FILLING METRIC SPACES

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Abstract. We prove an inequality conjectured by Larry Guth that relates the m-dimensional Hausdorff content of a compact metric space with its $(m-1)$-dimensional Urysohn width.

As a corollary, we obtain new systolic inequalities that both strengthen the classical Gromov's systolic inequality for essential Riemannian manifolds and extend this inequality to a wider class of non-simply connected manifolds.

The paper also contains a new version of isoperimetric inequality (Theorem 6.2). It asserts that for every positive integer m, Banach space B, compact subset X of B, and a closed subset Y of X there is a filling of Y by a continuous image of X with the $(m+1)$-dimensional Hausdorff content bounded in terms of the m-dimensional Hausdorff content of Y.

1. Introduction

1.1. **Main result.** A metric space X is called *boundedly compact*, if each closed metric ball in X is compact or, equivalently, each closed and bounded subset of X is compact. The *m-dimensional Hausdorff content* $\mathrm{HC}_m(U)$ of U, $U \subset X$, is the infimum of $\sum_i r_i^m$ among all coverings of U by countably many balls $B(x_i, r_i)$ in X. We say that X has *q-dimensional Urysohn width* $\leq W$ if there is a q-dimensional simplicial complex K and a continuous map $\pi : X \to K$ so that every fiber $\pi^{-1}(s)$ has diameter $\leq W$ in X.

In this paper we answer a question of L. Guth, relating Hausdorff content and Urysohn width of metric spaces [Gu17, Question 5.2].

Theorem 1.1. *For each positive integer m there exists $\varepsilon'_m > 0$, such that the following holds. If X is a boundedly compact metric space and there exists a radius R, such that every ball of radius R in X has m-dimensional Hausdorff content less than $\varepsilon'_m R^m$, then $UW_{m-1}(X) \leq R$. Here one can take $\varepsilon'_m = (10^{20}\ m)^{-m^2}$.*

Intuitively, the Urysohn width $UW_{m-1}(X)$ measures how well metric space X can be approximated by an $(m-1)$-dimensional space. The definition of Hausdorff content looks similar to the definition of Hausdorff measure, except that for Hausdorff content we do not take the limit over all coverings with the maximal radius of the ball in the covering tending to 0. In particular, the m-dimensional Hausdorff content

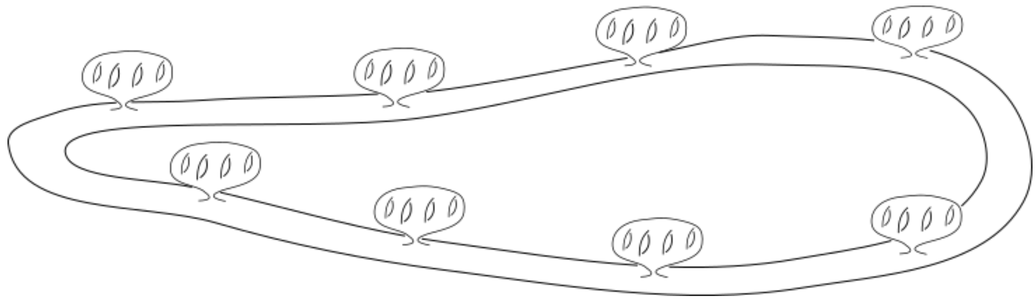

FIGURE 1. Surface Σ is a connected sum of a thin long torus and many copies of a surface of very large area and very small diameter. The area of every metric ball of radius 1 in Σ is large, but 2-dimensional Hausdorff content is small. By Theorem 1.1 the Urysohn width $UW_1(\Sigma)$ is also small.

of a set U is always less than or equal to the m-dimensional Hausdorff measure of U. When the Hausdorff dimension of a compact metric space U is greater than m the m-dimensional Hausdorff measure is infinite, but $\mathrm{HC}_m(U)$ is always finite and can be very small.

Theorem 1.1 generalizes a result of Guth in [Gu17], where X is assumed to be an m-dimensional Riemannian manifold, and HC_m replaced by the volume. This result has been previously conjectured by M. Gromov, and for a long time had been an open problem. Guth's proof is based on a clever construction of a covering by balls with controlled overlap from his previous paper [Gu11] and also uses S. Wenger's simplified version ([W]) of Gromov's proof of J.Michael-L.Simon isoperimetric inequality and its generalizations ([Gr]).

If X is compact, then choosing $R = \frac{\mathrm{HC}_m(X)}{\varepsilon'_m}^{\frac{1}{m}}$ and denoting $\frac{1}{(\varepsilon'_m)^{\frac{1}{m}}}$ as $c(m)$ we obtain:

Theorem 1.2. *For each m there exists a constant $c(m)$ such that each compact metric space X satisfies the inequality*

$$UW_{m-1}(X) \leq c(m)\,\mathrm{HC}_m^{\frac{1}{m}}(X).$$

Here one can take $c(m) = (10^{20}m)^m$.

As observed by Guth in [Gu17] Theorem 1.2 can be viewed as a quantitative version of the classical Szpilrajn theorem asserting that each compact metric space with zero m-dimensional Hausdorff measure has Lebesgue covering dimension $\leq m-1$. Indeed, if the m-dimensional Hausdorff measure of X is equal to zero, then, as HC_m does not exceed the m-dimensional Hausdorff measure, $\mathrm{HC}_m(X)$ also must be equal to zero. Now Theorem 1.2 implies that $UW_{m-1}(X) = 0$, which implies that the covering dimension of X is at most $m-1$ (see the proof of Lemma 0.8 in [Gu17]). Also, note that if X is a closed m-dimensional Riemannian manifold, then Theorem 1.2 improves the well-known Gromov's inequality relating the volume of a closed Riemannian manifold and its filling radius (as the filling radius does not exceed $\frac{1}{2}UW_{m-1}$ -see [Gr], pp. 128-129, where UW_{m-1} is denoted as $Diam'_{m-1}$).

1.2. **New systolic inequalities.** We observed that Theorem 1.2 has the following corollary.

Definition 1.3. A CW-complex X is m-essential if there exists a coefficient group G such that one of the homomorphisms $Q_{*i}: H_i(M^n; G) \longrightarrow H_i(K(\pi_1(M^n), 1))$ for some $i \geq m$, induced by the classifying map $Q: X \longrightarrow K(\pi_1(X), 1)$ is non-trivial (that is, has a non-zero image). X is m-"essential" if Q is not homotopic to a map that factors through a map to am $(m-1)$-dimensional CW-complex.

Of course, n-essential closed n-dimensional manifolds are exactly the essential manifolds as defined in [Gr]. The definition of m-"essential" CW-complexes generalizes the definition of "essential" polyhedra on p. 139 of [Gr]. Obviously, if X is m-essential, then X is also m-"essential". Therefore, one can regard m-essentiality as a sufficient condition of m-"essentiality".

Recall that a metric is called a length metric, if the distance between each pair of points is equal to the infimum of lengths of paths connecting these points, and a metric space such that its metric is a length metric is called a length space. Also, recall that if X is a non-simply connected length space, then $sys_1(X)$ denotes the infimum of lengths of non-contractible closed curves in X.

Theorem 1.4. *For each positive integer m there exist a constant $c(m)$ with the following property. Let X be an m-essential, or more generally, an m-"essential" finite CW-complex endowed with a length metric. Then*

$$sys_1(X) \leq C(m)\, \mathrm{HC}_m^{\frac{1}{m}}(X). \qquad (*)$$

Here one can take $C(m) = (10^{20}m)^m$.

Remarks. 1. Compact m-essential Riemannian manifolds with or without boundary of dimension $n \geq m$ constitute the most obvious example of path metric spaces satisfying the conditions of the theorem.

2. If m is the dimension of X, then this inequality improves Theorem B'_1 on p. 139 of [Gr] (as $\mathrm{HC}_n(X) \leq V_n(1)vol(X)$). When X is a closed m-dimensional Riemannian manifold, this inequality is the strengthening of the famous Gromov systolic inequality $sys_1(M^n) \leq c(n)vol^{\frac{1}{n}}(M^n)$.

3. Observe that if $k < m$ then $HC_k^{\frac{1}{k}}(X) \geq HC_m^{\frac{1}{m}}(X)$, (see Lemma 1.6 below). Therefore, disregarding the constants $c(m)$, these inequalities become stronger as m

increases. On the other hand, the assumption that X is m-essential also becomes stronger when m increases.
4. Note that our estimate for $C(m)$ grows as $(Cm)^m$ which is the same as in Wenger's version of the systolic inequality ([W]) (albeit with a much worse estimate for the constant C than in Wenger's paper), and is somewhat better than Gromov's $(Cn^{\frac{3}{2}})^n$. Whenever we wanted to obtain a specific value for the constant C, we did not try to optimize it. It is not difficult to see that our $C = 10^{20}$ can be significantly improved even staying within the framework of our approach.

We are going to prove this theorem in section 8. The proof combines the ideas from [Gr] with our main theorem.

Examples. 1. If E^m is an essential closed manifold (in the sense of [Gr]), then any product $E^m \times N^{n-m}$ with a closed manifold, or a connected sum $E^m \times N_1^{n-m} \# N_2^n$ is m-essential and satisfies the systolic inequality from the previous theorem. For example, if M^8 is a Riemannian manifold diffeomorphic to $T^3 \times S^5 \# S^4 \times S^4$, then $sys_1(M^8) \leq c(3)\,\mathrm{HC}_3(M^8)^{\frac{1}{3}}$.
2. The classical Hopf's theorem implies that if a closed Riemannian manifold M^n has a homology class in $H_2(M^n)$ that is not spherical (i.e. not in the image of the Hurewicz homomorphism $\pi_2(M^n) \longrightarrow H_2(M^n)$), then M^n is 2-essential. Therefore, $sys_1(M^n) \leq c(2)\sqrt{\mathrm{HC}_2(M^n)}$. Similarly, if a closed Riemannian manifold M^n satisfies $\pi_i(M^n) = 0$ for all $i \in \{2, \dots, m-1\}$ and there exists a non-spherical m-dimensional homology class of M^n, then M^n is m-essential and satisfies the systolic inequality (*).

The inequalities (*) can be restated as the assertion that there exists a constant $b(m) > 0$ such that if $\mathrm{HC}_m(X) \leq b(m)$ for an m-essential X, then $sys_1(X) \leq 1$. Of course, our main theorem implies more, namely, that the assumption about the Hausdorff content of the whole manifold can be replaced by the assumption about all metric balls of radius 1: Also, as Theorem 1.1 holds also for boundedly compact metric spaces, the previous theorem can be immediately generalized for locally finite CW-complexes. In particular, it holds for m-essential complete non-compact Riemannian manifolds with or without boundary.

Theorem 1.5. *Let X be an m-"essential" boundedly compact length space homeomorphic to a CW-complex, where $m \geq 2$. If for some $R > 0$ for each metric ball B of radius R in X $\mathrm{HC}_m(B) \leq b(m)R^m$, then $sys_1(X) \leq R$, where $b(m) = (10^{20}m)^{-m^2}$.*

Example. Consider a complete Riemannian $T^2 \times \mathbb{R}^N$. If HC_2 of each metric ball of radius 1 does not exceed $b(2)$, then there exists a non-contractible closed curve of length ≤ 1.

1.3. **Hausdorff content: basic properties and some examples.** The following properties of the Hausdorff content for compact metric spaces immediately follow from the definition:

1. Subadditivity. $\mathrm{HC}_m(\bigcup_i A_i) \leq \sum_i \mathrm{HC}_m(A_i)$.

2. Good behaviour under Lipschitz maps. If $f : X \longrightarrow Z$ is a L-Lipschitz map, and $Y \subset X$, then $\mathrm{HC}_m(f(Y)) \leq L^m \,\mathrm{HC}_m(Y)$.

Applying this to the inclusion map of Y into X (which is, obviously, 1-Lipschtiz we obtain:

3. Monotonicity. If $Y \subset X$, then $\mathrm{HC}_m(Y) \leq \mathrm{HC}_m(X)$.

4. Rescaling. If A is a subset of a Banach space, and λ is a scalar, then $\mathrm{HC}_m(\lambda A) = |\lambda|^m \,\mathrm{HC}_m(A)$.

5. For each m $\mathrm{HC}_m(X) \leq rad^m(X) \leq diam^m(X)$.

Indeed, a compact metric space X can always be covered by one metric ball of radius $diam(X)$.

6.

Lemma 1.6. *If $m > k$ then $\mathrm{HC}_k^{\frac{1}{k}}(X) \geq \mathrm{HC}_m^{\frac{1}{m}}(X)$.*

Indeed, choose a finite covering of X by metric balls with radii r_i so that $\sum_i r_i^k \leq \mathrm{HC}_k(X) + \varepsilon$ for an arbitrarily small ε. Now $\sum_i r_i^m = \sum_i (r_i^k)^{\frac{m}{k}} \leq (\sum_i r_i^k)^{\frac{m}{k}} \leq (\mathrm{HC}_k(X) + \varepsilon)^{\frac{m}{k}}$. Now the desired inequality follows when we take $\varepsilon \longrightarrow 0$.

7. Consider the Euclidean 2-ball B of radius 1. We already know that its HC_1 cannot exceed 1, but, in fact, it is equal to 1, as the sum of radii of any collection of balls covering a diameter of B (that has length 2) should be at least 1.

8. HC_m is **not additive**. Indeed, cut the ball B in the previous examples into two halves H_1, H_2 along a diameter. Note, that $\mathrm{HC}_1(H_i)$ is still equal to one. So, when we remove one of these halves, the values of HC_1 does not decrease. Moreover, if we will take the union of the remaining half with a metric ball of arbitrarily small radius centered at one of two points of the diameter, the value of HC_1 of the union will, in fact, become greater than 1. This example illustrates difficulties that one can encounter trying to make the value of HC_m of a metric space smaller by cutting out its subsets and replacing them by subsets with a smaller value of HC_m.

9. Let B_K be a two-dimensional metric ball of radius 1 in the hyperbolic space with constant negative curvature $K << -1$. The area of B_K behaves as $\exp(\sqrt{-K}) >> 1$, yet $\mathrm{HC}_2(B_k) \leq 1 << vol(B_k)$, as B_K can be covered by just one metric ball of radius 1. It is obvious that $\mathrm{HC}_2(B_K) = 1$, and that if we will cut a concentric metric ball of a small radius ε out of B_K, the value of HC_2 will not change. (This is another example of non-additivity of HC_m, this time in the situation when $m = dim(X)$.)

10.

Lemma 1.7. *Consider the unit cube $C = [0,1]^n$ in l^n_∞ (which is the n-dimensional linear space endowed with the* max-*norm). For each $m \leq n$ $\mathrm{HC}_m(C) = \frac{1}{2^m}$.*

Indeed, observe that the cube is the metric ball of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, \ldots, \frac{1}{2})$, and so $\mathrm{HC}_m(C) \leq \frac{1}{2}^m$.

To prove the opposite inequality consider a covering of C by metric balls with radii r_i so that $\sum_i r_i^m \leq \mathrm{HC}_m(C) + \varepsilon$ for an arbitrarily small ε. Observe that these metric balls are cubes with side length $2r_i$. As the Lebesgue measure of their union cannot be less than the Lebesgue measure of C, we conclude that $\sum_i (2r_i)^n \geq 1$, whence $\frac{1}{2^n} \leq \sum_i (r_i^m)^{\frac{n}{m}} \leq (\sum_i r_i^m)^{\frac{n}{m}} \leq (\mathrm{HC}_m(C) + \varepsilon)^{\frac{n}{m}}$. When $\varepsilon \longrightarrow 0$, we obtain the desired inequality.

Corollary 1.8. *If $P = [0, r_1] \times \ldots \times [0, r_n]$, is an n-dimensional parallelepiped in l^n_∞, where $r_1 \leq \ldots \leq r_n$, and $m \leq n$, then $\mathrm{HC}_m(P) \geq (\frac{r_1}{2})^m$.*

Indeed, by monotonicity, $\mathrm{HC}_m(P)$ is not less than HC_m of the n-cube $C \subset P$ with side length r_1, for which we have $\mathrm{HC}_m(C) = (\frac{r_1}{2})^m$.

1.4. **Some ideas of the proof of Theorem 1.1 and the plan of the paper.** Our proof of the main theorem essentially follows Guth's proof of the inequality $UW_{n-1}(M^n) < 1$ for closed Riemannian manifolds M^n such that for each metric ball B of radius ≤ 1 $vol(B) \leq \varepsilon_m$. So, we follow the exposition in [Gu11] and [Gu17], that, in turn, incorporates many ideas from [Gr] and [W]. Yet almost no element of Guth's proof works as is for compact metric spaces/Hausdorff content instead of Riemannian manifolds/the volume. Virtually every component of the proof needs to be done differently or at least significantly modified.

(1) We would like to start from defining "good balls" similarly to how it was done in [Gu11]. Assuming $R = 1$, for each $p \in X$ we consider the sequence of values $\frac{1}{7}, \frac{1}{7^3}, \ldots, \frac{1}{7^{2i+1}}, \ldots$ and look for the first (the maximal) value r in this sequence, such that the "reasonable growth" condition $\mathrm{HC}_m(7B) \leq 49^{m+1}\, \mathrm{HC}_m(\frac{1}{7}B)$ holds for the metric ball B of radius r centered at p. (Here and below if B is a metric ball of radius r, then λB denotes the concentric metric ball of radius $\lambda\, r$. Observe, that if X were the Euclidean space, $\mathrm{HC}_m(7B) = 49^m\, \mathrm{HC}_m(\frac{1}{7}B)$.) The resulting ball B will be, by definition, the *good ball* centered at p.

Yet observe, that the existence of a good ball requires the assumption that the lower HC_m-density at p is positive (compare with the proof of Lemma 1 in [Gu11]). Obviously, this is not true for an arbitrary compact metric space X.

Our remedy is to take the product X' of X with a very small round sphere $\mathbb{S}^m$ and observe that (a) X' has HC_m-density 1; and (b) If Theorem 1.1 holds for X' and some ε_m, then it holds for X and ε'_m. Thus, we reduce the general case to the case of positive density, and in this case the "good ball" centered at p always exists. The details of this reduction can be found in the next section.

(2) Once one has good balls centered at all points of X, the standard Vitali covering construction yields a "good covering" of X by good balls exactly as in Lemma 2 of [Gu11] - see section 3 for more details.

Our Lemma 3.3 is a somewhat generalized version of the adaptation of Lemma 3 from [Gu11] for HC_m. Yet the proof of Lemma 3 in [Gu11] relies on the additivity property of the Hausdorff measure, which does not hold for Hausdorff contents. Of course, the sole reason for failure of additivity for HC_m is the fact that given a cover of the union of two disjoint sets A and B by metric balls, the same metric ball can cover both a part of A and a part of B providing "savings" with the situation, when A and B are covered separately.

Our remedy involves the following observation: Assume that disjoint metric balls $B_1, \ldots, B_k$ have comparable radii (say, all their radii are in the interval $[\frac{s}{2}, s]$ for some s), and HC_m of their union is very small in comparison with s^m (say, $< (\frac{s}{1000})^m$). This means that metric balls used for an almost optimal (from the point of view of HC_m) covering of $\bigcup_i B_i$ can contain only metric balls of radius $\leq \frac{s}{1000}$. Now we can just throw away all metric balls in this covering that intersect more than one of the balls B_i, and observe that the remaining balls still cover $\bigcup_i (1 - \frac{1}{250}) B_i$, and no remaining ball can intersect more than one B_i. As the result, we can conclude that $\mathrm{HC}_m(\bigcup_i B_i)$ is at least as large as $\sum_i \mathrm{HC}_m(0.996 B_i)$. If, in addition, we know that for some $\theta > 0$ $\mathrm{HC}_m(0.996 B_i) \geq \theta \, \mathrm{HC}_m(B_i)$, then we have the inequality $\mathrm{HC}_m(\bigcup_i B_i) \geq \theta \sum_i \mathrm{HC}_m(B_i)$ that provides a replacement for the missing additivity.

As in [Gu11], Lemma 3.3 can be used to deduce Lemma 3.4 asserting that if d balls from the good covering intersect, then the smallest of their radii does not exceed $a(m) \exp(-b(m) d)$ for some constants $a(m)$ and $b(m)$.

(3) In [Gu11] the good covering is used to construct a map ϕ from the manifold to the parallelepiped $P = [0, r_1] \times \ldots \times [0, r_D]$, where $r_1 \leq r_2 \leq \ldots \leq r_D$ denote the radii of good balls in the good covering, and D the cardinality of the good covering. Each coordinate ϕ_i of ϕ corresponds to the respective ball B_i of the covering. The map ϕ_i sends all points outside of B_i to 0, all points in $\frac{1}{2} B_i$ to r_i, and between $\frac{1}{2} B_i$ and B_i it linearly decreases as the function of the distance to the centre of B_i.

In [Gu11] P was endowed with the Euclidean metric. In section 4 we make one small but absolutely crucial change: Our P introduced is endowed with the l^∞-metric. As the result, the Lipschitz constant for ϕ is bounded by an absolute constant that does not depend on D (over which we have no control). (This contrasts with the Euclidean case, where one gets the factor $\sqrt{D}$ in the Lipschitz constant.) As the result, we can use our Lemma 3.4 to prove the crucial inequality in Lemma 4.1 (our analog of Lemma 5 from [Gu11]) without the need to use an analog of Lemma 4 from [Gu11]. This fact is very fortunate, as some simple examples convinced us that there is no good analog of Lemma 4 for HC_m. (Technically speaking, the issue here is that the proof of Lemma 4 in [Gu11] uses the additivity of the volume three times. For two of these occurrences an argument similar to the argument outlined in (2) can be used to save the situation, yet the last use of the additivity seems to kill any possibility of adapting the lemma for the Hausdorff content.) Our Lemma 4.1 provides an upper bound for the ratio of $\mathrm{HC}_m(\phi(X)\bigcup F)$ to r_1^m, where F is a d-dimensional face of P, and r_1 is its smallest side length. This upper bound exponentially decreases with d.

Note that if X is not compact, but merely boundedly compact, D can be ∞. Yet the image of ϕ will be in a certain closed subcomplex of P, called the rectangular nerve of the cover, that we denote RN. All faces of RN are finite-dimensional, and RN is contained in the subset of P that consists of all points x such that for some i the ith coordinate of x is equal to r_i.

(4) The proof of the main theorem continues only in the last section. Section 5 is devoted to the proof of the coarea inequality and the cone inequality for general compact metric spaces and the Hausdorff content. These inequalities will then be used in section 6 to prove an isoperimetric inequality, which will be the crucial ingredient in the proof of the main theorem in section 7.

The cone inequality involves two compact metric spaces X and $Y\subset X$ inside of a ball of radius R in a Banach space. One wants to construct a continuous map ψ of X into B so that this map remains the identity map on Y, yet $\mathrm{HC}_{m+1}(\psi(X))\leq c(m)R\,\mathrm{HC}_m(Y)$. The obvious idea is to try to map X into the cone CY over Y with the tip at the center of $B(R)$. Yet how can one map X into CY? The mapping to the cone would involve mapping at least some neighbourhood of Y in X to Y, yet, there does not seem to be any general way to do this. As the result, our "cone" is *not* CY (although it lies at a small distance from it).

Also, in section 5 we present a concise proof of a very general version of the coarea inequality in metric spaces for Hausdorff content.

(6) Section 6 contains an adaptation of the Gromov isoperimetric inequality for Hausdorff contents. The original J. Michael-L. Simon isoperimetric inequality [MS] asserts that given an $(n-1)$-dimensional cycle Y in $\mathbb{R}^N$ Y bounds a n-chain with n-dimensional volume $\leq c(n)vol_{n-1}(Y)^{\frac{n}{n-1}}$. Gromov proved this using a different method that makes it possible to prove this inequality for L^∞ and other Banach spaces instead of $\mathbb{R}^N$ ([Gr]). Wenger simplified Gromov's proof and improved the value of the constant ([W]). In [Gu17] Guth adopted Wenger's proof to prove a version of the Michael-Simon inequality for maps, where given a n-manifold X with boundary Y and its map f into $\mathbb{R}^N$ one wants to alter f to a new map F that coincides with f on Y and satisfies the inequality $vol_n(F(X))\leq c(n)vol(f(Y))^{\frac{n}{n-1}}$. (Yet a similar theorem valid for maps to all Banach spaces and with a concrete value for constant $c(n)$ is stated (without proof) already in [Gr] in section (A$'''$) of Appendix 2. In this theorem stated by Gromov both X and Y are assumed to be polyhedra

rather than manifolds. In our Theorem 6.2 X and Y are assumed to be compact metric spaces, and we use Hausdorff contents instead of Hausdorff measures.) Proofs of Gromov, Wenger and Guth use induction with respect to n. Our proof also uses the induction with respect to m, despite the fact that unlike the situation in all these papers, our m has nothing to do with the dimension of a compact metric space Y.

We present two versions of the isoperimetric inequality for maps of compact metric spaces and Hausdorff contents. One version is very general; the second version which is actually used in the proof of the main theorem works only in the situation when $f(Y)$ that we want to "fill" is on the boundary of a parallelepiped U in a finite-dimensional Banach space. However, in this second version we prove an important additional property of the filling $F(X)$, namely, that it is located sufficiently close to the boundary of U.

We want to follow the approach of [W] that involves a sequence of local improvements of the cycle Y that we want to fill. To improve Y one finds a certain collection of disjoint metric balls, cuts them out, and replaces them by chains with smaller volumes. On the first glance this approach seems absolutely bound to fail in our situation. Indeed, consider examples 8 and 9 in the previous subsection. In both cases we cut out a large piece from some Y, yet HC does not decrease, and, moreover, can increase once one adds a set with an arbitrarily small Hausdorff content.

Yet a remedy exists: We combine the idea explained in (2) that involves using only the metric balls with HC_m that is much less than the mth power of the radius with a powerful new idea: Fix an almost optimal (from the point of view of HC_m) finite covering Q of Y by metric balls B_i. We introduce the concept of m-dimensional Hausdorff content, $\widetilde{\mathrm{HC}}_m$, with respect to the covering Q for subsets of Y. $\widetilde{\mathrm{HC}}_m$ is defined as HC_m but with the following important difference: This infimum of $\sum_i r^m$ is taken only over coverings by metric balls from Q. (That is, each ball of any covering of a subset of Y that we are allowed to consider must be one of the balls B_i.) Observe that: (a) $\mathrm{HC}_m \leq \widetilde{\mathrm{HC}}_m$, so an upper bound for $\widetilde{\mathrm{HC}}_m$ is automatically an upper bound for HC_m; (b) For Y $\widetilde{\mathrm{HC}}_m$ and HC_m are almost the same. Now the idea is to ran Wenger's argument by removing (and then replacing) only metric balls with $\widetilde{\mathrm{HC}}_m \sim (\frac{r}{A(m)})^m$, where r denotes the radius of the considered ball, and $A(m)$ is a sufficiently large constant. (It is not difficult to see that if there are not sufficiently many metric balls with this property to run this argument, then Y has a "round shape" (meaning that $\mathrm{HC}_{m-1}(Y)^{\frac{1}{m-1}} \sim$ the diameter of Y in the ambient metric), so that the isoperimetric inequality for Y follows from our version of the cone inequality.) The basic idea here is that now we are throwing out only "important" balls. Moreover, for each ball B that is being replaced, the balls from Q in the optimal covering of B yielding the value of $\widetilde{\mathrm{HC}}_m(B)$ are also important (despite being very small): removal of each of those balls reduces $\widetilde{\mathrm{HC}}_m(Y)$ and, therefore, its HC_m.

(7) Once the isoperimetric inequality is established we return to proving the main theorem. Our proof presented in section 7 has the same main ideas as the proof in [Gu17], yet is somewhat different. We would like to modify the $\phi : X \longrightarrow RN$ in stages, so that all new maps remain subordinate to the open cover. Eventually, we will produce a map from X to the $(m-1)$-skeleton of RN, which will still be subordinate to the cover, and, therefore, it will automatically have small values of the diameter of the inverse image of each point. Simultaneously, with ϕ we modify $RN = RN^{(0)}$, through a sequence of its closed subcomplexes $RN^{(i)}$, so that the image of each intermediate map $\phi^{(i)}$ will be in $RN^{(i)}$.

We assume that $\phi^{(0)}$ is ϕ. We define $\phi^{(i)}$ inductively. The new map $\phi^{(i+1)}$ will differ from $\phi^{(i)}$ only on inverse images of maximal open faces of $RN^{(i)}$ of dimension $\geq m$. Also, $RN^{(i+1)}$ is obtained from $RN^{(i)}$ by removing all its maximal open faces of dimension $\geq m$.

We consider all maximal open faces F of P of dimension $\geq m$. For each face F of dimension $k \geq m+1$, we first compose the restriction of $\phi^{(i)}$ to $(\phi^{(i)})^{-1}(F)$ with a map defined on $\phi^{(i)}(X) \bigcap F$ and whose image lies in a very small neighbourhood of ∂F. This map can be described as follows: We cut F by the boundary of a slightly smaller parallelepiped $F_1 \subset F$ and apply the previously proven isoperimetric inequality that enables us to replace the part of the image of $\phi^{(i)}(X)$ inside F_1 by a "small" m-chain "close" to the boundary of F_1 and, thereby, to the boundary of F. (This replacement is done at the level of maps.) Note, that HC_m can somewhat increase on this step. Then we compose the "amended" $\phi^{(i)}$ on $(\phi^{(i)})^{-1}(F)$ with the radial projection from the center of F to its boundary. The result will be $\phi^{(i+1)}$ on $(\phi^{(i)})^{-1}(F)$. This operation also increases HC_m. Yet as the result of both these operations the upper bound for HC_m provided by the inequality (4.1) increases by a factor of less than $1 + a_1(m)\varepsilon_m^{a_2(m)} \exp(-a_3(m)k)$. Therefore, despite the fact that D can be uncontrollably large, the product of this factors for $k, k-1, \dots, m+1$ never becomes too large, no matter how large k is. In fact, this product does not exceed 2 providing that ε_m is small enough.

If the dimension of F is m, we just observe that, as $\mathrm{HC}_m(F \bigcap \phi^{(i)}(X)) < \mathrm{HC}_m(F)$, there is an open metric ball β in F that does not intersect $\phi^{(i)}(X)$. We define $\phi^{(i+1)}$ on $(\phi^{(i)})^{-1}(F)$ as the composition of a retraction of $F \setminus \beta$ to ∂F with the restriction of $\phi^{(i)}$ on $(\phi^{(i)})^{-1}(F)$.

2. Reduction to the case of positive density

Theorem 1.1 was proven in [Gu17] for m-dimensional Riemannian manifolds and volume instead of HC_m. One fact about m-dimensional Riemannian manifolds used in the proof there was that $\lim_{r\to 0} \frac{Vol(B(x,r))}{r^m} = const(m)$. Of course, this is not true for Hausdorff content in arbitrary metric spaces. However, this property can be (partially) restored by taking a product with a very small round m-dimensional sphere. We are going to present the rigorous construction as the next Proposition 2.1. This proposition also incorporates the observation that since the conclusion of Theorem 1.1 is scale invariant, it is sufficient to require that the assumption of the theorem holds only for all metric balls of a fixed positive radius R, e.g. $R = 1$ or $R = 2$.

Proposition 2.1. *For each positive integer m there exists $\varepsilon_m > 0$, such that the following holds. Suppose X is a boundedly compact metric space, satisfying the following properties:*
1. every ball of radius 1 in X has m-dimensional Hausdorff content less than ε_m;
2. for every $x \in X$ we have $\liminf_{r\to 0} \frac{\mathrm{HC}_m(B(x,r))}{r^m} = 1$.
Then $UW_{m-1}(X) \le 1$.

Now we describe how to deduce Theorem 1.1 from Proposition 2.1. When X is a compact metric space we proceed as follows. By scaling the metric we may assume that $R = 2$ in the hypothesis of Theorem 1.1. Fix ε_m from Proposition 2.1 and let $\varepsilon'_m = \frac{1}{10}\frac{\varepsilon_m}{2^m}$. By the assumption of Theorem 1.1 we have $\mathrm{HC}_m(B) \le \varepsilon_m/10$ for every ball B of radius 2 in X.

Let $\mathbb{S}^m(\tau) \subset \mathbb{R}^{m+1}$ denote the round m-sphere with radius τ. Consider space $Y = \{(x,p) : x \in X, p \in \mathbb{S}^m(\tau)$ with product metric $d((x_1,p_1),(x_2,p_2)) = \max\{d_X(x_1,x_2), d_{\mathbb{S}^m(\tau)}(p_1,p_2)\}$. We claim that for a sufficiently small choice of τ space Y satisfies assumptions 1 and 2 of Proposition 2.1. To define τ fix a finite covering $\{B(x_i,2)\}$ of X by balls of radius 2, such that concentric balls of half the radius still cover X. For each i let $B(x^i_j, r^i_j)$ be a finite covering of $B(x_i,2)$, such that $\sum_j (r^i_j)^m \le \varepsilon_m/5$. We set $\tau = \frac{1}{10}\min_{i,j} r^i_j$.

Now we can show that the two assumptions are satisfied.

1. Consider a ball B of radius 1 in Y, $B = B(x,1)\times\beta(p,1)$. We have that $B(x,1) \subset B(x_i,2)$ for some i and so there exists a covering of $B(x,2)$ by balls $B(x^i_j, r^i_j)$. Since $\tau \le \frac{r^i_j}{10}$ we have that balls $B(x^i_j,r^i_j)\times\beta(p,r^i_j)$ cover B, so $\mathrm{HC}_m(B) \le \varepsilon_m/5$.

2. This follows since $\mathrm{HC}_m(B(x,r)\times\beta(p,r)) \ge \mathrm{HC}_m(\beta(p,r))$.

Applying Proposition 2.1 to Y we obtain a map $\pi' : Y \to S$ into an $(m-1)$-dimensional simplicial complex S with fibers of diameter ≤ 1. Composing with the inclusion map we obtain the desired map from X into S with fibers of diameter ≤ 1.

When X is boundedly compact, but not compact, we need to modify the above argument by constructing a "product" metric with variable radius of $\mathbb{S}^m(\tau)$. More precisely this can be done as follows. Fix a point $x_0 \in X$. Let S denote a surface of revolution obtained by rotating the graph of a monotone decaying function τ :

$[0,\infty) \to (0,1)$ around the non-negative x_{m+1}-axis in $\mathbb{R}^{m+1}$. Topologically S is $\mathbb{S}^m \times [0,\infty)$. Consider $X \times S$ with the product metric and define $\tilde{X} \subset X \times S$ to be the set $\tilde{X} = \{(x,p,t), x \in X, p \in \mathbb{S}^m, t \in [0,\infty) : d(x,x_0) = t\}$ with the restriction metric from $X \times S$. We have a homeomorphism $\Psi : X \times \mathbb{S}^m \to \tilde{X}$ given by $\Psi(x,p) = (x,p,dist(x,x_0))$.

Let ε_m be as defined above. By the assumption of Theorem 1.1 we have $\mathrm{HC}_m(B) \leq \varepsilon_m/10$ for every ball B of radius 2 in X. We will show that if $\tau(0)$ is small enough and $\tau(t)$ decays sufficiently fast then $\tilde{X}$ satisfies assumptions of Proposition 2.1 and Theorem 1.1 follows as in the compact case.

Fix a locally bounded covering $\{B(x_i,2)\}$ of X, so that balls of half the radius still cover X. For each i let $\{B(x^i_j, r^i_j)\}$ be a finite covering of $B(x_i,2)$, such that $\sum_j (r^i_j)^m \leq \varepsilon_m/5$. Given a non-negative integer n let $N(n)$ be such that $\frac{1}{N(n)} < r^i_j$ for all $x^i_j \in X$ with $dist_X(x^i_j, x_0) \leq n+2$. We choose $\tau(t)$ so that $\tau(t) \leq \frac{1}{10N(\lfloor t \rfloor)}$. Then for any ball $B((x,p,dist(x,x_0)),1) \subset \tilde{X}$ we have that it is contained in $\bigcup_j B(x^i_j, r^i_j) \times \beta((p, dist(x^i_j, X_0)), r^i_j)$ with $\sum_j (r^i_j)^m \leq \varepsilon_m/5$. This shows that the proof of Theorem 1.1 can be reduced to the proof of Proposition 2.1.

In the rest of the paper we prove Proposition 2.1.

3. Construction of good covering

We will say that $B(p,R)$ is a *good ball* if it satisfies the following properties.

A. Reasonable growth: $\mathrm{HC}_m(B(p,7R)) \leq 49^{m+1}\,\mathrm{HC}_m(B(p,\frac{1}{7}R))$.

B. Volume bound: $\mathrm{HC}_m(B(p,R) \leq \mathrm{HC}_m(B(p,7R)) \leq 7^{m+1}\varepsilon_m R^{m+1}$.

C. Small radius: $R \leq \frac{1}{7}$.

This definition is a direct adaptation of the definition in [Gu11] for HC_m instead of the volume.

Lemma 3.1. *Let X be a boundedly compact metric space satisfying assumptions 1 and 2 of Proposition 2.1. Then for every $x \in X$ there is a radius R so that $B(x,R)$ is a good ball. Moreover, if $\varepsilon_m < \frac{1}{700^{m+1}}$, then* $\mathrm{HC}_m(B(p,7R)) \leq (\frac{R}{100})^{m+1}$.

Proof. The proof essentially repeats the proof of Lemma 1 in [Gu11]. We consider the sequence of radii $R_i = 7^{-1-2i}$ with i ranging from 0 to ∞ and are looking for the first value that satisfies the reasonable growth condition. Observe that each time, when this condition is *not* satisfied for $R = R_i$, the density $D(p,7R) = \frac{\mathrm{HC}_m(B(p,7R))}{(7R)^m}$ drops by the factor $\frac{1}{49}$, when we pass from $R = R_i$ to $R = R_{i+1} = \frac{R_i}{49}$. Therefore, sooner or later some R_i will satisfy this condition. (Indeed, if this never happens, the lower density will be dropping all the way to 0 contradicting the assumed property 2 of X.) The same calculation implies that each time, when condition A is not satisfied for $R = R_i$, and we pass to $R = R_{i+1}$, the ratio $\frac{D(p,7R)}{7R}$ does not increase. Now note that

$\frac{D(p,7R_0)}{7R_0} \leq \varepsilon_m$. (This is equivalent to our assumption that $\mathrm{HC}_m(B(p, R_0)) \leq \varepsilon_m$.) Therefore, if i_0 is the smallest value of i such that R_i satisfies the reasonable growth condition, then $\frac{D(p,7R_i)}{7R_i} \leq \varepsilon_m$. This means that $\mathrm{HC}_m(B(p, 7R_i)) \leq 7^{m+1} R_i^{m+1} \varepsilon_m$. Therefore, $B(p, R_{i_0})$ is a good ball. Condition C follows by construction. □

Because of Lemma 3.1, we can cover X with good balls. Following [Gu11] we now use the Vitali covering lemma to choose a convenient sub-covering with some control of the overlaps. More precisely, we call an open cover $\{B_i\}$ *good* if it obeys the following properties.

1. Each open set B_i is a good ball.
2. The concentric balls $(1/2)B_i$ cover M.
3. The concentric balls $(1/6)B_i$ are disjoint.

(Recall that if B_i is short-hand for $B(p_i, r_i)$, then $(1/2)B_i$ is short-hand for $B(p_i, (1/2)r_i)$.)

Lemma 3.2. *Let X be a boundedly compact metric space satisfying assumptions 1 and 2 of Proposition 2.1, then it has a locally finite good cover. If X is compact, then this cover is finite.*

Proof. For each point $p \in M$, pick a good ball $B(p)$. Then look at the set of balls $\{(1/6)B(p)\}_{p \in M}$. These balls cover M. Choose a cover of X by good balls $\frac{1}{6}B_i$, where B_i are good balls, and only finitely many of these balls intersect B for any closed metric ball B in X. Now the same argument as in the Vitali covering lemma implies the existence of a locally finite collection of pairwise non-intersecting ball $\frac{1}{6}B_i$ (from the chosen locally finite covering), such that the concentric balls $\frac{1}{2}B_i$ cover the whole X. □

We now fix a good cover for our metric space X, which we will use for the rest of the paper. The following lemma is an analog of Lemma 3 in [Gu11], yet the proof of Lemma 3 in [Gu11] uses the additivity of the volume, that we do not have for HC_m. We circumvent this difficulty using an idea sketched in subsection 1.4.

Lemma 3.3. *Assume that $\varepsilon_m < 700^{-m-1}$. Then there exists a constant $C = C(m)$ depending only on m with the following property: For any positive s and any metric ball B of radius $\leq s$, not necessarily in our cover, the number of balls B_i from our cover, with radii in the range $s \leq r_i \leq 2s$, intersecting B, is less than C. One can take $C(m) = 49^{m+1}$.*

Proof. Let $\{B_i\}_{i=1}^k$ be the set of balls in our cover that intersect B and have radii in the indicated range. We number them so that B_1 has the property $\mathrm{HC}_m(1/7B_1) \leq \mathrm{HC}_m(1/7B_i)$ for all $i > 1$. Now, all of the balls B_i and B are contained in the ball $7B_1$. On the other hand, all the $(1/6)B_i$ are disjoint.

Fix a covering by balls $\beta(j)$ of $7\ B_1$ with radii ϱ_j, so that for some very small ε $\mathrm{HC}_m(7\ B_1)+\varepsilon \geq \sum \varrho_j^m$. As $\mathrm{HC}_m(7\ B_1) \leq (\mathrm{radius}(B_1)/100)^{m+1}$, the radii ϱ_j are less than the radius $(B_1)/100 \leq \frac{s}{50}$.

We claim that for each j there is at most one i such that β_j intersects $(1/7)B_i$.

Indeed, if β_j intersects $(1/7)B_j$, it is contained in $(1/6)B_j$, and, therefore, it can not intersect any other $(1/7)B_i$.

It follows that $\mathrm{HC}_m(7B_1) \geq \sum_i \mathrm{HC}_m((1/7)B_i) \geq k\,\mathrm{HC}_m((1/7)B_1)$. Because of reasonable growth (property A) of B_1 we conclude that $k \leq C(m) = 49^{m+1}$.

□

The following lemma was stated as an observation in [Gu11] (in the case of Riemannian manifolds/volume). Our proof essentially repeats the proof in [Gu11].

Lemma 3.4. *If some d good balls $B_1, \ldots, B_d$ have a non-empty intersection, and r is the smallest radius of these balls, then $r \leq \frac{2}{7}\exp(-b(m)d)$, where $b(m) = \frac{\ln 2}{49^{m+1}}$.*

Proof. Assume that the balls are numbered so that their radii $r_i = \mathrm{radius}(B_i)$ form an increasing sequence. Thus, $r = r_1$. Let $r_0 = \frac{1}{7}$ (as in the definition of good balls). Define the interval $[r, r_0]$ into at most $\log_2 \frac{r_0}{r}+1$ subintervals $[r, 2r],\ [2r, 4r], \ldots, [2^K r, r_0]$ that we are going to call *scales*. The previous lemma implies that there are at most $C(m)$ balls B_i with radii within each scale, where one can take $C(m) = 49^{m+1}$. So the total number d of balls does not exceed $C(m)(\log_2 \frac{r_0}{r}+1)$. Solving this inequality for r we obtain the assertion of the lemma. □

4. Rectangular nerve of a good cover

Let $\{B_i\}$ be a good cover. Consider the parallelepiped $P = \prod_{i=1}^D [0, r_i]$ where r_i is the radius of the ball B_i. (If X is not compact but only boundedly compact , then $D = \infty$.) We are going to call P *the parallelepiped of the cover.* We choose the metric on P induced by the l^∞ norm on the ambient space $\mathbb{R}^D$. (In other words, $dist(x, y) = \max_{i=1}^D |x_i - y_i|$.) This is different from [Gu11], where P was considered with the Euclidean metric.

We are going to define a continuous map $\phi : X \to P$. To define ϕ, we let ϕ_i be a real-valued function supported on B_i with $\phi_i(x) = r_i$ for $x \in \frac{1}{2}B_i$ and $\phi_i(x) = r_i - 2dist(x, p_i)$ on $B_i \setminus \frac{1}{2}B_i$, where p_i denotes the center of B_i. Taking ϕ_i as coordinates, we get a map $\phi : X \longrightarrow \prod_{i=1}^D [0, r_i]$. Our definition of ϕ and our choice of the metric on $\prod_{i=1}^D [0, r_i]$ imply that the Lipschitz constant of ϕ is 2. It is important for us here that this upper bound does not depend on D (that we cannot control), as in the case for the Euclidean metric on $\prod_i [0, r_i]$.

The *rectangular nerve* of the cover had been defined in [Gu11] as some closed subcomplex of P that includes all open faces of P having a non-empty intersection with the image of ϕ. We denote the rectangular nerve as RN, and following [Gu11] define it as follows. Observe that each open face F of the rectangle can be determined by dividing the dimensions $1, ..., D$ into three sets: I_0, I_1, and $I_{(0,1)}$ with $i \in I_0$ if $\phi_i = 0$ for all points in F, $i \in I_1$ when $\phi_i = r_i$ for all points in F, and $i \in I_{(0,1)}$ when $0 < \phi_i < r_i$. Define I_+ as the union of I_1 and $I_{(0,1)}$. By definition, an open face F is contained in the rectangular nerve RN if and only if the set $I_1(F)$ is not empty, and the intersection $\cap_{i \in I_+(F)} B_i$ is not empty. Note that all faces in RN are finite-dimensional.

So defined rectangular nerve has two important properties. First, the construction of a good cover implies that each point is in $\frac{1}{2}B_i$ for some good ball B_i. Also, each point of $\phi(X) \bigcap F$ must be in the intersection $\cap_{i \in I_+(F)} B_i$. So, each point of X will be mapped by F into one of open faces of P that were included in RN. Thus, as previously claimed, $\phi(X) \subset RN$. Second, RN is contained in the union of all closed hyperfaces of P defined by the equations $x_i = r_i$, where x_i denotes the i-th coordinate of a point of P. Thus, for each point x in RN, there is some k such that $x_k = r_k$.

From now on we regard ϕ as a map not into P but to its closed subset RN.

In section 7 we will inductively deform the image of $\phi = \phi^{(0)}$, by pushing it into lower dimensional skeleta of RN until it lands in the $(m-1)$-skeleton of RN. Simultaneously, we will be changing $RN = RN^{(0)} \supset RN^{(1)} \supset \dots$ by removing on each step all maximal open faces of dimension $\geq m$. More precusely, for each map $\phi^{(i)}$ our goal will be the following: We start from the collection of all maximal open faces F_j in $RN^{(i)}$. For each F_j such that $dimF_j > m-1$, we will alter $\phi^{(i)}$ on $(\phi^{(i)})^{-1}(F_j)$ so that this map remains continuous, but its image will be in the boundary of F_j. We denote the resulting map (altered on all sets $(\phi^{(i)})^{-1}(F_j)$ for all maximal faces F_j of dimension $\geq m$) as $\phi^{(i+1)}$. Simultaneously, we change $RN^{(i)}$ by removing all maximal open faces of $RN^{(i)}$ of dimensions $\geq m$, and denote the result as $RN^{(i+1)}$.

Observe that if $(\phi^{(i)}(y))_k = r_k$ for some y and k, then the same will be true for $\phi^{(i+1)}$. Indeed, we need to check this only in the case, when $\phi^{(i)}(y)$ is in some maximal open face F_j in $RN^{(i)}$, where we alter $\phi^{(i)}$, but in this case the kth coordinate must be r_k for all points of F_j, and, therefore, for all points of its boundary. Similarly, if $(\phi^{(i)}(y))_k = 0$ for some y and k, then the same will be true for $\phi^{(i+1)}$. It is clear that for each face F_j in RN $\phi^{(i)}$ will map F_j into its $(m-1)$-dimensional skeleton, if $i > dimF_j - m$. Moreover, for $i > dimF_j - m$ the restrictions of all continuous maps $\phi^{(i)}$ to the closure $(\phi^{(i)})^{-1}(F_j)$ will coincide. Define $\phi^{(\infty)}$ as follows. For each x consider the open cell F in RN containing $\phi(x)$. Let F_j denotes a maximal open cell in RN containing F in its closure with the maximal possible dimension. (The existence of such a cell follows from the local finiteness of the good cover.) Now define $\phi^{(\infty)}(X)$ as $\phi^{(i)}(x)$, where $i = \max\{0, dimF_j - m + 1\}$. The map $\phi^{(\infty)}$ will be the desired map with diameter of each fiber $(\phi^{(\infty)})^{-1}(x) \leq \frac{2}{7} < 1$. Indeed, let $\phi^{(\infty)}(y) = x$. By our construction, $\phi^{(\infty)}(y) = \phi^{(i)}(y)$ for some i. Consider the (non-empty) set I of indices k such that $(\phi(y))_k = (\phi^{(0)}(y)_k = r_k$. Our observation implies that the similarly defined set J of indices k such that $x_k = \phi^{(i)}(y)_k = r_k$ will contain I, and, therefore, will be non-empty. In the opposite direction, if $k \in J$, then $\phi^{(\infty)}(y)_k = \phi^{(\infty)}(y)_k \neq 0$. This means, that 1) Each $y_1 \in (\phi^{(\infty)})^{-1}(x)$ is in $\frac{1}{2}B_k$ for a good ball B_k for some $k = k(y_1) \in J$; 2) For each other point $y_2 \in (\phi^{(\infty)})^{-1}(x)$ $y_2 \in B_{k(y_1)}$. This means that the distance between y_1 and y_2 in X does not exceed $2r_{k(y_1)} \leq \frac{2}{7}$, as claimed.

In order to perform these deformations we need to keep track of the m-dimensional Hausdorff content of the images. Note that given a face F of RN we can push $\phi(X) \cap F$ into ∂F whenever $\mathrm{HC}_m(\phi(X) \cap F) < \mathrm{HC}_m(F)$. In order to apply this reasoning repeatedly we need the following analog for HC_m of Lemma 5 from [Gu11].

Given an open face $F \subset RN$, we let $\mathrm{Star}(F)$ denote the union of F and all open faces F' such that $F \subset \bar{F}'$. If F has dimension k, then each open face in $F' \subset \mathrm{Star}(F)$ has dimension $\geq k$, and the only k-face in $\mathrm{Star}(F)$ is F. We let $\varrho(F)$ denote the shortest length of any of the sides of F, and B_F is the corresponding ball in the good covering with radius $\varrho(F)$.

Lemma 4.1. *There are constants $C_1(m)$ and $b(m) > 0$ depending only on m so that the following estimate holds: Suppose $\{B_i\}$ is a good cover. Then there is a map $\phi^{(0)} : X \to N$ subordinate to the cover so that that following volume estimate holds:*

For any face $F \subset RN$ of dimension $d(F)$,

$$\mathrm{HC}_m[\phi^{(0)}(X) \cap \mathrm{Star}(F)] \leq 14^{m+1} \varepsilon_m \varrho(F)^m e^{-b(m)d(F)}. \tag{4.1}$$

where $b(m) = \frac{\ln 2}{49^{m+1}}$.

Proof. The map $\phi^{(0)}$ is just the map ϕ from X to RN constructed above. Observe that $\phi^{(D)}(X) \bigcap \mathrm{Star}(F) = \phi^{(0)}(\bigcap_{i \in I_+(F)} B_i) \subset \phi^{(0)}(B_F)$. As we are using the max-norm on the target space, the Lipschitz constant of $\phi^{(0)}$ does not exceed 2 regardless of the dimension (in contrast with the Euclidean case, where it can behave as $\sqrt{D}$). Therefore, $\mathrm{HC}_m(\phi^{(0)}(X) \bigcap \mathrm{Star}(F)) \leq 2^m \mathrm{HC}_m(B_F) \leq 14^{m+1}\varepsilon_m \varrho(F)^{m+1}$. But Lemma 3.4 implies that $\varrho(F) \leq \frac{2}{7} \exp^{-b(m)d(F)}$. Trading one $\varrho(F)$ for the right hand side of the last inequality we obtain the inequality in the lemma. $\square$

We will continue our proof in section 7, after establishing in section 6 an isoperimetric inequality for HC_m. In the next section we are going to state and prove versions of the cone inequality and the coarea inequality that are required for our purposes.

5. Cone and coarea inequalities for Hausdorff content

Lemma 5.1 (Cone inequality 1). *Let $B(R)$ be a closed metric ball of radius R in a Banach space S. Let X be a subset of S, and Y be a proper compact subset of X, such that $Y \subset B(R)$. Then there exists a subset Z of $B(R)$ such that:*
(1) Y is a subset of Z;
(2) $\mathrm{HC}_m(Z) \leq m(1+\frac{1}{m})^m R\,\mathrm{HC}_{m-1}(Y) < emR\,\mathrm{HC}_{m-1}(Y)$;
(3) There exists a continuous map $\Psi : X \longrightarrow Z$ such that its restriction on Y is the identity map.

Here $e = lim_{m\to\infty}(1+\frac{1}{m})^m$ is the Euler's number.

Proof. Let $\{B_i(r_i)\}_{i\in I}$ be a finite covering of Y by closed metric balls such that $\sum_i r_i^{m-1} \leq \mathrm{HC}_{m-1}(Y) + \varepsilon$ for some ε that can be taken arbitrarily small. Let r be any positive number $\leq \min_i r_i$.

Let p denotes the center of the ball $B(R)$ from the text of the theorem. For each set V let $Cone_p(V)$ denote the union of the closed straight line segments in the ambient Banach space connecting p with all points of V. Observe, that for each metric ball $B_i(r_i)$ the set $Cone_p(B_i(r_i))$ can be covered by at most $m\frac{R}{r_i}$ metric balls of radius $(1+\frac{1}{m})r_i$. (The first of these balls is $B_i((1+\frac{1}{m})r_i)$; the centers of all subsequent balls are spaced along the segment connecting the center of $B_i(r_i)$ and p at distances $\frac{r_i}{m}$ apart. It is straightforward to check, using triangle inequality, that this collection of balls covers $Cone_p(B_i(r_i))$). Therefore, $Cone_p(N_r(V))$ can be covered by a collection of at most $m\frac{R}{r_i}$ metric balls of radius $(1+\frac{1}{m})r_i$, where i ranges over I. Therefore, $\mathrm{HC}_m(Cone_p(N_r(V)) \leq \frac{R}{r_i}\sum_i m((1+\frac{1}{m})r_i)^m \leq m(1+\frac{1}{m})^m R(\mathrm{HC}_{m-1}(Y)+\varepsilon)$, where ε is arbitrarily small.

Let $\phi : [0,\infty] \to [0,1]$ be a continuous monotone function with $\phi(0) = 1$ and $\phi(t) = 0$ for all $t \geq r$ defined as $1 - \frac{1}{r}x$ for $x \in [0,r]$ and zero for all $x \in [r,\infty)$. We define a map $\Phi : B(R) \to Cone_p(N_r(Y)) \bigcap B(R)$ by the formula $\Phi(x) = \phi(dist(x,Y))x + (1-\phi(dist(x,Y))p$. (It is obvious that Φ maps all points of $B(R)$ at distance $\geq r$ from Y to p, and that the restriction of Φ to Y is the identity map.) As X is not assumed to be in $B(R)$, the image of Φ can (somewhat) stick out of $B(R)$. This can be remedied by composing Φ with the 1-Lipschitz retraction $S \longrightarrow B(R)$ that sends each point of S to the nearest point of $B(R)$. Denote the restriction of this composition to X by Ψ. Now we can set $Z = \Psi(X)$. As $Z \subset Cone_p(N_r(Y))$, $\mathrm{HC}_m(Z) \leq \mathrm{HC}_m(Cone_p(N_r(Y)) \leq m(1+\frac{1}{m})^m R(\mathrm{HC}_{m-1}(Y)+\varepsilon)$ for arbitrarily small ε, which implies the lemma, when $\varepsilon \longrightarrow 0$. $\square$

Observe that Z is not a cone over Y, although it is very close to $Cone_p(Y)$. In fact, in the proof above we can replace the chosen value of r by any smaller positive value, and the proof would still works. So, if desired we could choose Z arbitrarily close to an actual cone over Y in S. Below we will be referring to Z as the cone over Y, and will call this construction *the coning of* Y. Also, note that if X and Y are given as images under a continuous map $\tau : \tilde{X} \longrightarrow S$ from a metric space $\tilde{X}$ to S of $\tilde{X}$ itself and its proper subset $\tilde{Y}$, then we can consider the composition F of the the map Ψ with τ, and obtain the following version of the previous lemma:

Lemma 5.2 (Cone inequality 2)**.** *Let $B(R)$ be a metric ball of radius R in a Banach space S. Let X be a metric space, and Y its proper compact subset. Given a continuous map $\tau : X \longrightarrow S$, such that $\tau(Y) \subset B(R)$ there exists a continuous map $F : X \longrightarrow B(R)$ such that:*
(1) The restriction of F to Y coincides with τ;
(2) $\mathrm{HC}_m(F(X)) \leq m(1+\frac{1}{m})^m R\,\mathrm{HC}_{m-1}(\tau(Y)) < emR\,\mathrm{HC}_{m-1}(\tau(Y))$;
and
(3) Assume that both the center of $B(R)$ and $\tau(X)$ are in a convex set U. Then one can additionally ensure that $F(X) \subset U$, and so F maps X to $B(R)\bigcap U$.

Remark. It might be of interest to note that if the convex set U in the previous lemma is compact, and $F(X \setminus Y)\bigcap \partial U = \emptyset$, then one can choose F so that it is the identity map not only on $\tau(Y)$ but also everywhere outside of the interior of U. Indeed, as in the previous lemma, it is sufficient to consider the particular case, when τ is an inclusion map of $X \subset U$. We are assuming that $X \bigcap \partial U \subset Y$.

Choose a positive $r < r_0$ as in the proof of Lemma 5.1. Choose a positive $r_* \leq r$ such that $dist(X \setminus Cone_p(N_r(Y)), \partial U) > r_*$. Define ϕ as in the proof of Lemma 5.1, but for r_* instead of r, and then define Φ by the formula $\Phi(x) = \phi(dist(x, Y \bigcup \partial U))x + (1 - \phi(dist(x, Y))p$. We will consider Φ as the map of U (rather than $B(R)$, as in the proof of Lemma 5.1). It is obvious that its image will also be in U. Also, the restriction of Φ to ∂U is the identity map. Therefore, we can extend Φ to a map $F : S \longrightarrow S$ by defining it as the identity map outside of U.

In the next lemma $B(R)$ denotes a metric ball of radius R in a (boundedly) compact metric space X, $B(R_2) \setminus B(R_1)$ is the annulus between two concentric metric balls, S_R denotes the metric sphere of radius R.

Lemma 5.3. *(Co-area inequality) Let $U \subset B(R_2) \setminus B(R_1)$ be a closed set. Then,*

$$\int_{R_1}^{*R_2} \mathrm{HC}_{m-1}(S_R \bigcap U)\ dR \leq 2\,\mathrm{HC}_m(U),$$

where $\int^*$ denotes the upper Lebesgue integral. Therefore, there exists $R \in [R_1, R_2]$, such that $\mathrm{HC}_{m-1}(S_R \cap U) \leq \frac{2}{R_2 - R_1}\mathrm{HC}_m(U)$.

Proof. Let $\{B_{r_i}(p_i)\}$ be a covering of U with $\sum_i r_i^m \leq \mathrm{HC}_m(U) + \varepsilon$, where $i \in \{1, \dots, N\}$ for some N. The desired inequality would follow from the inequality $\int_{R_1}^{*R_2} \mathrm{HC}_{m-1}(S_R \bigcap U)\ dR \leq 2\sum_i r_i^m$. We are going to prove a stronger inequality, where $\mathrm{HC}_{m-1}(S_R \bigcap U)$ is replaced by the following quantity that is obviously not less than $\mathrm{HC}_{m-1}(S_R \bigcap U)$, namely, $\sum_{i\in I(R)} r_i^{m-1}$, where $I(R)$ denotes the set of all indices i such that the intersection of $B_{r_i}(p_i)$ and $S(R)$ is not empty. The left hand side of the desired inequality becomes

$$\int_{R_1}^{R_2} \sum_{i\in I(R)} r_i^{m-1} dR = \int_{R_1}^{R_2} \sum_{i=1}^{N} r_i^{m-1}\chi_i(R)dR = \sum_{i=1}^{N} r_i^{m-1} \int_{R_1}^{R_2} \chi_i(R)dR,$$

where the characteristic function $\chi_i(R)$ is equal to 1 for all $R \in [R_1, R_2]$ such that S_R and $B_{r_i}(p_i)$ have a non-empty intersection, and to 0 otherwise. Finally, observe that $\int_{R_1}^{R_2} \chi_i(R)dR \leq 2r_i$, which implies the desired inequality. $\square$

Let $\Sigma_R,\ R \in [R_1, R_2]$ be a family of closed sets in a boundedly compact metric space X with the following property (C): For each pair of points $x \in \Sigma_t$ and $y \in \Sigma_s$ $dist_X(x, y) \geq |s - t|$. (For example, this property holds for a family of equidistant surfaces such that $dist(\Sigma_t, \Sigma_s) = |t - s|$.) Let $A(R_1, R_2)$ denote $\bigcup_{R\in[R_1,R_2]} \Sigma_R$.

It is obvious that the above proof generalizes (verbatim) to the following somewhat more general situation:

Lemma 5.4. *(Co-area inequality 2) Let Σ_R be a family that satisfies the property (C), $U \subset A(R_1, R_2)$ a closed set. Then,*

$$\int_{R_1}^{*R_2} \mathrm{HC}_{m-1}(\Sigma_R \bigcap U)\ dR \leq 2\,\mathrm{HC}_m(U).$$

Therefore, there exists $R \in [R_1, R_2]$, such that $\mathrm{HC}_{m-1}(\Sigma_R \cap U) \leq \frac{2}{R_2-R_1}\,\mathrm{HC}_m(U)$.

For example, this lemma now applies to the situation, when X is the parallelepiped $P = [0, r_1] \times [0, r_2] \times \dots [0, r_D]$ endowed with $\max(l^\infty)$-norm, $0 \leq R_1 < R_2 \leq \min_i r_i/2$, and Σ_R is the boundary of the paralellipiped $P_R = [R, r_1 - R] \times [R, r_2 - R] \times \dots [R, r_D - R]$, or, more generally, to similar situations, when the common center of the family of boundaries of parallelipipeds P_R does not coincide with the center of P.

6. Isoperimetric extension inequality.

Theorem 6.1. *Assume that $U \subset \mathbb{R}^n$ is a closed n-dimensional parallelepiped with sides parallel to the coordinate axes and with l_∞ metric. Let $f : X \longrightarrow U$ be a continuous map from a compact metric space X to U, and Y denote $f^{-1}(\partial U)$. Assume that m is an integer number between 2 and $n-1$, and that $\mathrm{HC}_{m-1}(f(Y)) > 0$. Then there exist constants $I_1(m)$, $I_2(m)$, and a map $F : X \longrightarrow U$ with the following properties:*
(1) The restriction of F on Y coincides with f;
(2) $\mathrm{HC}_m(F(X)) \leq I_1(m)\,\mathrm{HC}_{m-1}(f(Y))^{\frac{m}{m-1}}$;

(3) Let $R = I_2(m)\,\mathrm{HC}_{m-1}(f(Y))^{\frac{1}{m-1}}$*. Then* $F(X)$ *is contained in* R*-neighbourhood of* ∂U*. Here one can take* $I_1(m) = (100m)^m$ *and* $I_2(m) = (4000m)^m$*.*

Remark. The proof of Theorem 6.1 given below yields (1), (2) (but not (3)) for arbitrary Banach spaces S, arbitrary closed subsets Y of X (not necessarily $Y = f^{-1}(\partial U)$), and arbitrary values of $m \geq 2$. Thus, we obtain the following theorem that can be compared with Theorem A$'''$ in Appendix 2 of [Gr].

Theorem 6.2. *Let* S *be a Banach space,* X *a compact metric space,* Y *a closed subset of* X*,* $f : X \longrightarrow S$ *a continuous map, and* $m \geq 2$ *an integer number. Assume that* $\mathrm{HC}_{m-1}(f(Y)) > 0$*.*

Then there exist a constant $I_1(m)$*, and a map* $F : X \longrightarrow S$ *with the following properties:*
(1) The restriction of F *on* Y *coincides with* f*;*
(2) $\mathrm{HC}_m(F(X)) \leq I_1(m)\,\mathrm{HC}_{m-1}(f(Y))^{\frac{m}{m-1}}$*;*

Remark. If $\mathrm{HC}_{m-1}(f(Y)) = 0$, our proof gives that for each positive δ there exists $F : X \longrightarrow S$ that coincides with f on Y and such that $\mathrm{HC}_m(F(X)) \leq \delta$.

Theorem 6.1 immediately follows from its particular case, when $X \subset U$, and f is the inclusion, as we can just apply this particular case to $f(X)$ and then compose the resulting F with f. So, we will present the proof only for this particular case.

We will proceed by induction on m. The proof given below will work for both theorems with very minor differences at the very end of section 6.1 and the proof of the base of the second induction in section 6.2. In the situation of Theorem 6.2 we define U as S.

6.1. **The base case** $m = 2$**.** We will start from a general overview of our strategy. We are going to partition X into several pieces and define maps form each piece to U that can be combined into a single continuous map F from X to U that can then be restricted to X, and will have the desired properties.

All but one of these "pieces" will be in small closed neighbourhoods of connected components of some union of metric balls in S providing an almost optimal covering of Y (from the perspective of HC_1). These pieces will be mapped using the coning construction, so that their boundaries will be mapped to points p_i. Finally, it would remain the last "piece", i.e. the closure of the complement to the (finite) union of all previously described "pieces". The map on its boundary is already defined; it sends the boundary to a finite collection of points, and one can extend it to a map into an arc connecting all these points using the Tietze extension theorem. Alternatively, one could just cone the points p_i into a tree with one new vertex (the tip of the cone) and use the well known generalization of Tietze theorem for the case when the target space is a contractible ANR (in particular, a contractible CW-complex).

Let $\{B_i\}_{i=1}^N$ be a finite collection of closed metric balls covering Y with $\sum r_i \leq \mathrm{HC}_1(Y) + \delta$. Let U_1, ..., U_k be those connected components of $\bigcup B_i$ that contain at least one point of $f(Y)$. Note that each U_j is the union of B_i for all $i \in E(j)$, where the sets $E(j)$ form a partition of $\{1, \ldots, N\}$. Therefore, $\sum diam(U_j) \leq \sum_i 2r_i \leq 2\,\mathrm{HC}_1(Y) + 2\delta$. For each j choose a closed ball $B(R_j)$ of radius $R_j = diam(U_j)$ with the center at $p_j \in U_j \bigcap Y$. By Lemma 5.2 applied to U, $B(R_i)$, $X \bigcap U_i$ as X, and $Y \bigcap U_i$ as Y, there exists a map $F_i : B(R_i) \longrightarrow S$ such that F_i is the identity on $U_i \bigcap Y$.Moreover, we have $\mathrm{HC}_2(F_i(X \bigcap U_i)) \leq 4.5 diam(U_i)\,\mathrm{HC}_1(U_i \bigcap Y) \leq 4.5\ diam^2(U_i)$. Summing over all i and noticing that $\sum_i diam(U_i)^2 \leq (\sum_i diam(U_i))^2 \leq 4(\mathrm{HC}_1(Y) + \delta)^2$ we see that $\sum_i \mathrm{HC}_2(F_i(X \bigcap U_i)) \leq 20\,\mathrm{HC}_1(Y)^2$, if δ is sufficiently small. Also, it is obvious that F_i maps $B(R_i) \bigcap U$ to itself.

Recall that the construction of each F_i involves choosing a positive parameter r. The value of r must be sufficiently small, but it can be chosen arbitrarily close to 0. This parameter plays a role in the construction of a map $\Phi_i : U \longrightarrow B(R_i)$. Note that Φ_i and its extension to the whole $B(R_i)$, F_i, map all points of X at distances $\geq r$ from U_i to p_i. Choose $r < \frac{1}{2} \min_{i,j} dist(U_i, U_j)$. Then for each i the set $X \bigcap \partial N_r(U_i)$ is mapped by F_i to p_i. We are going to define F on $X \bigcap N_r(U_i)$ as F_i.

Let $G \subset U$ be an arc with no self-intersections, so that $\bigcup p_i \subset G$. So far F has been defined on $X \bigcap (\bigcup_i N_r(U_i))$. It maps each set $X \bigcap \partial N_r(U_i)$ to p_i. Using Tietze extension theorem we can continuously extend F to a map of $X \setminus$ interior$(\bigcup_i N_r(U_i)$ into G. As the image of this map is 1-dimensional, it has zero two-dimensional Hausdorff measure, and, therefore, zero HC_2. Thus, F satisfies property (2) of the theorem with $I_1(2) = 20$.

To prove property (3) in Theorem 6.1 define Q_2 as G, and Q_1 as $\bigcup_i F_i(X \setminus \bigcup$ interior$(N_r(U_i))$. For each point x in the image of $F(N_r(U_i))$ there exists a point $y \in Y$ such that the distance between x and y does not exceed $diam(U_i) \leq 3\,\mathrm{HC}_1(Y)$. We are going to somewhat modify our construction of F on the inverse image of Q_2 to ensure property (3). Note that we can assume without any loss of generality that R is less than the distance from the center of the paralellepiped U to its boundary. It is easy to find a ball β with a center c near the center of the parallelepiped U with a very a small positive radius that does not intersect G. Now we can retract $U \setminus \beta$ to the R-neighbourhood of ∂U. This retraction leaves Q_1 intact, but maps Q_2 inside the R-neighbourhood. As the dimension of the image of Q_2 is still one, its HC_2 is zero. We change F on $X \setminus$ interior$(\bigcup_i N_r(U_i)$ by composing it with the retraction $U \setminus \beta \longrightarrow \partial U$. This completes the proof of property (3) with $I_2(2) = 3$.

6.2. **Inductive step: set up.** By induction we assume the conclusions of Theorem 6.1 to be true for all dimensions less than or equal to m. We will now prove it for $m + 1$.

Let $\varepsilon = \varepsilon(n, Y, U) > 0$ be a small constant to be determined later.

We will consider compact sets $Y' \subset X' \subset U$.

We will first show that the conclusions of the theorem hold for X' and Y', whenever $\mathrm{HC}_m(Y') \leq \varepsilon$, and $\max_{y\in Y'} dist(y, Y) \leq R_1(Y)$ for some $R_1(Y)$ that will be defined later. (Recall that according to the assumption of Theorem 6.1 $Y \subset \partial U$.) Then we inductively prove the result for larger and larger values of $\mathrm{HC}_m(Y')$ until we obtain it for X, Y and f. Formally speaking, the induction will be with respect to $k > 1$ such that $\mathrm{HC}_m(Y') \in ((1 + \frac{1}{20^m})^{k-1}\varepsilon, (1 + \frac{1}{20^m})^k\varepsilon]$. At each step of the induction we are going to increase the allowed upper bound for $\mathrm{HC}_m(Y')$ by the factor of $1 + \frac{1}{20^m}$, and to prove that the desired extension is possible for all such sets Y' that are also sufficiently close to Y. More precisely, we require that $\max_{y\in Y'} dist(y, Y) \leq R_k(Y)$, where $R_k(Y) > 0$ that will be defined later decreases with k. Obviously, after finitely many induction steps we will obtain the assertion of the lemma for Y.

More precisely, we will show that there exists $F' : X' \longrightarrow U$, such that

(1) The restriction of F' on Y' is the identity map;

(2) $\mathrm{HC}_{m+1}(F'(X')) \leq I'_1(m + 1)\,\mathrm{HC}_m(Y')^{\frac{m+1}{m}} + \varepsilon_0$;

In the situation of Theorem 6.1

(3) $\max_{x\in X'} dist(F'(x), \partial U) \leq R_k(Y) + I_2(m)\,\mathrm{HC}_m(Y')^{\frac{1}{m}}$.

The value of $\varepsilon_0 > 0$ in (2) can be chosen to be arbitrarily small, assuming that $\varepsilon > 0$ is small enough. Setting $I_1(m) = 2I'_1(m)$ and choosing Y, X and f for Y', X' and f' we obtain the result of the theorem.

First, we prove the base of the induction with respect to k. Assume that $\mathrm{HC}_m(Y') \leq \varepsilon$. Let $R_0(Y) = diam(Y) + R_1(Y)$. Observe that $Y' \subset B_{R_0}(p)$ for some $p \in U$. By Lemma 5.1 there exists a map $F_0 : X \longrightarrow B_{R_0}(p) \cap U$, so that

$$\mathrm{HC}_{m+1}(F_0(X')) \leq emR_0\,\mathrm{HC}_m(Y'),$$

and the restriction of F' on Y' coincides with f'. This is sufficient to prove (1) and (2) (that is, for Theorem 6.2), as one can take $F' = F_0$.

In the situation of Theorem 6.1 we need also to prove (3). We are going to modify F_0. We apply projection from an average point argument similar to the proof of Deformation Theorem of Federer and Fleming ([FF], see also Lemma 7.2 in [Gu13] and Lemma 2.5 in [Y] for more general versions of this argument adapted to Hausdorff content). Given a point $p \in U_r = U \setminus N_r(U)$ let $R_{p,r}$ denote a radial projection map $R_{p,r} : U_r \setminus \{p\} \longrightarrow \partial U_r$. Denote the length of the shortest edge of U by r_1. Extend $R_{p,r}$ to a map $R_{p,r} : U \longrightarrow \overline{N_r(U)}$ by setting $R_{p,r}(x) = x$ for all $x \in U \setminus U_r$. For any $x, y \in U_r \setminus \{p\}$ we have

$$dist(R_{p,r}(x), R_{p,r}(y)) \leq \frac{const(U, n)dist(x, y)}{\min\{|x - p|, |y - p|\}} \tag{6.1}$$

Here we can take *dist* to be l_∞ or Euclidean distance since the two differ not more than by a factor $const(n)$.

Cover $F_0(X')$ by a finite collection of balls $\{\beta(p_i,\varrho_i)\}$ satisfying $\sum_i \varrho_i^{m+1} < 2\,\mathrm{HC}_{m+1}(F_0(X')) \leq 2emR_0(Y)\,\mathrm{HC}_m(Y') \leq c(m,Y)\varepsilon$. If $R_1(Y) \geq \frac{r_1}{2}$ we have nothing to prove. Assume that $R_1(Y) < \frac{r_1}{2}$, and define $R_* = R_*(Y) = R_1(Y) + I_2(m)\varepsilon^{\frac{1}{m}}$, $r_* = r_*(Y) = \frac{r_1}{2} - R_*(Y)$. If ε is sufficiently small, r_* is positive. Let $E = \bigcup_i \beta(p_i, 2\varrho_i)$. Note the the (Euclidean) volume of E does not exceed $2^n \sum_i (\varrho_i^{m+1})^{\frac{n}{m+1}} \leq 2^n\varepsilon$. Therefore, if $\varepsilon = \varepsilon(n,Y,U)$ is sufficiently small, $V = U_{R_1} \setminus E$ is a non-empty set of volume greater than $\frac{r_*^n}{2}$.

We will choose a point $p \in V$ and define $F'(x) = R_{p,R_*} \circ F_0(x)$. By (6.1) we have that $R_{p,R_*}(\beta(p_i,\varrho_i))$ is contained in a ball of radius $\leq \frac{const(U,n)\varrho_i}{|p-p_i|-\varrho_i}$. Hence, we have $\mathrm{HC}_{m+1}(R_{p,R_*} \circ F_0(X')) \leq \sum_i \frac{const(U,n)\varrho_i^{m+1}}{(|p-p_i|-\varrho_i)^{m+1}}$. Consider the integral of this quantity over V:

$$\begin{aligned}\int_V \mathrm{HC}_{m+1}(R_{p,R_*} \circ F_0(X'))dp &\leq \int_V \sum_i \frac{const(U,n)\varrho_i^{m+1}}{(|p-p_i|-\varrho_i)^{m+1}}dp \\ &\leq const(U,n)\sum_i \varrho_i^{m+1}\int_V \frac{dp}{(|p-p_i|-\varrho_i)^{m+1}} \\ &\leq const(U,n)\,\mathrm{HC}_m(Y')\end{aligned} \tag{6.2}$$

The exact value of the constant $const(U,n)$ changes from line to line; the last integral is bounded since $m+1 \leq n-1$. (This inequality is one of the assumptions of Theorem 6.1.)

It follows that there exists a point $p \in V$ with $\mathrm{HC}_{m+1}(R_{p,R'} \circ F_0(X')) \leq \frac{const(U,n)\,\mathrm{HC}_m(Y')}{r_*^n/2} = const(U,n,Y)\varepsilon$. Also, observe that, as $Y' \subset N_{R_1(Y)}(\partial U)$, R_{p,R_*} is the identity map on Y'. We choose $\varepsilon_0 = const(U,n,Y)\varepsilon$. Choosing ε sufficiently small, we can assume $\varepsilon_0 < \frac{I_1'(m+1)}{\mathrm{HC}_m(Y)^{\frac{m+1}{m}}}$. This is our last restriction on ε, and it is not difficult to choose a positive $\varepsilon(n,Y,U)$ satisfying all the constraints above.

This finishes the proof of the initial step of the induction, when $\mathrm{HC}_m(Y') \leq \varepsilon$ (or, $k=1$).

Now, we are going to explain the idea of the proof of the inductive step. Suppose $(1+\frac{1}{20^m})^k\varepsilon < \mathrm{HC}_m(Y') \leq (1+\frac{1}{20^m})^{k+1}\varepsilon$ and $\max_{y\in Y'} dist(y,Y) \leq R_k(Y)$, and that we already established the desired assertion for all $\tilde{Y}'$ such that $\mathrm{HC}_m(\tilde{Y}') \leq (1+\frac{1}{20^m})^k\varepsilon$ and $\max_{y\in\tilde{Y}'} dist(y,Y) \leq R_k(Y)$.

In this case we are going to find certain $Y'' \subset X'$ such that 1) For each $y \in Y''$ $dist(y,Y') < R_k(Y) - R_{k+1}(Y)$; 2) $\mathrm{HC}_m(Y'') \leq \frac{\mathrm{HC}_m(Y')}{1+\frac{1}{20^m}}$. The first property implies that for each $y \in Y''$ $dist(y,Y) \leq R_k(Y)$, and now 2) will imply that Y'' can be "filled" as desired. We will be filling Y'' by the image of a certain subset X'' of X' that contains Y''; the rest of X' will be used to "fill" the "gap" between Y' and Y''.

Here are some details of how this will be accomplished. In the next subsection we are going to define a certain finite system of closed metric balls $\tilde{B}_j$ in S intersecting Y', and establish some useful properties of these balls. Then we will remove all the intersections of the interior of $\tilde{B}_j$ with Y' from Y'. Instead we will attach "fillings" of $\partial\tilde{B}_j \bigcap Y'$ obtained by the application of the induction assumption. More precisely, we will apply our theorem to m, Y defined as $\partial\tilde{B}_j \bigcap Y'$, and X defined as $\partial\tilde{B}_j \bigcap X'$. The desired filling will be $F(\partial\tilde{B}_j \bigcap X')$. (The resulting F will be denoted τ_j below.) The result will be Y''. In other words, $Y'' = (Y' \setminus \bigcup_j(\tilde{B}_j \bigcap Y')) \bigcup \bigcup_j \tau_j(\partial\tilde{B}_j \bigcap X')$. The subset X'' will be the closure of $X' \setminus \bigcup_j \tilde{B}_j \bigcap X'$. The "rest of X'" will be $\bigcup_j \tilde{B}_j \bigcap X'$. "Filling the gap between Y' and Y''" is accomplished by applying the coning (Lemma 5.2) to $B(R)$ defined as $\tilde{B}_j$, Y defined as $(Y' \bigcap B_j) \bigcup (X' \bigcap \partial\tilde{B}_j)$, and τ defined as the identity map on $Y' \bigcap \tilde{B}_j$, and τ_j on $X' \bigcap \partial\tilde{B}_j$. In section 6.4 we will provide more details of this construction and verify that so constructed F' satisfies property (2). The proof will use various properties of chosen balls $\tilde{B}_j$ established in section 6.3.

The distances between points $y \in Y''$ and Y' will be less than $2\max_j rad(\tilde{B}_j)$. As we will see in the next subsection, the radii of all $\tilde{B}_j$ will be less than $2A(m)\,\mathrm{HC}_m(Y)^{\frac{1}{m}}(1+\frac{1}{20^m})^{-l/m}$, where $A(m)$ is a specific constant (later we will choose $A(m)$ as $(60mI_1(m))^{\frac{m-1}{m}}$), and $l = \lfloor \log_{1+\frac{1}{20^m}} \frac{\mathrm{HC}_m(Y)}{\mathrm{HC}_m(Y')} \rfloor$. Now we can *define* all $R_i(Y)$ by recurrent relations $R_i(Y) = R_{i-1}(Y) - 4A(m)\,\mathrm{HC}_m(Y)^{\frac{1}{m}}[(1+\frac{1}{20^m})^{\frac{1}{m}}]^{-l}$, and $R_1(Y)$ can be defined so that all $R_i(Y)$ for $i \le K$ remained positive, where K is the minimal k such that $\mathrm{HC}_m(Y) \le (1+\frac{1}{20^m})^k \varepsilon(n, Y, U)$. So, one can take $R_1(Y) = 4A(m)\frac{1}{1-(1+\frac{1}{20^m})^{-\frac{1}{m}}}\,\mathrm{HC}_m(Y)^{\frac{1}{m}}$. Therefore, one will get property (3) in Theorem 6.1 with $I_2(m+1) = 4A(m)\frac{1}{1-(1+\frac{1}{20^m})^{-\frac{1}{m}}} \le (4000m)^m$. (Here we are using $I_1(m) = (100m)^m$.) Thus, one can take $I_2(m) = (4000m)^m$.

6.3. **Definition of covering $\{\tilde{B}_j\}$ and some useful estimates.** Fix a covering of Y' by closed balls β_j, $j = 1, \dots, N$ of radius r_j centered in S so that $\sum_j r_j^m \le \mathrm{HC}_m(Y')(1+\delta)$, where δ can be chosen arbitrarily small. We denote this collection of balls Q. For each subset W of Y' define m-dimensional Hausdorff content $\widetilde{\mathrm{HC}}_m(W)$ with respect to Q as the infimum of $\sum_{j\in J} r_j^m$ over all subsets $J \subset \{1, \dots, N\}$ such that $W \subset \bigcup_{j\in J} \beta_j$. In other words, we calculate the Hausdorff content with respect to only balls from the collection Q. Clearly, for each W we have $\widetilde{\mathrm{HC}}_m(W) \ge \mathrm{HC}_m(W)$, so any upper bound for $\widetilde{\mathrm{HC}}_m$ will be automatically an upper bound for HC_m.

Define $A(m)$ as $(60mI_1(m))^{\frac{m-1}{m}}$. For a point $p \in Y'$ consider quantity $\frac{\widetilde{\mathrm{HC}}(B(p,r)\bigcap Y')}{r^m}$. Observe that as $r \to 0$ this quantity approaches positive infinity (since $\widetilde{\mathrm{HC}}$ of any non-empty subset of Y' is bounded from below by the m-th power of the radius of the smallest ball), and, as $r \to \infty$, it approaches 0. Hence, for each p we can define $r'(p) = \sup\{r | \widetilde{\mathrm{HC}}(B(p,r) \bigcap Y_i) \geq \frac{r^m}{A(m)^m}\}$. It will be more convenient to work with a slightly larger radius $r(p) = (1+\frac{1}{m})r'(p)$. Assuming δ is sufficiently small, $r(p) < 2A(m)\,\mathrm{HC}_m(Y')^{\frac{1}{m}}$. If $\mathrm{HC}_m(Y') \leq (1+\frac{1}{20^m})^{-l}\,\mathrm{HC}_m(Y)$, then $r(p) < 2A(m)\,\mathrm{HC}_m(Y)^{\frac{1}{m}}[(1+\frac{1}{20^m})^{\frac{1}{m}}]^{-l}$.

Now we would like to define concentric balls $\tilde{B}_j(p, \tilde{r}(p))$ for some somewhat larger radii $\tilde{r}(p) \geq r(p)$. For this purpose consider the annulus $A = B(p, (1+\frac{1}{m})r(p)) \setminus B(p, r(p))$. Applying the coarea inequality (Lemma 5.3) to $A \bigcap Y'$ we see that there exists $\tilde{r}(p) \in [r(p), (1+\frac{1}{m})r(p)]$ such that

$$\mathrm{HC}_{m-1}(\partial B(p, \tilde{r}(p) \cap Y') \leq \frac{2m}{r(p)}\,\mathrm{HC}_m(B(p, (1+\frac{1}{m})r(p)) \cap Y') \tag{6.3}$$

Use the Vitali covering construction to find a finite system of disjoint balls $\tilde{B}_j = B(q_j, \tilde{r}(q_j))$, $j \in \{1, \ldots, L\}$ for some L such that $\bigcup_i B(q_j, 3\tilde{r}(q_j))$ covers all Y'.

From the definition of $\tilde{B}_j$ we will derive four useful inequalities, namely (6.7), (6.8), (6.9), and (6.10), relating various quantities to $\mathrm{HC}_m(Y')$.

Inequality (6.7). Observe that from the definition of $r(q_j)$ there exists a sequence of radii r_l approaching $\frac{r(q_i)}{1+\frac{1}{m}}$ from below with $\widetilde{\mathrm{HC}}_m(B(q_j, r_l) \cap Y') \geq \frac{r(q_j)^m}{(1+\frac{1}{m})^m A(m)^m}$; on the other hand, $\widetilde{\mathrm{HC}}_m(B(q_j, r) \cap Y') < \frac{r^m}{A(m)^m}$ for every $r > \frac{r(q_j)}{1+\frac{1}{m}}$. Since $\widetilde{\mathrm{HC}}_m(B(q_j, r) \cap Y')$ is monotone, it follows that

$$\widetilde{\mathrm{HC}}_m(B(q_j, \frac{r(q_j)}{1+\frac{1}{m}}) \cap Y') = \frac{r(q_j)^m}{(1+\frac{1}{m})^m A(m)^m} \tag{6.4}$$

Also, for every $\theta \geq \frac{1}{1+\frac{1}{m}}$ we have

$$\widetilde{\mathrm{HC}}_m(B(q_j, \theta r(q_j)) \cap Y') \leq (1+\frac{1}{m})^m \theta^m \widetilde{\mathrm{HC}}_m(B(q_j, \frac{r(q_j)}{1+\frac{1}{m}}) \cap Y') \tag{6.5}$$

Let $\{\beta_{jl}\}$ denote the covering of $B(q_j, \frac{r(q_j)}{1+\frac{1}{m}}) \cap Y'$ realizing its Hausdorff content with respect to Q, $\widetilde{\mathrm{HC}}_m(B(q_j, \frac{r(q_j)}{1+\frac{1}{m}}) \cap Y') = \sum_l rad(\beta_{jl})^m$. Since $\frac{1}{(1+\frac{1}{m})^m A(m)^m} < \frac{1}{(2m)^m}$ we have $rad(\beta_{jl}) < \frac{r(q_j)}{2m}$ for each l. In particular, if $i \neq j$, then none of balls β_{jl} can appear as a ball $\beta_{il'}$ in a covering that realizes $\widetilde{HC}_m(B(q_i, \frac{r(q_i)}{1+\frac{1}{m}}) \cap Y')$. Therefore,

$\sum_j \widetilde{\mathrm{HC}}_m(B(q_j, \frac{r(q_j)}{1+\frac{1}{m}}) \cap Y') = \widetilde{\mathrm{HC}}_m(\cup_j B(q_j, \frac{r(q_j)}{1+\frac{1}{m}}) \cap Y') \leq \widetilde{\mathrm{HC}}_m(Y')$. Also, β_{j_l} does not intersect $Y' \setminus B(q_j, r(q_j))$. It follows that

$$\widetilde{\mathrm{HC}}_m(Y' \setminus \bigcup \tilde{B}_j) \leq \widetilde{\mathrm{HC}}_m(Y') - \sum_j \widetilde{\mathrm{HC}}_m(B(q_j, \frac{r(q_j)}{1+\frac{1}{m}}) \cap Y'). \tag{6.6}$$

On the other hand, since $Y' \subset \bigcup B(q_i, 3r(q_i))$ and utilizing (6.5) we have

$$\sum_j \widetilde{\mathrm{HC}}_m(B(q_j, \frac{r(q_j)}{1+\frac{1}{m}}) \cap Y') \geq \sum_j \frac{1}{(1+\frac{1}{m})^m 3^m} \widetilde{\mathrm{HC}}_m(B(q_j, 3r(q_j)) \cap Y') \geq \frac{1}{(1+\frac{1}{m})^m 3^m} \widetilde{\mathrm{HC}}_m(Y').$$

Denote $\left(\frac{\sum_j \widetilde{\mathrm{HC}}_m(B(q_j, \frac{r(q_j)}{1+\frac{1}{m}}) \cap Y')}{\widetilde{\mathrm{HC}}_m(Y')}\right)^{\frac{1}{m}}$ by α.

We just established that $\alpha \geq \frac{1}{3(1+\frac{1}{m})} > \frac{1}{5}$; earlier we saw that $\alpha \leq 1$.

Using the definition of α in (6.6) we obtain

$$\widetilde{\mathrm{HC}}_m(Y' \setminus \bigcup \tilde{B}_j) \leq (1 - \alpha^m) \widetilde{\mathrm{HC}}_m(Y') \leq (1 - \frac{1}{5^m}) \mathrm{HC}_m(Y'). \tag{6.7}$$

Now we are going to deduce three more inequalities:

$$\sum_j r(q_j) \mathrm{HC}_{m-1}(\partial \tilde{B}_j \cap Y')^{\frac{m}{m-1}} \leq \frac{30(m+1)}{A(m)^{\frac{1}{m-1}}} \alpha^{m+1} \mathrm{HC}_m(Y')^{\frac{m+1}{m}} \tag{6.8}$$

$$\sum_j \mathrm{HC}_{m-1}(\partial \tilde{B}_j \cap Y')^{\frac{m}{m-1}} \leq \frac{30m}{A(m)^{\frac{m}{m-1}}} \alpha^m \mathrm{HC}_m(Y') \tag{6.9}$$

$$\sum_j r(q_j) \mathrm{HC}_m(\tilde{B}_j \cap Y') \leq 12A(m) \alpha^{m+1} \mathrm{HC}_m(Y')^{\frac{m+1}{m}} \tag{6.10}$$

Proof of inequality (6.8). From inequalities $\widetilde{\mathrm{HC}}_m((1+\frac{1}{m})B_j \cap Y') \leq \frac{(1+\frac{1}{m})^m r(q_j)^m}{A(m)^m}$, (6.3) and (6.5) we obtain

(6.11)

$$\begin{aligned}
\mathrm{HC}_{m-1}(\partial\tilde{B}_j\cap Y') &\leq \frac{2m}{r(q_j)}\,\mathrm{HC}_m((1+\frac{1}{m})B_j\cap Y')\\
&= \frac{2m}{r(q_j)}\widetilde{\mathrm{HC}}_m((1+\frac{1}{m})B_j\cap Y')^{\frac{1}{m}}\,\mathrm{HC}_m((1+\frac{1}{m})B_j\cap Y')^{\frac{m-1}{m}}\\
&\leq \frac{4m}{A(m)}\,\mathrm{HC}_m((1+\frac{1}{m})B_j\cap Y')^{\frac{m-1}{m}}\\
&\leq \frac{4m*(1+\frac{1}{m})^{2m-2}}{A(m)}\widetilde{\mathrm{HC}}_m(B(q_j,\frac{r(q_j)}{1+\frac{1}{m}})\cap Y')^{\frac{m-1}{m}}\\
&\leq \frac{4e^2m}{A(m)}\widetilde{\mathrm{HC}}_m(B(q_j,\frac{r(q_j)}{1+\frac{1}{m}})\cap Y')^{\frac{m-1}{m}}
\end{aligned}$$

We use (6.4) and (6.11) to bound from above the the left hand side of (6.8).

(6.12)

$$\begin{aligned}
\sum_j r(q_j)\,\mathrm{HC}_{m-1}(\partial\tilde{B}_j\cap Y')^{\frac{m}{m-1}} &\leq \sum_j \frac{4e^2mr(q_j)}{A(m)^{\frac{m}{m-1}}}\widetilde{\mathrm{HC}}_m(B(q_j,\frac{r(q_j)}{1+\frac{1}{m}})\cap Y')\\
&\leq \frac{4e^2m(1+\frac{1}{m})}{A(m)^{\frac{1}{m-1}}}\sum_j\widetilde{\mathrm{HC}}_m(B(q_j,\frac{r(q_j)}{1+\frac{1}{m}})\cap Y')^{\frac{m+1}{m}}\\
&\leq \frac{4e^2(m+1)}{A(m)^{\frac{1}{m-1}}}\Big(\sum_j\widetilde{\mathrm{HC}}_m(B(q_j,\frac{r(q_j)}{1+\frac{1}{m}})\cap Y')\Big)^{\frac{m+1}{m}}\\
&\leq \frac{4e^2(m+1)}{A(m)^{\frac{1}{m-1}}}\alpha^{m+1}\widetilde{\mathrm{HC}}_m(Y')^{\frac{m+1}{m}}\\
&\leq \frac{30(m+1)}{A(m)^{\frac{1}{m-1}}}\alpha^{m+1}\,\mathrm{HC}_m(Y')^{\frac{m+1}{m}}
\end{aligned}$$

This finishes the proof of (6.8).

Proof of inequalities (6.9) and (6.10). By (6.11) and arguing as in the last step of the proof of (6.8) we obtain

$$\begin{aligned}
\sum_j \mathrm{HC}_{m-1}(\partial\tilde{B}_j\cap Y')^{\frac{m}{m-1}} &\leq \frac{4e^2m}{A(m)^{\frac{m}{m-1}}}\sum_j\widetilde{\mathrm{HC}}_m(B(q_j,\frac{r(q_j)}{1+\frac{1}{m}})\cap Y'))\\
&\leq \frac{30m}{A(m)^{\frac{m}{m-1}}}\alpha^m\,\mathrm{HC}_m(Y')
\end{aligned}$$

Using (6.4) and (6.5) we get

$$\begin{aligned}\sum_j r(q_j)\operatorname{HC}_m(\tilde{B}_j\cap Y') &\le \sum_j (1+\frac{1}{m})A(m)\widetilde{\operatorname{HC}}_m(B(q_j,\frac{r(q_j)}{1+\frac{1}{m}})\cap Y')^{\frac{1}{m}}\widetilde{\operatorname{HC}}_m(\tilde{B}_j\cap Y')\\ &\le (1+\frac{1}{m})*(1+\frac{1}{m})^{2m}A(m)\sum_j \widetilde{\operatorname{HC}}_m(B(q_j,\frac{r(q_j)}{1+\frac{1}{m}})\cap Y'))^{\frac{m+1}{m}}\\ &\le (1+\frac{1}{m})*(1+\frac{1}{m})^{2m}A(m)(\sum_j \widetilde{\operatorname{HC}}_m(B(q_j,\frac{r(q_j)}{1+\frac{1}{m}})\cap Y')))^{\frac{m+1}{m}}\\ &\le 12A(m)\alpha^{m+1}\operatorname{HC}_m(Y')^{\frac{m+1}{m}}\end{aligned}$$

6.4. **Inductive step: construction of F'.** Here's a brief description of the strategy of the proof. Map F' will be defined in a different manner on different parts of the domain X'. First we will define F' on $\bigcup_j X'\cap\partial\tilde{B}_j$ in such a way that $F'(X'\cap\partial\tilde{B}_j)$ will have controlled m-dimensional Hausdorff content. This is accomplished by applying (using our inductive assumption) Theorem 6.1 one dimension lower with $X'\cap\partial\tilde{B}_j$ playing the role of X, $Y'\cap\partial\tilde{B}_j$ playing the role of Y. and $\tilde{B}_j$ playing the role of U. The next step is to extend F' to $\bigcup_j X'\cap\tilde{B}_j$. We do this using the cone construction (Lemma 5.2) and utilizing the bound we obtained for the image of $X'\cap\partial\tilde{B}_j$ in the previous step. Finally, if we define X'' by throwing out $\bigcup_j X'\cap\tilde{B}_j$ from the domain X' and taking the closure, and then define Y'' by replacing $\bigcup_j Y'\cap\tilde{B}_j\subset Y'$ with $F'(X'\cap\partial\tilde{B}_j)$, we will show (using inequalities proved in the previous section) that the Hausdorff content of Y'' has decreased by a multiplicative constant in comparison with the Hausdorff content of Y'. This allows us to apply the inductive assumption on k, extending the definition of F' to X'' and, thus, to the whole of X.

We start by defining F' on $\bigcup_j X'\cap\partial\tilde{B}_j$. For each j we apply inductive assumption for the dimension m to sets $X_j=\partial\tilde{B}_j\cap X'$, $Y_j=\partial\tilde{B}_j\cap Y'$ and convex set $\tilde{B}_j$. We obtain a map $\tau_j: X_j\to\tilde{B}_j$ with the properties

$$\operatorname{HC}_m(\tau_j(X_j))\le I_1(m)\operatorname{HC}_{m-1}(\partial\tilde{B}_j\cap Y')^{\frac{m}{m-1}} \tag{6.13}$$

Let $\tilde{F}$ denote a map, such that $\tilde{F}(x)=x$ for $x\in Y'$ and $\tilde{F}(x)=\tau_j(x)$ for $x\in X_j$. We extend τ_j to a map $\tilde{F}: X\longrightarrow U$ using Tietze extension theorem.

Now we would like to modify $\tilde{F}$ on $\bigcup_j X'\cap\tilde{B}_j$. Let $Z_j=(Y'\cap\tilde{B}_j)\cup\tau_j(X_j)$. We apply Lemma 5.1 to define $F_j: X'\cap\tilde{B}_j\longrightarrow U$, such that $F_j=\tilde{F}$ on X_j and $F_j(x)=x$ on $Y'\cap\tilde{B}_j$. The resulting map satisfies the inequality $\operatorname{HC}_{m+1}(F_j(X'\cap\tilde{B}_j))\le emr(q_j)\operatorname{HC}_m(Z_j)$.

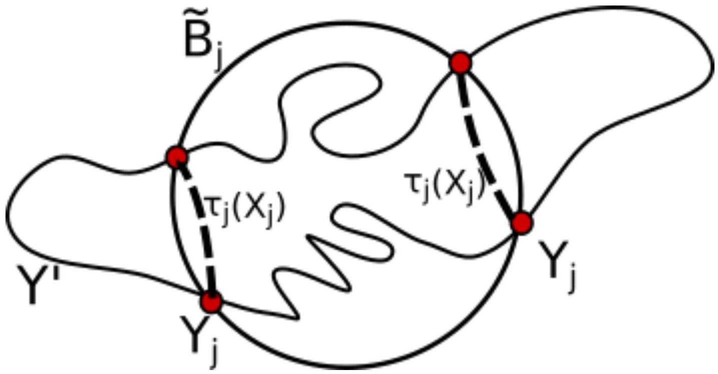

FIGURE 2

Using (6.13), (6.8) and (6.10) we estimate

(6.14)
$$\begin{aligned}
\sum_j \mathrm{HC}_{m+1}(F_j(X'\cap \tilde{B}_j)) &\le \sum_j emr(q_j)\big(\mathrm{HC}_m(\tau_j(X_j)) + \mathrm{HC}_m(Y'\cap \tilde{B}_j)\big)\\
&\le \sum_j emr(q_j)\big(I_1(m)\,\mathrm{HC}_{m-1}(\partial\tilde{B}_j\cap Y')^{\frac{m}{m-1}} + \mathrm{HC}_m(Y'\cap\tilde{B}_j)\big)\\
&\le \frac{30em(m+1)I_1(m)}{A(m)^{\frac{m}{m-1}}}\alpha^{m+1}\,\mathrm{HC}_m(Y')^{\frac{m+1}{m}} + 12emA(m)\alpha^{m+1}\,\mathrm{HC}_m(Y')^{\frac{m+1}{m}}\\
&\le [\frac{30em(m+1)I_1(m)}{A(m)^{\frac{m}{m-1}}} + 12emA(m)]\alpha^{m+1}\,\mathrm{HC}_m(Y')^{\frac{m+1}{m}}
\end{aligned}$$

Finally, we use the inductive assumption for k to modify $\tilde{F}$ on $X'' = X'\setminus\bigcup\tilde{B}_j$. We estimate the Hausdorff content of $Y'' = (\bigcup_j \tau_j(X_j))\cup(Y'\setminus\bigcup_j\tilde{B}_j)$. Combining inequalities (6.13), (6.7) and (6.9) we get

$$\begin{aligned}
\mathrm{HC}_m(Y'\setminus\bigcup_j\tilde{B}_j) + \sum_j \mathrm{HC}_m(\tau_j(X_j)) &\le (1-\alpha^m)\,\mathrm{HC}_m(Y') + \frac{30mI_1(m)}{A(m)^{\frac{m}{m-1}}}\alpha^m\,\mathrm{HC}_m(Y')\\
&\le (1-\alpha^m(1-\frac{30mI_1(m)}{A(m)^{\frac{m}{m-1}}}))\,\mathrm{HC}_m(Y')\\
&= (1-\frac{\alpha^m}{2})\,\mathrm{HC}_m(Y') \le (1-\frac{1}{2*5^m})\,\mathrm{HC}_m(Y')\\
&\le (1-\frac{1}{10^m})\,\mathrm{HC}_m(Y') < (1+\frac{1}{20^m})^k\varepsilon.
\end{aligned}$$

by our choice of $A(m)$. (Recall that $\alpha \geq \frac{1}{5}$.) It follows that we can apply the inductive assumption on k for X'', Y'' and the restriction of $\tilde{F}$ to X''.

We obtain a map $F'' : X'' \longrightarrow U$, which is an identity on Y' and satisfies

$$\mathrm{HC}_{m+1}(F''(X'')) \leq I_1'(m+1)(1-\frac{\alpha^m}{2})^{\frac{m+1}{m}} \mathrm{HC}_m(Y')^{\frac{m+1}{m}} + \varepsilon_0 \tag{6.15}$$

Observe that F'' and $\tilde{F}$ agree on Y' and $\partial X''$. We define $F'(x) = F''(x)$ for $x \in X''$ and $F'(x) = F_j(x)$ for $x \in X_j$. Combining (6.14) with (6.15) we get

$$\begin{aligned}
\mathrm{HC}_{m+1}(F'(X')) \leq \big(&(\frac{30em(m+1)I_1(m)}{A(m)^{\frac{m}{m-1}}} + 12emA(m))\alpha^{m+1} + \\
&I_1'(m+1)(1-\frac{\alpha^m}{2})^{\frac{m+1}{m}}\big) \mathrm{HC}_m(Y')^{\frac{m+1}{m}} + \varepsilon_0 \\
&\leq ((\frac{e(m+1)}{2} + 12emA(m))\alpha^m + I_1'(m+1)(1-\frac{1}{2}\alpha^m)) \mathrm{HC}_m(Y')^{\frac{m+1}{m}} + \varepsilon_0 \\
&= ((\frac{e(m+1)}{2} + 12e60^{\frac{m-1}{m}} m^{2-\frac{1}{m}} I_1(m)^{\frac{m-1}{m}})\alpha^m + \\
&I_1'(m+1)(1-\frac{1}{2}\alpha^m)) \mathrm{HC}_m(Y')^{\frac{m+1}{m}} + \varepsilon_0 \\
&\leq I_1'(m+1) \mathrm{HC}_m(Y')^{\frac{m+1}{m}},
\end{aligned}$$

providing that $I_1(m+1) = 2I'(m+1) \geq 4[\frac{e(m+1)}{2} + 12e60^{\frac{m-1}{m}} m^{2-\frac{1}{m}} I_1(m)^{\frac{m-1}{m}}] + 1$, and ε_0 is sufficiently small. (Here we were using the inequality $\alpha \leq 1$.) It is easy to see that the last inequality holds, if we take $I_1(m) = (100m)^m$.

This finishes the proof of the theorem.

7. Proof of Proposition 2.1 (and the main Theorem 1.1).

As the case $m = 1$ easily follows from the definitions, we assume that $m \geq 2$. Now the proof is similar to the proof in [Gu11], [Gu17] but with some twist. We start from the map of X to the rectangular nerve RN corresponding to the chosen good covering of X that was constructed in section 4. Then, as it was explained in section 4, we proceed by induction starting from the maximal cells of RN and going to their skeleta of lower and lower dimension. We will construct a sequence of continuous maps $\phi = \phi^{(0)} \sim \phi^{(1)} \sim ... \sim \phi^{(i)} \sim \ldots$ subordinate to the cover, where $\phi^{(i)}$ will map X to the union of $j(F)$-skeleta of of all maximal cells F of RN, where $j(F) = \max\{m-1, dimF - i\}$. Recall, that $\phi^{(i+1)}$ differs from $\phi^{(i)}$ only on the union of inverse images $(\phi^{(i)})^{-1}(F_j)$ for all maximal open faces F_j in $RN^{(i)}$ of dimension $\geq m$, and for each F_j the restriction of $\phi^{(i+1)}$ to $(\phi^{(i)})^{-1}(F_j)$ has its image in the closure of F_j (see section 4). Besides this requirement, we are going to construct

$\phi^{(k)}$, $k = 1, 2, \ldots$ so that these maps will obey the following estimate, slightly weaker than the estimate that $\phi = \phi^{(0)}$ obeys: For each face F of dimension $\geq m$ in RN

$$\mathrm{HC}_m[\phi^{(k)}(X) \cap \mathrm{Star}(F)] < 2C_1(m)\varepsilon_m \varrho(F)^m e^{-\beta d(F)}. \qquad (**)$$

Here $C_1(m) = 14^{m+1}$ and $\beta = b(m) = 49^{-m-1}\ln 2$ (see Lemma 4.1). This inequality will be needed to be able to continue constructing the subsequent maps $\phi^{(k+1)}$, $\phi^{(k+2)}$, etc. Once we are able to construct a sequence $\phi^{(i)}$ as above, we will construct $\phi^{(\infty)}$ exactly as in section 4. Then, as it was proven in section 4, we are done, as every two points of X mapped to the same point of the $(m-1)$-skeleton of RN are at the distance $\leq \frac{2}{7} < 1$.

On each step assume that $\phi^{(k)}$ was already constructed. To construct $\phi^{(k+1)}$ we consider all maximal open faces F_j of $RN^{(k)}$ of dimension $\geq m$. For each face F_j we will be altering the $\phi^{(k)}$ on the inverse image of F_j under $\phi^{(k)}$.

If $dim F_j = m$, we first observe that $\mathrm{HC}_m(\phi^{(k)}(X) \bigcap F_j) \leq \mathrm{HC}_m(F_j)$. To see this we compare the right hand side of the inequality (**) with the lower bound $(\frac{r_1}{2})^m$ for $\mathrm{HC}_m(F_j)$ provided by Corollary 1.9. This implies the existence of an open metric ball β in the complement of $\phi^{(k)}(X) \bigcap F_j$ in F_j. Consider a retraction λ of $F_j \setminus \beta$ to ∂F_j. Define $\phi^{(k+1)}$ on $(\phi^{(k)})^{-1}(F_j)$ as the composition of the restriction of $\phi^{(k)}$ to the same set and λ. It is clear that the validity of (**) is not affected.

If $i = dim F_j \geq m+1$, then the procedure will consist of two steps. On the first step we improve the image of $\phi^{(k)}$ in the considered cell using Theorem 6.1. As the result the image of the map will be very close to the boundary of the cell. Our upper bound for the m-dimensional Hausdorff content can increase but only by a factor very close to 1. On the second step we compose our map with the radial projection from the center of the ball. Again, our upper bound for the HC_m of the image can increase, but as the image is very close to the boundary, the factor will be very close to 1. Moreover, in both cases the difference between the factor and 1 exponentially decreases with the dimension of the cell. As a corollary, the product of all these factors for different values of the dimension remains uniformly bounded by an expression that does not depend on the dimension of the initial maximal open cell in RN. (Recall that we do not have a control over their dimensions.) Finally, choosing ε_m sufficiently small, we see that the product of all these factors for the different dimensions will be less than 2, as desired.

Here is the description of these two steps:

Step 1. Denote the dimension of F_j by i, and redenote F_j as F^i. Consider a system of equidistant surfaces in F^i for ∂F^i at distances ranging from 0 to $r_1 \varepsilon_m^{\frac{\gamma}{m}} \exp(-\frac{\gamma\beta}{m}\, i)$, where r_1 is the smallest side length of F^i, and $\gamma = \gamma(m) \in (0,1)$ will be chosen later. Use Lemma 5.4 to find an equidistant that we will denote ∂F^i_ϱ such that

$\mathrm{HC}_{m-1}(\partial F_\varrho \bigcap \phi^{(k)}(X) \leq 4\, C_1(m)\varepsilon_m^{\frac{m-\gamma}{m}}\, r_1^{m-1}e^{-\beta\frac{m-\gamma}{m}\ i}$. Observe that ∂F^i_ϱ is a boundary of a parallelepiped that we denote F^i_ϱ. Here ϱ denotes the distance between ∂F^i and ∂F^i_ϱ. (Recall that the considered metric on F^i is l^∞.) Denote the annular domain in F^i between ∂F^i and ∂F^i_ϱ by A^ϱ_0. Now we would like to improve the restriction of $\phi^{(k)}$ on $(\phi^{(k)})^{-1}(F^i_\varrho)$ by applying Theorem 6.1 to U defined as F^i_ϱ, X defined as $(\phi^{(k)})^{-1}(F^i_\varrho)$, f defined as the restriction of $\phi^{(k)}$ on $(\phi^{(k)})^{-1}(F^i_\varrho)$, Y defined as the inverse image of $\partial F^i_\varrho \bigcap \phi^{(k)}(X)$ under this f, the dimension n equal to i, and m in Theorem 6.1 defined as our m (so that the condition $m+1 \leq n$ in Theorem 6.1 holds).

As as result, we are going to get a new map $\bar{f}$ on X (replacing f) that coincides with our old map f (that is, $\phi^{(k)}$) on $\partial F^i_\varrho \bigcap \phi^{(k)}(X)$, has its image in the R-neighbourhood of ∂F^i_ϱ, for R defined as in the text of Theorem 6.1, and has HC_m of its image bounded by $I_1(m)(4C_1(m))^{\frac{m}{m-1}}\varepsilon_m^{\frac{m-\gamma}{m-1}} r_1^m \exp(-\beta\frac{m-\gamma}{m-1}i) \leq C_1(m)\varepsilon_m r_1^m \exp(-\beta\ i)[I_1(m)4^{\frac{m}{m-1}}C_1(m)^{\frac{1}{m-1}}\varepsilon_m^{\frac{1-\gamma}{m-1}}\exp(-\frac{(1-\gamma)\beta}{m-1}i)]$.

Observe that $R \leq r_1 I_2(m)(4C_1(m))^{\frac{1}{m-1}}\varepsilon_m^{\frac{m-\gamma}{m(m-1)}} r_1^{\frac{2}{m-1}}\exp(-\frac{\beta\ (m-\gamma)}{m(m-1)}\ \ i) \leq r_1\varepsilon^{\frac{\gamma}{m}}\exp(-\frac{\gamma\beta}{m}\ i)$, if ε_m is sufficiently small. For this purpose it is sufficient to take any $\varepsilon_m \leq (4C_1(m))^{-\frac{1}{1-\gamma}}I_2(m)^{-\frac{m-1}{1-\gamma}}$, where $C_1(m) = 14^{m+1}$. Therefore the image of $\tilde{f}$ is in $R_1 = 2r_1\varepsilon_m^{\frac{\gamma}{m}}\exp(-\frac{\gamma\beta}{m}\ i)$-neighbourhood of ∂F.

Now merge $\bar{f}$ and the restriction of $\phi^{(i)}$ to the inverse image of A^ϱ_0 under $\phi^{(i)}$ into one continuous map Φ of $(\phi^{(k)})^{-1}(F^i)$ into F^i. Its image will be in the R_1-neighbourhood of ∂F^i. Its HC_m will not exceed the HC_m of $\phi^{(k)}(X)\bigcap F^i$ plus HC_m of the image of $\bar{f}$, that is, it will not exceed $C_1(m)\varepsilon_m r_1^m \exp(-\beta\ \ i)[1 + I_1(m)4^{\frac{m}{m-1}}C_1(m)^{\frac{1}{m-1}}\varepsilon_m^{\frac{1-\gamma}{m-1}}\exp(-\frac{(1-\gamma)\beta}{m-1}\ \ i)] = C_1(m)\varepsilon_m r_1^{m+2}\exp(-\beta\ i)(1+const(m)\varepsilon_m^{\frac{1-\gamma}{m-1}}\exp(-\frac{(1-\gamma)\beta}{m-1}\ i))$, where $const(m)$ denotes $4^{\frac{m}{m-1}}C_1(m)^{\frac{1}{m-1}}I_1(m)$.

Step 2. On this step we compose the map Φ defined at the end of step 1 with the radial projection from the center of F^i to ∂F_i. The Lipschitz constant will not exceed $(1-\frac{2R_1}{r_1})^{-1} < (1-4\varepsilon_m^{\frac{\gamma}{m}}\exp(-\frac{\gamma\beta}{m}\ i))^{-1} < 1 + 8\varepsilon_m^{\frac{\gamma}{m}}\exp(-\frac{\gamma\beta}{m}\ i)$ for a sufficiently small ε_m.

The resulting map will be $\phi^{(k+1)}$ defined on $(\phi^{(k)})^{-1}(F^i)$. Note that when we pass from $\phi^{(k)}$ to $\phi^{(k+1)}$ our upper bound for the HC_m of the image increases by not more than the factor of $(1+const(m)\varepsilon_m^{\frac{1-\gamma}{m-1}}\exp(-\frac{(1-\gamma)\beta}{m-1}\ i))(1+8\varepsilon_m^{\frac{\gamma}{m}}\exp(-\frac{\gamma\beta}{m}\ i))^m$. For our idea to work, the product $\prod_{i=m}^D(1+const(m)\varepsilon_m^{\frac{1-\gamma}{m-1}}\exp(-\frac{(1-\gamma)\beta}{m-1}\ \ i))(1+$

$8\varepsilon_m^{\frac{\gamma}{m}}\exp(-\frac{\gamma\beta}{m}\ i))^m$ must converge and be less than 2. Replacing this product by the infinite product, taking the natural logarithm, and using the inequality $\ln(1+x) < x$ for all positive x, we can replace this requirement by a stronger requirement that

$$\varepsilon_m^{\frac{1-\gamma}{m-1}} const(m)\sum_{i=m}^{\infty} e^{-\frac{(1-\gamma)\beta}{m-1}\ i} + 8\varepsilon_m^{\frac{\gamma}{m}} m\sum_{i=m}^{\infty} e^{-\frac{\gamma\beta}{m}\ i} < \ln 2 \qquad (***)$$

. Clearly, the series in this expression converge to some finite $\tau(m)$, and choosing ε_m sufficiently small we can ensure that this inequality is valid.

Choose $\varepsilon_m = (c^*m)^{-m^2}$ and $\gamma = \gamma(m) = \min\{\frac{2}{3}, \frac{4}{\ln m}\}$ for some sufficiently large constant c^*. To see there exists a choice of c_*, c^* that makes the inequality above valid, it is sufficiently to consider only the case, when m is sufficiently large.

We are going to replace the inequality (***) by the system of two inequalities (that taken together are obviously stronger than (***)): We are going to require that each of the two summands in the left hand side of (***) is less than $\frac{1}{2}\ln 2$. We are going to substitute $49^{-(m+1)}\ln 2$ for β, and observe that the sums of two geometric progressions in the left hand side of (***) are very close to $\frac{m-1}{1-\gamma}49^{m+1}\log_2 e$ and, correspindingly, $\frac{m}{\gamma}49^{m+1}\log_2 e$. Replacing $\frac{1}{2}\ln 2$ by a smaller value 0.25, substituting the expression for $const(m)$ and multiplying both sides by 4 we obtain:

$$4\varepsilon_m^{\frac{1-\gamma}{m-1}}4^{\frac{m}{m-1}}14^{\frac{m+1}{m-1}}I_1(m)\frac{m-1}{1-\gamma}49^{m+1}\log_2 e \le 1,$$

and

$$32\varepsilon_m^{\frac{\gamma}{m}}m\frac{m}{\gamma}\log_2 e 49^{m+1} \le 1.$$

We consider two cases: $m < e^6$, when $\gamma = \frac{2}{3}$, and $m > e^6$, when $\gamma = \frac{4}{\ln m}$. In the first case taking the logarithm, we obtain the inequalities:

$$(1+\frac{m}{m-1})\ln 4 + \frac{m+1}{m-1}\ln 14 + m(\ln 100 + \ln m) + \ln(\log_2(e)) + \ln(m-1) + \ln(3) +$$
$$(m+1)\ln 49 - \frac{m^2}{3(m-1)}(\ln m + \ln c^*) < 0,$$

and

$$\ln(48\log_2 e) + 2\ln m + (m+1)\ln 49 - \frac{2}{3}m(\ln m + \ln c^*) < 0.$$

Solving for $\ln c^*$ and taking the maximum over the set of all integer m between 2 and $403 = \lfloor e^6 \rfloor$, we see that both inequalities are always satisfied if $c^* > e^{38}$.

Now consider the case, when $m \ge 404 = \lceil e^6 \rceil$ and $\gamma = \frac{4}{\ln m}$.

Again, taking the logarithm, and rounding up the logarithms of constants to the nearest integers we obtain somewhat stronger inequalities,

$$6.34 + \ln m + 4(m+1) + m(4.61 + \ln m) + \frac{1 - \frac{4}{\ln m}}{m-1}(-m^2)(\ln c^* + \ln m) < 0,$$

and

$$4 + 4(m+1) + 2\ln m + \frac{4}{m \ln m}(-m^2)(\ln c^* + \ln m) < 0.$$

Now the second inequality holds for $c^* \geq 2$; the first inequality holds for $c^* \geq e^{38}$. We conclude that all four inequalities hold for all $c^* > e^{38}$ and, in particular, for $c^* \geq 10^{17}$.

Finally, recall that $\varepsilon'_m = \frac{\varepsilon_m}{10*2^m}$. This completes the proof of the main theorem.

8. The proof of systolic inequalities.

Here we prove the systolic inequalities stated in subsection 1.2. We will be using the upper bounds for UW_{m-1} provided by Theorem 1.1.

The proof is modelled on the argument from [Gr] used there to deduce the inequality $sys_1(M^n) \leq c(n)vol^{\frac{1}{n}}(M^n)$. First, observe that according to [Gr], Appendix 1, Proposition (D) on p. 128, the Kuratowski embedding $f: X \longrightarrow L^\infty(X)$ is at the distance $\frac{1}{2}UW_{m-1}(X)$ from some $(m-1)$-degenerate map $g: X \longrightarrow L^\infty(M^n)$, that is, a map g which is a composition of a map g_1 of X into a $(m-1)$-dimensional polyhedron K, and a map g_2 of K into $L^\infty(X)$. Let W denote the quotient space of the cylinder $X \times [0,1]$ by the quotient map $g_1 : X \times \{1\} \longrightarrow K$. Define $F: W \longrightarrow L^\infty(X)$ as f on the "bottom" $X \times \{0\}$ of W, g (or, equivalently, g_2) on the "top", and as straight line segments connecting $f(x)$ and $g(x)$ in $L^\infty(X)$ on all "vertical" segments of W "above" $x \in X \times \{0\}$.

Exactly as in the proof of Lemma 1.2.B from [Gr] one can prove that if $sys_1(X) \geq 3UW_{m-1}(X)$, then the classifying map $Q: X \longrightarrow K(\pi_1(X), 1)$ can be extended to W by first mapping it as above to $L^\infty(X)$ and then extending the classifying map defined on $X \times \{0\} \subset W \subset L^\infty(X)$. As in the proof of Lemma 1.2.B from [Gr] one considers a very fine triangulation of W and performs the extension to 0-dimensional, then 1-dimensional, then 2-dimensional skeleta of the chosen triangulation of W. All new vertices of the triangulation are being mapped first to the nearest points of $f(X)$, and then to $K(\pi_1(X), 1)$ via the classifying map Q. All 1-dimensional simplices are first mapped to minimal geodesics between the images of their endpoints in $f(X)$, and then to $K(\pi_1(X), 1)$ using Q. Observe that the triangle inequality implies that their images in $f(X)$ have length $\leq UW_{m-1}(X) + \varepsilon$, where ε can be made arbitrarily small by choosing a sufficiently fine initial triangulation of W. We observe that an easy compactness argument implies that there exists a positive δ such that each closed curve of length $\leq 3UW_{m-1}(X)+\delta$ is still contractible. We choose ε above as $\frac{\delta}{3}$. Now the boundary of each new 2-simplex in W has been already mapped to a closed curve of length $\leq 3(UW_{m-1}(X)+\varepsilon)$ in X that is contractible in X. So, we can map the corresponding 2-simplex in W by, first, contracting the image of its boundary in $f(X)$ to a point, and then mapping the resulting 2-disc in $f(X)$ to $K(\pi_1(X),1)$ using the classifying map Q.

Finally, one argues that the extension to the skeleta of all higher dimensions is always possible as $K(\pi_1(X),1)$ is aspherical, as the corresponding obstructions live in homology groups of the pair $(W, f(X))$ with coefficients in trivial (higher) homotopy groups of the target space $K(\pi_1(X),1)$.

It remains to notice that this extension is impossible as the inclusion $X\times\{0\}\longrightarrow W$ is homotopic to $g=g_1\circ g_2$, and, therefore, induces trivial homomorphisms of all homology groups in dimensions $\geq m$. Therefore, the existence of such an inclusion would contradict the assumption that X is m-essential.

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