inequalities. Does the manifold $S^2 \times S^2$ admit metrics of arbitrarily small volume such that every noncontractible surface inside it has at least unit area? This question is still open, but the answer is affirmative for its analogue in the case of $S^n \times S^n$, $n \geq 3$. Our point of departure is M. Gromov's metric on $S^1 \times S^3$, and more general examples, due to C. Pittet, of metrics on $S^1 \times S^n$ with 'voluminous' homology. We take the metric product of these metrics with a sphere S^{n-1} of a suitable volume, and perform surgery to obtain the desired metrics on $S^n \times S^n$.

Mathematics Subject Classifications (1991): 53C23

0. Introduction

In 1949 Loewner proved that for any metric g on the 2-torus, one has

$$\frac{\operatorname{area}(g)}{\operatorname{sys}_1^2(g)} \geq \frac{\sqrt{3}}{2},$$

where the 1-systole $\operatorname{sys}_1(g)$ is defined to be the length of the shortest noncontractible curve in g.

Berger ([3]) introduced the notion of the k-systole of an n-dimensional Riemannian manifold (V^n, g) in 1972. Here $\operatorname{sys}_k(g)$ is the infimum of volumes of k-dimensional integer cycles representing nonzero homology classes (for $k = 1$, this gives the same definition as above, in the case of abelian fundamental group).

Loewner's inequality leads one to consider an analogous inequality for the 1-systole of an n-dimensional manifold, and an inequality for the middle-dimensional systole:

A. $\dfrac{\operatorname{vol}_n(g)}{\operatorname{sys}_1^n(g)} \geq$ positive constant, and

B. $\dfrac{\operatorname{vol}_{2n}(g)}{\operatorname{sys}_n^2(g)} \geq$ positive constant.

Gromov ([10]) proved inequality A for n-dimensional manifolds V admitting a map to a $K(\pi, 1)$ space such that the induced homomorphism in n-dimensional homology sends the fundamental class $[V]$ to a nonzero class. For example, a negatively curved manifold trivially satisfies A, since in this case the 1-systole equals twice the injectivity radius r (cf. [6, p. 276]), and a ball of radius r has at least the Euclidean volume growth r^n. However, the fact that the same manifold satisfies A when it is endowed with an arbitrary metric is highly nontrivial. The tools that Gromov uses to prove A include the coarea (i.e. Eilenberg's) inequality ([5, p. 101]) and the isoperimetric inequality of Federer–Fleming ([7]). We will use the same tools in the construction of counterexamples to the middle-dimensional inequality B. Let z be a k-cycle in $\mathbf{R}^N$.

In the simplest case, the coarea inequality can be stated as follows. Let $H \subset \mathbf{R}^N$ and let $f(x) = \mathrm{dist}(x, H)$, where $x \in \mathbf{R}^N$. Then

$$\mathrm{vol}_k(z) \geq \int_0^\infty \mathrm{vol}_{k-1}(z \cap f^{-1}(t))\, dt.$$

To state the isoperimetric inequality of Federer–Fleming, we introduce the notion of the filling volume of z.

DEFINITION. The filling volume $\mathrm{FillVol}(z)$ is defined to be the infimum of volumes of $(k+1)$-chains $c \subset \mathbf{R}^N$ satisfying $\partial c = z$.

The isoperimetric inequality then asserts the existence of a constant $C = C(N)$ such that for every k-cycle z of $\mathbf{R}^N$ one has

$$\mathrm{FillVol}(z) \leq C(N)\, \mathrm{vol}_k(z)^{(k+1)/k}.$$

Gromov's theorem combined with the work of Babenko ([1, p. 30]) gives a complete understanding of manifolds satisfying A. Meanwhile, recent examples of Gromov indicate that inequality B is probably false, whatever the underlying topological type. This paper is an effort to understand some of these examples obtained by surgery (sketched in [4, p. 302]).

Note that inequality B becomes valid for $S^n \times S^n$ if one replaces the systole by the stable systole (cf. [12, p. 60]), which may be defined by taking the infimum, not over integer cycles, but over rational cycles representing integer homology classes. A necessary feature of the counterexamples is thus the different asymptotics of the systole and the stable systole.

We construct counterexamples to the middle-dimensional isosystolic inequality B via counterexamples to the following intersystolic inequality involving systoles of dimension and codimension 1 of $(S^1 \times S^n, g)$:

$$\frac{\mathrm{vol}(g)}{\mathrm{sys}_1(g)\,\mathrm{sys}_n(g)} > \text{positive constant.} \tag{$*$}$$

THEOREM 1. *The failure of the intersystolic inequality* $(*)$ *implies the failure of the isosystolic inequality in middle dimension for* $S^n \times S^n$, *for all* $n \geq 3$.

For odd-dimensional spheres, $(*)$ fails even for homogeneous metrics, due to Gromov (cf. Section 1 and [4, p. 302]). Bérard Bergery and Katz ([2]) detect the failure of $(*)$ for $S^1 \times S^2$ by using NIL geometry, and Pittet ([14]) detects it for all $n \geq 2$ by using SOL geometry. Note that in all of these examples, it is the 1-systole that is 'unstable' (i.e. the estimates for the 1-systole are false for the stable 1-systole). Combining the results of Pittet or Gromov with the theorem, we obtain the following corollary.

COROLLARY 2. *The manifold $S^n \times S^n$ for all $n \geq 3$ admits metrics with arbitrarily small ratio* $\mathrm{vol}/\mathrm{sys}_n^2$. *Such metrics can be obtained by surgery on a product metric $V \times S^{n-1}$. Here V is diffeomorphic to $S^1 \times S^n$ and is endowed with metrics with* $\mathrm{vol}/(\mathrm{sys}_1\,\mathrm{sys}_n) \to 0$, *while S^{n-1} is a round sphere of volume* $\mathrm{sys}_n(V)/\mathrm{sys}_1(V)$.

Note that no product metric on $S^n \times S^n$ can have such a property. Our construction does not work for $n = 2$ because we cannot control the 4-dimensional volume of the handle attached in order to pass from $S^1 \times S^2 \times S^1$ to $S^2 \times S^2$ (cf. 3.2).

QUESTION 3. *Does the manifold $S^2 \times S^2$ admit metrics of arbitrarily small volume such that every noncontractible surface inside it has at least unit area?*

If the homotopy groups π_i vanish for $i \leq n-1$, then every noncontractible n-dimensional submanifold represents a nonzero homology class. Thus by renormalizing the metric of Corollary 2 to unit n-systole, we can affirm that for $n \geq 3$, the manifold $S^n \times S^n$ admits metrics of arbitrarily small volume such that every noncontractible n-dimensional submanifold has at least unit volume.

We will describe suitable metrics on $S^n \times S^n$ and look for lower bounds for their n-systole. The geometric ingredients in the proof of the lower bound are the coarea inequality, the isoperimetric inequality of Federer–Fleming, and the technique of calibration (cf. section 1). The topological ingredients in our proof are the excision homomorphism and the homological invariance of the Lefschetz transverse intersection of cycles (cf. [12] and [8]).

More precisely, we find a lower bound for the n-volume of the 'unstable' class $S^1 \times S^{n-1}$ in $V \times S^{n-1}$ by integration, using the coarea inequality. The key here is that the 1-cycle obtained by transverse intersection of cycles is homologically invariant and hence, under suitable conditions, represents a nonzero class in $H_1(S^1 \times S^n \times S^{n-1})$. After performing the surgery to go from $V \times S^{n-1}$ to $S^n \times S^n$, we encounter the difficulty of estimating the 'unstable' class in $S^n \times S^n$. This is done by imitating the argument with the coarea inequality. To obtain a homological interpretation of a suitable 1-dimensional class in the context of the (simply connected!) manifold $S^n \times S^n$, we resort to relative homology.

In Section 1, we describe metrics g_R on $S^1 \times S^3$ satisfying

$$\frac{\mathrm{vol}(g_R)}{\mathrm{sys}_1(g_R)\,\mathrm{sys}_3(g_R)} \to 0 \quad \text{as } R \to \infty.$$

These are Gromov's 'parallelogram' metrics on $S^1 \times S^3$. In Section 2, we exhibit the failure of the middle-dimensional isosystolic inequality for the metric product $V \times S^{n-1}_{\text{radius}}$ for an $(n-1)$-sphere of a suitable radius, where the manifold V, diffeomorphic to $S^1 \times S^n$, disobeys the $(1, n)$-intersystolic inequality (e.g. the parallelogram metrics). In Section 3, we perform a metric surgery on $V \times S^{n-1}$ to obtain suitable metrics on $S^n \times S^n$ and prove Corollary 2, modulo the control of the 'unstable' class. The latter is obtained in Section 4, which explains the use of relative homology.

1. Two Descriptions of the Parallelogram Metric

We construct a metric g_R on $S^1 \times S^3$ as the quotient of the product metric $\mathbf{R} \times S^3_R$, where S^3_R is the round sphere of radius R, by a suitable covering transformation T (cf. [4, p. 302]). Here T acts on a point $(x, z) \in \mathbf{R} \times S^3_R$ by

$$T(x, z) = \left(x + \frac{1}{R}, e^{2\pi i/R} z\right),$$

where S^3_R is viewed as a sphere of radius R in $\mathbf{C}^2$. One can think of T as a 'translation' of $\mathbf{R} \times S^3_R$ by $(1/R, 1)$. Here we write '1' for the second component because a point of the Hopf fiber is displaced a unit geodesic distance by T (the fiber is rotated by an angle which is the reciprocal of the radius of the 3-sphere).

There is an alternative description of this metric in terms of the matrix of metric coefficients. Let (e_1, e_2, e_3, e_4) be a basis of the tangent space at $a \in S^1 \times S^3$ satisfying the following properties:

(i) the e_i are orthonormal with respect to the standard metric (i.e. the metric product of unit spheres);
(ii) e_1 is tangent to S^1;
(iii) e_2 is tangent to the Hopf fiber of S^3 passing through the point a.

The inner products of these vectors with respect to the new metric g_R are given by the following matrix, which we will denote the same way by abuse of notation:

$$g_R = \begin{bmatrix} 1 & R & 0 & 0 \\ R & 1 + R^2 & 0 & 0 \\ 0 & 0 & 1 + R^2 & 0 \\ 0 & 0 & 0 & 1 + R^2 \end{bmatrix}$$

i.e. by the quadratic form $x_1^2 + 2Rx_1x_2 + (1 + R^2)(x_2^2 + x_3^2 + x_4^2)$. Here we took the radius of S^3 to be $\sqrt{1 + R^2}$ instead of R to simplify the nondiagonal coefficient. Note that if R is an integer then the Hopf fiber becomes the $(1, R)$-curve on the 'unit square' torus spanned by the fiber and S^1.

With respect to the new metric, the vectors e_1 and e_2 form a highly deformed parallelogram. For this reason we will refer to g_R as the parallelogram metric.

LEMMA 1.1. *The volume of* g_R *grows as* R^2, *while* $\mathrm{sys}_1 \sim 1$ *and* $\mathrm{sys}_3 \sim R^3$.

Proof. The 1-systole equals the minimum distance between two distinct points of an orbit $\{T^k(b)\}$, where $b \in \mathbf{R} \times S^3_R$. If $k \geq R$ then the $\mathbf{R}$-component of b increases by at least 1. If $1 \leq k \leq R$ then the S^3-component is at least a unit distance away from the starting point, having not had time to complete a circle. Note that the stable 1-systole is $O(1/R)$ since $\mathrm{dist}(b, T^{[R]}(b)) = O(1)$.

To estimate the 3-systole, we use a calibration argument (cf. [11, p. 38]). Let ω be the 3-form on $\mathbf{R} \times S^3_R$ obtained as the pull-back of the volume form of S^3_R by the projection to the second factor. Then ω is invariant by the covering translation T, closed, and has norm 1. Note that the latter condition fails if we pull back by any projection to the second factor of $(S^1 \times S^3, g_R)$. This can be explained heuristically by saying that the 'slanted' S^1 is not in the kernel of ω, but $\mathbf{R}$ is. Let z be a 3-cycle in $S^1 \times S^3$ representing a nonzero class $k[S^3] \in H_3(S^1 \times S^3) = \mathbf{Z}$. Then by Cauchy–Schwarz and Stokes,

$$\mathrm{vol}(z) \geq \left| \int_z \omega \right| = |k| \int_{S^3} \omega \sim R^3.$$

2. Middle-Dimensional Isosystolic Inequality

We explain how the failure of the $(1, n)$-intersystolic inequality for $S^1 \times S^n$ implies the failure of the middle-dimensional isosystolic inequality for $S^1 \times S^n \times S^{n-1}$. Let

$$V = (S^1 \times S^n, g_R) \quad \text{where } \frac{\mathrm{vol}(g_R)}{\mathrm{sys}_1(g_R)\,\mathrm{sys}_n(g_R)} \to 0 \text{ as } R \to \infty.$$

Let g_0 be a fixed metric of unit volume on S^{n-1}, and let λg_0 be a metric of volume

$$\mathrm{vol}(\lambda g_0) = \frac{\mathrm{sys}_n(V)}{\mathrm{sys}_1(V)}.$$

For example, for the metric g_R of Section 1 we choose $\lambda(R) = R^3$ so that $\mathrm{area}(S^2, \lambda g_0) \sim R^3$.

PROPOSITION 2.1. *The metric product* $M = V \times (S^{n-1}, \lambda g_0)$ *satisfies*

$$\frac{\mathrm{vol}_{2n}(M)}{\mathrm{sys}_n^2(M)} \to 0 \quad \text{as } R \to \infty.$$

Proof. We have

$$\mathrm{vol}(M) = \mathrm{vol}(V)\,\mathrm{vol}(\lambda g_0) = \frac{\mathrm{vol}(V)\,\mathrm{sys}_n(V)}{\mathrm{sys}_1(V)}.$$

If z represents a class in $H_n(M) = H_n(S^1 \times S^n \times S^{n-1}) = \mathbf{Z} \times \mathbf{Z}$ with a nonzero $[S^n]$ component, then the projection of z to V gives a cycle nonhomologous to 0, and is distance-decreasing. Hence $\mathrm{vol}(z) \geq \mathrm{sys}_n(V)$, and

$$\frac{\mathrm{vol}_{2n}(M)}{\mathrm{vol}_n^2(z)} \leq \frac{\mathrm{vol}_{2n}(M)}{\mathrm{sys}_n^2(V)} = \frac{\mathrm{vol}_{n+1}(V)}{\mathrm{sys}_1(V)\,\mathrm{sys}_n(V)} \to 0.$$

The lower bound on the volume of an n-cycle representing $[S^1 \times S^{n-1}] \in H_n(S^1 \times S^n \times S^{n-1})$ is obtained using the Lefschetz transverse intersection of cycles and the coarea inequality. More precisely, the set-theoretic formula

$$(S^1 \times S^n \times \{pt\}) \cap (S^1 \times \{pt\} \times S^{n-1}) = S^1 \times \{pt\} \times \{pt\}$$

makes sense at the homological level. Given a $(n+1)$-cycle $C_{n+1} \in [S^1 \times S^n] \in H_{n+1}(M)$, and an n-cycle $C_n \in [S^1 \times S^{n-1}] \in H_n(M)$, we will have, under the hypothesis of transversality,

$$C_{n+1} \cap C_n \in [S^1] \in H_1(M),$$

since Lefschetz's transverse intersection of cycles ([8]) is dual to the cup product in cohomology, so that the class of the intersection is independent of the choice of the cycles C_n and C_{n+1}.

In particular, the length of $C_{n+1} \cap C_n$ is greater than or equal to $\mathrm{sys}_1(M) = \mathrm{sys}_1(V)$. Let $p : M \to S^{n-1}$ be the projection to the second factor. The intersection $C_n \cap p^{-1}(x)$ is transverse for a set of $x \in S^{n-1}$ of full measure by Sard's theorem. Hence on this set, it represents a nonzero multiple of the class $[S^1]$ and so has length bounded below by $\mathrm{sys}_1(V)$. Since the metric is a product, we apply the coarea inequality to find

$$\begin{aligned}\mathrm{vol}(C_n) &\geq \int_{(S^{n-1},\lambda g_0)} \mathrm{length}(C_n \cap p^{-1}(x))\,\mathrm{d}x \\ &\geq \int_{(S^{n-1},\lambda g_0)} \mathrm{sys}_1(V)\,\mathrm{d}x = \mathrm{sys}_n(V).\end{aligned}$$

This proves the volume bound for cycles representing $[S^1 \times S^{n-1}]$. The argument is similar for any class proportional to $[S^1 \times S^{n-1}]$.

3. The Metric on $S^n \times S^n$

Let $M = V \times S^{n-1}$, with the metric of Proposition 2.1, where $V = S^1 \times S^n$. We would like to control the n-systole of M after the surgery replacing it by

$M' = S^n \times S^n$. Recall that the standard surgery along S^1 in $S^1 \times S^{n-1}$ produces S^n (cf. [9, p. 33]). We denote by C the n-sphere obtained by surgery on $S^1 \times S^{n-1}$, to distinguish it from S^n, and write $M' = S^n \times C$.

We fill the S^1-factor in an S^n-equivariant way, i.e. we attach an $(n+2)$-dimensional blob $B^2 \times S^n$ to M, by glueing its boundary $\partial(B^2 \times S^n) = V$ to a fixed $V \subset M$. We thicken the blob to obtain the simplicial complex

$$P = M \bigcup_{\alpha} (B^2 \times S^n \times B_\varepsilon^{n-1}). \tag{3.1}$$

Here $B_\varepsilon^{n-1} \subset S^{n-1}$ is a small ball, and the attaching map α is the identity map of $V \times B_\varepsilon^{n-1}$. Then M' can be thought of as the boundary of P.

We need a lower bound, after surgery, for the volume of a cycle representing the class $[S^n] \in H_n(M') = \mathbf{Z} \times \mathbf{Z}$ (or, more generally, a class with a nonzero $[S^n]$-component). The case of $[C]$ will be treated in the next section.

The idea is to endow the disc B^2, which fills S^1, with the metric of a long cone. If the cycle runs along the whole length of the cone, then the coarea inequality will produce a narrow place where we can cut the cycle into two pieces. The cut can be filled using an isoperimetric inequality. Thus the cycle splits into the sum of two cycles. One of them lies entirely in the family of cones, and the other can be pushed out of the cones. In both cases the volume can be bounded from below by an easy projection argument.

More precisely, let $L > 0$ and let

$$B^2 = (S^1 \times [0, L]) \cup \Sigma,$$

where the 'cap' Σ (a disc with a fixed metric) is attached to the cylinder $S^1 \times [0, L]$ along the boundary component $S^1 \times \{L\}$, while $\partial B = S^1 \times \{0\}$.

Note that prior to performing the surgery, we deform the metric on M near V so it becomes a product in an ε-neighborhood $U_\varepsilon = S^1 \times S^n \times B_\varepsilon^{n-1}$ of V. We perform the deformation without decreasing the metric of M, and so that length$(S^1) > 2$ and vol$(S^n) > 2$ (cf. Remark 3.5). We obtain M' by attaching a handle

$$H = B^2 \times S^n \times S_\varepsilon^{n-2}$$

to $M \backslash U_\varepsilon$ along the identity map on the boundary $\partial H = S^1 \times S^n \times S_\varepsilon^{n-2} = \partial U_\varepsilon$.

PROPOSITION 3.2. *Let $n \geq 3$. Then the manifold $M' = (M \backslash U_\varepsilon) \cup H$ has the following four properties:*

(i) *if ε is sufficiently small, then* vol$(M') \leq 2$ vol(M)*;*
(ii) *the handle H admits a distance-decreasing projection to its factor S^n, whose volume is at least* $\mathrm{sys}_n(V)$*;*
(iii) *if we remove the cap Σ from B^2, leaving the long cylinder, the remaining part of the handle H admits a distance-decreasing retraction to M;*

(iv) *both* (ii) *and* (iii) *remain true if* M' *is replaced by the polyhedron* P, i.e. *if* H *is replaced by* $B^2 \times S^n \times B_\varepsilon^{n-1}$.

Proof. To prove (i), note that the volume of the handle H

$$\mathrm{vol}(H) = \mathrm{vol}(B^2)\,\mathrm{vol}(S^n)\,\mathrm{vol}(S_\varepsilon^{n-2}) \sim L\,\mathrm{length}(S^1)\,\mathrm{vol}(S^n)\varepsilon^{n-2}$$

can be made arbitrarily small provided $n \geq 3$. The dimension restriction is due to the fact that the blob $B^2 \times S^n$ must have smaller dimension than M to avoid increasing its volume. The remaining parts of the proposition are immediate from the construction.

PROPOSITION 3.3. *Let* $M = V \times (S^{n-1}, \lambda g_0)$ *where* $\lambda^{(n-1)/2} = \mathrm{sys}_n(V)/\mathrm{sys}_1(V)$. *We perform the surgery replacing* M *by* $M' = (M \backslash U_\varepsilon) \cup H$, *diffeomorphic to* $S^n \times C$, *where* C *is diffeomorphic to the* n*-sphere. Let* z *be an* n*-cycle in* M' *representing a class with a nonzero* $[S^n]$*-component. Then* $\mathrm{vol}(z) \geq \frac{1}{2}\mathrm{sys}_n(V)$.

Proof. We first consider the following two cases.

(a) The cycle z stays away from the caps $\Sigma \subset B^2$. Then it can be pushed off the handle back into M without increasing its volume. Once in M, it projects to an n-cycle in V of smaller volume, which is not homologous to 0 by the hypothesis of the proposition.
(b) The cycle z is contained entirely within the handle H. Then z projects to the S^n factor with nonzero degree and by a distance-decreasing map.

In either case, we get the lower bound of $\mathrm{sys}_n(V)$ for the volume of z in view of Proposition 3.2(ii) and (iii).

Now let z be an arbitrary cycle in M', and suppose $\mathrm{vol}(z) < \frac{1}{2}\mathrm{sys}_n(V)$. Let f be the distance to $M \backslash U_\varepsilon \subset M'$:

$$f : M' \to \mathbf{R},\ f(x) = \mathrm{dist}(x, M \backslash U_\varepsilon).$$

If the cylinder is long enough, e.g. $L \geq \mathrm{sys}_n(V)$, then the coarea inequality produces a slice $z \cap f^{-1}(t)$ of small area:

$$\mathrm{area}(z \cap f^{-1}(t)) < 1.$$

Note that $f^{-1}(t) = S^1 \times S^n \times S_\varepsilon^{n-2}$ where by construction $\mathrm{length}(S^1) > 2$ (S^1 is the cross-section of the cylinder portion of B^2) and $\mathrm{vol}(S^n) > 2$. We now consider the intersection $z \cap f^{-1}(t)$ as an $(n-1)$-cycle in the polyhedron P of (3.1). We apply the isoperimetric inequality of Lemma 3.7 below to $S^1 \times S^n \times B_\varepsilon^{n-1} \subset P$, and conclude that the $(n-1)$-cycle $z \cap f^{-1}(t)$ can be filled in P by an n-chain c_t of small volume. Note that the lemma does not apply directly in M', since the factor S_ε^{n-2} of $f^{-1}(t)$ is too small to satisfy the hypotheses of the lemma. Moreover, different connected components of $z \cap f^{-1}(t)$ may not even be homologous to 0 in $H_{n-1}(S^1 \times S^n \times S^{n-2}) = \mathbf{Z}$.

Let $z_t = z \cap f^{-1}([0,t])$. If the cycle $z_t + c_t$ represents a nonzero homology class in P, then $\mathrm{vol}(z_t + c_t) > \mathrm{sys}_n(V)$ by (a) and Proposition 3.2(iv). Therefore

$$\mathrm{vol}(z) \geq \mathrm{vol}(z_t) \geq \mathrm{sys}_n(V) - \mathrm{vol}(c_t) \sim \mathrm{sys}_n(V), \tag{3.4}$$

contradicting the initial assumption $\mathrm{vol}(z) < \frac{1}{2}\mathrm{sys}_n(V)$. Otherwise we apply (b) to the cycle $z - z_t - c_t$.

Remark 3.5. Let us make explicit the parameter ε in the case of the parallelogram metric g_R of Section 1. We choose a metric on $S^1 \times S^3 \times S^2$ which is bigger than the original metric of M, as follows. Let h be a product metric on $V = S^1 \times S^3$ which is bigger than g_R. Define a function ϕ so that $\phi(\rho) = 1$ if $\rho < \varepsilon$ and $\phi(\rho) = 0$ if $\rho > 2\varepsilon$. We define a new metric on $M = V \times S^2$ by the formula

$$\phi(\rho)h + (1 - \phi(\rho))g + \mathrm{d}\rho^2 + a^2(\rho)\,\mathrm{d}\theta^2$$

where $\rho(x) = \mathrm{dist}(x, S^1 \times S^3)$ is defined by the radial distance from a fixed point $b \in S^2$, $\rho \in [0, \pi R^{3/2}]$, while θ is the polar angle, and $a(\rho) = R^{3/2}\sin(\rho/R^{3/2})$.

Take the length of the 'long cylinder' to be $L = R^3$. If the cycle z has volume $o(R^3)$ then for some t we will have $\mathrm{area}(z \cap f^{-1}(t)) = o(1)$ and hence $\mathrm{vol}(c_t) \sim o(1)$ by Lemma 3.7, and the calculation (3.4) applies. Let h_R be the metric on $S^1 \times S^3$ defined as the product of a circle of radius $\sqrt{2}$ and a 3-sphere of radius $\sqrt{1+2R^2}$, i.e.

$$h_R = \mathrm{Diag}(2, 1+2R^2, 1+2R^2, 1+2R^2)$$

in terms of the basis described in Section 1. Then $h_R \geq g_R$ since $2x^2 + (1+2R^2)y^2 - (x^2 + (1+R^2)y^2 + 2Rxy) = (x - Ry)^2 \geq 0$. Note that $\mathrm{vol}(S^1 \times S^3, h_R)$ grows as R^3. Thus the handle $H = B^2 \times S^3 \times S^1_\varepsilon$ has $\mathrm{vol}(H) \sim LR^3\varepsilon = \varepsilon R^6$. Since $\mathrm{vol}(M) \sim R^5$, it suffices to choose $\varepsilon = o(R^{-1})$ to justify the assertion (i) of Proposition 3.2.

DEFINITION 3.6. We say that V satisfies the isoperimetric inequality for small k-cycles if there exists a constant $C > 0$ such that for every k-cycle z of $\mathrm{vol}(z) < 1$ one has

$$\mathrm{FillVol}(z) \leq C\,\mathrm{vol}(z)^{(k+1)/k}.$$

LEMMA 3.7. *A metric product of spheres of volumes at least 2 satisfies the isoperimetric inequality for small cycles. This conclusion remains true if we further multiply by a convex Euclidian domain.*

Proof. Let S^n be the unit sphere and let $V = \mathbf{R}^p$. We first prove the isoperimetric inequality for small cycles in $V \times S^n$. Note that z is homologous to 0 by the volume assumption, even if $k = n$. Similarly, if $k = 1$, the 1-cycle z can be filled by 'coning' each connected component from any point in this component.

The proof is by induction on n. Let S_t be the distance sphere of radius t in S^n, $0 < t < \pi$. By the coarea inequality, there exists a t such that $\mathrm{vol}_{k-1}(z \cap (V \times S_t)) < 1$, where $t \in [\pi/2 - \frac{1}{2}, \pi/2 + \frac{1}{2}]$. Then the $(n-1)$-sphere S_t has radius bigger than $\cos(\frac{1}{2})$. We apply the inductive hypothesis to $V \times S_t$ and conclude that there exists a chain c in $V \times S_t$ such that $z \cap (V \times S_t) = \partial c$ and whose volume is bounded by the isoperimetric inequality.

We decompose z as the sum $z = z_1 + z_2$ where $z_1 = z_t + c$ and $z_2 = z - z_1$. Then $z_i, i = 1, 2$ lie in the (closures of) the connected components of $V \times (S^n \backslash S_t)$. But the components of $S^n \backslash S_t$ are spherical caps which are Bilipschitz equivalent to the unit ball $B \subset \mathbf{R}^n$. We are thus reduced to the isoperimetric inequality in $V \times B \subset \mathbf{R}^{p+n}$. Applying the Federer–Fleming inequality in $\mathbf{R}^{p+n}$ we obtain a suitable chain in $\mathbf{R}^{p+n}$. But $\mathbf{R}^n$ admits a distance-decreasing retraction to B. (Note that the same retraction takes care of the extra convex factor of the lemma.) This produces the desired chain in $V \times B$ and proves the isoperimetric inequality for small cycles in $\mathbf{R}^p \times S^n$.

To prove this for $S^p \times S^n$, we exchange the factors and apply the previous argument to $V \times S^p$ where $V = S^n$.

4. Bounding the Unstable Class

Denote by C the n-sphere obtained by surgery on $S^1 \times S^{n-1}$. In the previous section we described a metric surgery to go from $S^1 \times S^n \times S^{n-1}$ to $S^n \times C$. So far we have found lower bounds for volumes of cycles representing classes with nonzero $[S^n]$-component in $H_n(S^n \times C)$. Note that the argument of Section 3 actually estimates the mass of such classes (i.e. 'stable' volume defined as the infimum over rational cycles representing them), provided the estimate on $\mathrm{sys}_n(V) = \mathrm{sys}_n(S^1 \times S^n)$ is stable. This cannot be done for $[C]$ since the stable isosystolic inequality is true. Thus the proof given below necessarily relies upon the unstable estimate of the 1-systole (cf. Lemma 1.1).

The proof becomes more readable when generalized as follows. Let $A \times B$ be the metric product of two compact Riemannian manifolds, while $A = A_1 \times A_2$ is a topological product. let $k = \mathrm{dim} A_1$ and assume

$$\mathrm{dim} A_1 + \mathrm{dim} B = \mathrm{dim} A_2 = n.$$

LEMMA 4.2. *Suppose $H_n(A \times B)$ is generated by $[A_2]$ and $[A_1 \times B]$. Then for any metric on A, one has*

$$\mathrm{sys}_n(A \times B) = \min\{\mathrm{sys}_k(A)\,\mathrm{vol}_{n-k}(B), \mathrm{sys}_n(A)\}.$$

Proof. This is a reformulation of Proposition 2.1.

Assume that A_1 bounds a $(k+1)$-manifold Σ, and denote by C the result of applying the standard surgery to $A_1 \times B$ along an $A_1 \subset A_1 \times B$. Let $B_\varepsilon \subset B$ be a small ball centered at $b \in B$.

LEMMA 4.3. *In $A \times B$, we perform a surgery along $A \times \{b\}$ (inside $A \times B_\varepsilon$) by attaching a handle $\Sigma \times A_2 \times \partial B_\varepsilon$, to obtain $A_2 \times C$. Consider a cycle z representing a nonzero multiple of $[C] \in H_n(A_2 \times C)$. Then*

$$\mathrm{vol}(z) \geq \tfrac{1}{2}\,\mathrm{sys}_k(A)\,\mathrm{vol}_{n-k}(B).$$

Proof. Let $p: A \times B \to B$ be the projection to the second factor. By the coarea inequality,

$$\mathrm{vol}(z) \geq \int_{B \setminus B_\varepsilon} \mathrm{vol}(z \cap p^{-1}(x))\,\mathrm{d}x.$$

We need a homological interpretation of $z \cap p^{-1}(x)$, where $x \notin B_\varepsilon$, as a nontrivial representative of a class in $H_k(p^{-1}(x)) = H_k(A)$, in order to find lower bounds for volume as before. Let $U_\varepsilon = A \times B_\varepsilon$. Let $X = (A \times B)\backslash U_\varepsilon$ and $Y = \partial U_\varepsilon$. The cup product in cohomology

$$H^{n-k}(X,Y) \times H^n(X) \to H^{2n-k}(X,Y)$$

is dual to the product in homology defined at the level of cycles by transverse intersection

$$H_{n+k}(X) \times H_n(X,Y) \to H_k(X), \tag{4.4}$$

which provides a homological interpretation for the transverse intersection $A \cdot (A_1 \times B) = A_1$. Here we view $A_1 \times B$ as a relative class in $H_n(X,Y) = H_n(A \times B, U_\varepsilon)$ (excision). Note that the isomorphism can be realized at the level of cycles without deforming the cycle representing $[A_1 \times B]$ outside a small neighborhood of U_ε. Since we have excluded U_ε from the picture, (4.4) is also a homological interpretation of $A \cdot C = A_1$.

Acknowledgements

The author is grateful to M. Gromov for a number of helpful discussions of this paper, and to A. Reznikov for a simplified proof of Lemma 4.3.

References

1. Babenko, I.: Asymptotic invariants of smooth manifolds, *Russian Acad. Sci. Izv. Math.* **41** (1993), 1–38.
2. Bérard Bergery, L. and Katz, M.: Intersystolic inequalities in dimension 3, *Geom. Funct. Anal.* **4** (6) (1994), 621–632.
3. Berger, M.: Du côté de chez Pu, *Ann. Sci. Ec. Norm. Sup.* **5** (1972), 1–44.
4. Berger, M.: Systoles et applications selon Gromov, exposé 771, *Seminaire N. Bourbaki, Astérisque*, 216, Société Mathématique de France, 1993, pp. 279–310.
5. Burago, Yu. and Zalgaller, V.: *Geometric Inequalities*, Springer, 1988.
6. do Carmo M.: *Riemannian Geometry*, Birkhäuser, 1992.

7. Federer, H. and Fleming, W.: Normal and integral currents, *Ann. Math.* **72** (1960), 458–520.
8. Glezerman, M. and Pontryagin, L.: *Intersections in Manifolds*, Amer. Math. Soc. Translations No. 50, 1951. (Translation of *Uspehi Matematicheskih Nauk* (N.S.) **2** (1)(17) (1947), 50–155.)
9. Godbillon, C.: *Topologie algébrique*, Hermann, Paris, 1971.
10. Gromov, M.: Filling Riemannian manifolds, *J. Differential Geom.* **18** (1983), 1–147.
11. Gromov, M.: Systoles and intersystolic inequalities, IHES preprint, 1992.
12. Gromov, M., Lafontaine, J. and Pansu, P.: *Structures métriques pour les variétés riemanniennes*, Cedic, 1981.
13. Lefschetz, S.: Intersections and transformations of complexes and manifolds, *Trans. Amer. Math. Soc.* **28** (1926), 1–49.
14. Pittet, C.: Systoles on $S^1 \times S^n$, *Diff. Geom. Appl.* (to appear).