ALGORITHMIC ASPECTS OF IMMERSIBILITY AND EMBEDDABILITY

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Abstract. We analyze an algorithmic question about immersion theory: for which m, n, and $CAT = \mathbf{Diff}$ or $\mathbf{PL}$ is the question of whether an m-dimensional CAT-manifold is immersible in $\mathbb{R}^n$ decidable? As a corollary, we show that the smooth embeddability of an m-manifold with boundary in $\mathbb{R}^n$ is undecidable when $n-m$ is even and $11m \geq 10n+1$.

1. Introduction

The problem of classifying immersions of one smooth manifold M in another N was, in a sense, solved by Smale and Hirsch [Sma59] [Hir59], who reduced the question to one in homotopy theory. This is now viewed as an important example of the philosophy of h-principles [Gro86] [EM02]. While embedding seems to be much harder, many relevant questions have likewise been reduced to algebraic topology at least in principle, with, in our view, the signal achievements due to Whitney, Haefliger [Hae62, e.g.], and Goodwillie–Klein–Weiss [GKW03, e.g.].

Analogous work has been done, to a less complete degree, in the PL category, with an analogue of the Smale–Hirsch theorem given by Haefliger and Poénaru [HP64].

In this paper we discuss, mainly in the case $N = \mathbb{R}^n$ (or, equivalently, S^n), whether these classifications can actually be performed algorithmically given some finite data representing the pair of manifolds. This has consequences not only for computational topology but also for geometry.

In several papers, Gromov emphasized that topological existence results do not directly enable us to understand the geometric object that is supposed to exist. Indeed, the eversion of the sphere took quite a while to make explicit (and is now visible in several nice animations). A basic question is: "how complicated are embeddings or immersions, when they exist?"

In [CDMW18], an analogous problem was studied in the case of cobordism, which, since Thom, has had a similar structure. In that case, we showed that the problem has an at worst slightly superlinear solution, if it has one at all, with respect to a natural measure of the complexity of the manifold. Here we show that for immersion and embedding, in contrast, there is no computable upper bound to the complexity in immersion theory.

Corollary 1.1. *Consider smooth m-manifolds M with sectional curvature $|K| \leq 1$, injectivity radius ≥ 1, and volume $\leq V$ which immerse in S^{m+c}. By results of Cheeger and Gromov, there is a function $F_c(V)$ which bounds the norm of the second fundamental form of an immersion of such an m. On the other hand, if $c \leq m/4$ and is even, this function is not bounded by any computable function.*

Results of this sort derive from the idea of Nabutovsky [Nab95] wherein logical complexity of decision problems is reflected in lower complexity bounds for solutions of related variational problems. The difference between the above theorem and the situation in [CDMW18] is that in the case of cobordism, the relevant algebraic topology is stable homotopy theory, while for immersions the relevant problems are unstable.

1.1. **Immersion vs. embedding.** Some prior work has been done on the decidability of various questions involving embeddings. In a pair of papers from the 1990's, Nabutovsky and the second author [NW99] [NW96] considered the problem of recognizing embeddings, that is, deciding whether two embeddings of a manifold M in a manifold N are isotopic. When M and N are both simply connected, this is decidable as long as the codimension is not 2; in codimension 2, even equivalence of knots (embeddings of S^{n-2} in $\mathbb{R}^n$) for $n \geq 5$ is not decidable.

A related result asserted in [NW99] says that for closed (even simply-connected) manifolds the problem of embedding is in general undecidable, as in our paper, for reasons related to Hilbert's tenth problem. Here, we study the special case where the target is a sphere, and do not know what to expect for the case of closed manifolds embedding in the sphere.

More recent work has considered the problem of embedding simplicial complexes in $\mathbb{R}^n$. In [MTW11] it is shown that this problem is undecidable in codimensions zero and one, when $n \geq 5$; in [ČKV17], that it is decidable in the so-called *metastable range*, when the dimension of the complex is at most roughly $\frac{2}{3}n$.

Between codimension 3 and the metastable range, embedding theory is best described via the *calculus of embeddings*, due to Goodwillie, Klein and Weiss. This describes smooth embeddings of manifolds via a rather complicated homotopical construction which nevertheless can be arbitrarily closely approximated via finite descriptions (see e.g. [GKW03]); unlike in the metastable range, where work of Haefliger shows that immersion theory is essentially irrelevant, immersions form the "bottom layer" of this construction. Thus, understanding immersion theory from a computational point of view seems to be a good first step towards solving this set of problems. As we show, it also directly leads to some results regarding embeddings.

While a similar construction for PL embeddings of simplicial complexes is not currently in the literature, it seems plausible that such a construction can be developed and will be quite similar to the smooth version. We believe that many variations of the embeddability question can eventually be shown to be undecidable using this correspondence.

1.2. **Summary of old and new results.** The properties of embedding and immersion questions, including their decidability, depend heavily on the ratio between the dimensions m and n of the two objects considered. Our main result concerns the decidability of immersibility in $\mathbb{R}^n$; here we give a summary in different ranges.

The *stable range*, $m < \frac{1}{2}n + 1$: The Whitney immersion theorem states that every manifold in this range has an immersion in $\mathbb{R}^n$.

The *metastable range*, $\frac{1}{2}n + 1 \leq m < \frac{2}{3}n$: For manifolds in this range, both smooth and PL immersibility are always decidable.

$\frac{2}{3}n \leq m < \frac{4}{5}n$: In this range, PL immersibility of manifolds in $\mathbb{R}^n$ is decidable, as is smooth immersibility as long as $n - m$ is odd. We do not know whether smooth immersibility in even codimension is decidable.

$\frac{4}{5}n \leq m \leq n - 3$: In this range, PL immersibility of manifolds is decidable, whereas smooth immersibility is decidable if and only if $n - m$ is odd.

$m = n - 2$: In codimension 2, there are two notions of PL immersion: in a *locally flat* immersion, links of vertices are always unknotted in the ambient space; but one may also study PL immersions which are not necessarily locally flat. Here, smooth immersibility is undecidable at least when $n \geq 10$, as is PL locally flat immersibility, which is equivalent; PL not necessarily locally flat immersibility is decidable.

$m = n - 1$: In codimension 1, immersibility is decidable.

This parallels the overall picture for embedding theory, about which we still know much less. Note that the stable and metastable ranges are slightly different here compared to the immersion case.

The stable range, $m \leq n/2$: The Whitney embedding theorem states that every manifold in this range has an embedding in $\mathbb{R}^n$. For simplicial complexes, one needs $m < n/2$; for $n = 2m$, embeddability is obstructed by the Van Kampen obstruction.

The metastable range, $m \leq \frac{2}{3}n - 1$: Here the embeddability of simplicial complexes is decidable; this is a theorem of Čadek, Krčál and Vokřínek [ČKV17]. Moreover, PL embeddings in this range are smoothable, so smooth embeddability is decidable as well.

$\frac{2}{3}n \leq m \leq \frac{10}{11}n$: In this range, nothing is known about whether embeddability is decidable; however, see [MTW11] for some lower bounds on computational complexity. Moreover, ongoing work of Filakovský, Wagner and Zhechev on the embedding extension problem (is it possible to extend an embedding of a subcomplex to an embedding of the whole space?) suggests that the more general problem of *classifying* embeddings of simplicial complexes up to isotopy is undecidable in the vast majority of this range.

$\frac{10}{11}n < m \leq n - 2$: The state of the art on embeddability of simplicial complexes is much the same here as in the previous range. However, our results on immersions are enough to show:

Theorem 4.2. When $\frac{10}{11}n < m \leq n - 2$ and $n - m$ is even, the embeddability of a smooth m-manifold with boundary in $\mathbb{R}^n$ is undecidable.

The examples we create are always PL embeddable, however; the construction relies on the smooth structure of the manifold. Moreover, we do not know whether embeddability is decidable when restricted to closed manifolds; as discussed in 4.1, the method of Theorem 4.2 cannot work in that case.

$n = m - 1$: Here, PL embeddability is undecidable, as shown in [MTW11].

1.3. **Methods.** Questions of immersibility and embeddability are classically handled by reducing them first to pure homotopy theory and then reducing the homotopy theory to algebra. To resolve any particular instance, then, one has to do the corresponding algebraic computation. To decide whether the answers can be obtained algorithmically, one has to (1) find an algorithm to perform the reduction and (2) determine whether the resulting algebra problem is decidable.

The homotopy-theoretic side of these questions is fairly well-studied. Novikov showed in 1955 that it is undecidable whether a given finite presentation yields the trivial group; in particular, this means that whether a given simplicial complex is simply connected is undecidable. This was extended by Adian to show that many other properties of groups are likewise undecidable. Soon after, Brown [Bro57] showed, by way of contrast, that the higher homotopy groups of a simply connected space are computable.

Much more recently, Čadek, Krčál, Matoušek, Vokřínek and Wagner [ČKM+14a] showed that the set of homotopy classes $[X, Y]$ is in general uncomputable, even when Y is a simply connected space. This is because the problem of determining which rational invariants can be attained is tantamount to resolving a system of diophantine equations; this is the famously undecidable Hilbert's tenth problem.

It seems as if fundamental group issues and Hilbert's tenth problem are the only obstructions to computability in homotopy theory. The same group of authors, along with Filakovský, Franek, and Zhechev, have authored a number of papers [ČKM+14b] [FV13] [Vok17] [ČKV17] [FFWZ17] describing algorithms for various problems in homotopy theory that do not encounter these. While some open questions do remain, all of the homotopy-theoretic problems encountered in this paper can easily be reduced to ones covered by their results.

The main issue, then, is that of the reduction. The h-principles of Hirsch–Smale [Hir59] and Haefliger–Poénaru [HP64], respectively, show that immersions of codimension k in the smooth and PL categories are classified via lifts of the stable tangent bundle to the classifying spaces $BO(k)$ and $BPL(k)$. While $BO(k)$ can easily be approximated by a Grassmannian of k-planes in a high-dimensional Euclidean space, and therefore classifying maps are also not difficult to compute, $BPL(k)$ is more recalcitrant. While it is known to be of finite type, that is, homotopy equivalent to a complex with finite skeleta, this equivalence is inexplicit and it is not clear how to algorithmically reduce the tangential data of a PL manifold to a finite amount of data. In this paper, we employ various workarounds; the question of understanding BPL more directly remains open and is also relevant to the quantitative topology of PL manifolds.

1.4. **Complexity.** Our algorithms do not give any information about the complexity of the computations. In many cases, we perform a construction by iterating through all objects of a given form until we find the needed one; this uses the fact that its existence is known and that it is algorithmically recognizable. However, often such an object only exists when the input is a manifold; this means that the algorithm will not terminate if presented with an invalid input (for example, a simplicial complex all of whose links are homology spheres, but not spheres.)

We believe that this issue can be circumvented and that these algorithms can be made much more efficient, but this is beyond the scope of this paper.

1.5. **Acknowledgements.** The authors would like to thank the Israel Institute for Advanced Studies for a stay in fall 2017 during which this work was conceived.

2. Effective representation

In this section, we discuss algorithms and data structures to represent certain objects which have not been worked out in detail in the past.

2.1. **Smooth manifolds.** There are several possible ways of representing smooth manifolds computationally; as far as we know, this topic has not been thoroughly explored. According to the Nash–Tognoli theorem, every smooth n-manifold embedded in $\mathbb{R}^{2n+1}$ is closely approximated by a smooth real algebraic variety. This is one way of specifying smooth manifolds, however it is not clear whether it can be computed from other possible representations.

For our purposes, smooth n-manifolds will be specified via C^1 triangulations. That is, we take a triangulation and specify a polynomial map of each top-dimensional simplex to $\mathbb{R}^N$, for some N, such that the tangent spaces coincide where simplices meet. For a sufficiently fine triangulation, the polynomials can be taken to be of degree depending only on n.

This structure gives a way to easily read off the classifying map to $\mathrm{Gr}_n(\mathbb{R}^N)$ of the tangent bundle. This lets us compute the Pontryagin classes as pullbacks of cohomology classes of the Grassmannian, obtaining simplicial representatives by integrating over each simplex.

We note that there are a number of other ways of specifying a smooth manifold via a combinatorial structure. We list some of these here; the extent to which they can be transformed into each other requires further study.

A more general way of specifying smooth (that is, C^1) n-dimensional submanifolds of $\mathbb{R}^N$ is by patching them together from smooth real semialgebraic sets, with a consistent derivative along the boundaries of the patches. This includes the case of a single smooth variety; a triangulated smooth manifold with semialgebraic simplices; and a handle decomposition with semialgebraic handles.

One can dispense with the explicit embedding by taking a triangulated manifold and assigning an element of $\mathrm{Gr}_n(\mathbb{R}^N)$ to each vertex. If the triangulation is sufficiently fine, we can send adjacent vertices close enough to each other (at most some constant distance depending on n and N) that one can interpolate linearly over the simplices, uniquely determining a smooth structure on the manifold. One must ensure, of course, that this structure is compatible with the PL structure.

Finally, one can specify a manifold via an atlas of coordinate patches and transition functions; for example one may require the patches to be real semialgebraic and the transition functions to be rational functions (in one direction).

2.2. **Classifying spaces for spherical fibrations.** Manifold topology makes use of a variety of classifying spaces, the most familiar of which are the classifying spaces BO_n and BSO_n for unoriented and oriented vector bundles. These have relatively straightforward models as Grassmannians of n-planes in $\mathbb{R}^\infty$. The classifying spaces BPL_n for PL structures are much more complicated to model combinatorially; while some work in this direction has been done by Mnëv [Mnë07], in this paper we do not attempt to make BPL_n and classifying maps to it concrete enough to manipulate algorithmically. Instead, we use some well-known computations to avoid talking about BPL_n at all and focus instead on BG_n, the classifying space for the much weaker structure of S^{n-1}-fibrations.

Transition functions between fibers in an S^{n-1}-fibration are chosen from the topological monoid G_n of homotopy automorphisms of S^{n-1}; this fits into a fiber sequence

$$\Omega^{n-1}S^{n-1} \to G_n \to S^{n-1},$$

where the latter map is induced by evaluation at the basepoint. This monoid has a classifying space BG_n; as $n \to \infty$, this converges to a stable object BG. In order to compute with BG_n and BG, we need to construct finite models for its skeleta as well as the tautological bundles over them.

Lemma 2.1. *(i) There is an algorithm which, given natural numbers m and n, constructs a finite simplicial set $B_{m,n}$ with an m-connected map to BG_n, together with any stage of the relative Postnikov tower of the pullback to $B_{m,n}$ of the tautological bundle over BG_n.*

(ii) There is an algorithm that constructs the map $B_{m,n} \to B_{m,m+1}$ induced by suspension. Note that this models the stabilization $BG_n \to BG$, since the map $BG_{m+1} \to BG$ is m-connected.

(iii) There is an algorithm which, given a stable range PL embedding $M^m \to \mathbb{R}^{2m+k}$, $k \geq 1$, constructs the classifying map $M \to B_{m,m+k}$ of the normal bundle.

We note that the model we compute is extremely inexplicit in how it classifies fibrations.

Proof. We will need to cite the existence of a number of algorithms in computational homotopy. These are:

- The isomorphism type of homotopy groups of spheres can be computed [Bro57].
- Moreover, for any element of $\pi_k(S^n)$, an explicit simplicial representative may be computed [FFWZ17].
- Given a map $Y \to B$ between simply connected spaces, its relative (or Moore–)Postnikov tower may be computed to any finite stage [ČKV17].
- Given two maps $X \to S^n$, for any simplicial complex X, there is an algorithm to determine whether they are homotopic [FV13].
- Moreover, given a map $X \to Y$ known to be nullhomotopic, we can compute an explicit nullhomotopy. This can be done through an exhaustive search for maps from increasingly fine subdivisions of CX.

We start by outlining the algorithm for (i). We first show that we can compute $\pi_k(G_n)$, given n and k. From the aforementioned fiber sequence

$$\Omega^{n-1}S^{n-1} \xrightarrow{i} G_n \xrightarrow{j} S^{n-1},$$

we obtain the homotopy exact sequence

$$\pi_{n+k-1}(S^{n-1}) \xrightarrow{i_*} \pi_k(G_n) \xrightarrow{j_*} \pi_k(S^{n-1}) \xrightarrow{\phi_k} \pi_{n+k-2}(S^{n-1})$$

Thus it is enough to perform the following steps:

(1) Compute representatives of generators for $\pi_k(S^{n-1})$ and $\pi_{k+1}(S^{n-1})$.
(2) Compute the obstruction theoretic map $\phi_k : \pi_k(S^{n-1}) \to \pi_{n+k-2}(S^{n-1})$. Given a map $f : S^k \to S^{n-1}$, $\phi_k(f)$ is the obstruction to extending the map

$$f \vee \mathrm{id} : S^k \vee S^{n-1} \to S^{n-1}$$

to $S^k \times S^{n-1}$. That is, $\phi_k(f)$ is the *Whitehead product* $[f, \mathrm{id}_{S^{n-1}}]$, the composition

$$S^{n-k-2} \to S^k \vee S^{n-1} \to S^{n-1}$$

where the first map is homotopic to the attaching map of the top cell of $S^k \times S^{n-1}$. From this map, we compute its homotopy class as an element of $\pi_{n-k-2}(S^{n-1})$; doing this for a representative of each generator gives a finite description of the map ϕ_k.

(3) Now $\pi_k(G_n)$ is generated by lifts of $\ker \phi_k$ and the image of $\pi_{n-k-1}(S^{n-1})$. We compute, via exhaustive search, a homotopy lift of each generator of $\ker \phi_k$ to a map $S^k \times S^{n-1} \to S^{n-1}$; the generators of $\pi_{n-k-1}(S^{n-1})$ determine maps $S^k \times S^{n-1} \to S^{n-1}$ by precomposing with the map collapsing $S^k \vee S^{n-1}$. We then determine the isomorphism type of $\pi_k(G_n)$ by computing all relations.

When both groups are finite, this is a finite computation. There remain the following cases:

$k = n-2$, n odd: Then $\pi_{k+n-1}(S^{n-1}) \cong \mathbb{Z}$, but the map

$$\phi_{k+1} : \pi_{k+1}(S^{n-1}) \to \pi_{n+k-1}(S^{n-1})$$

is either surjective or has cokernel $\mathbb{Z}/2\mathbb{Z}$, depending on the resolution of the Hopf invariant one problem in that dimension, since $[\mathrm{id}, \mathrm{id}]$ always has Hopf invariant two. Thus $\mathrm{im}\, i_*$ is finite; this can be hardcoded into the computation.

$k = n-1$, n even: In this case $\ker \phi_k = \mathbb{Z}$, whereas $\pi_{k+n-1}(S^{n-1})$ is finite. Therefore

$$\pi_k(G_n) = \mathbb{Z} \oplus \mathrm{im}(i_*),$$

and we only need to compute the relations within the image of i_*.

$k = 2n-3$, n odd: In this case $\ker \phi_k = \mathbb{Z}$ and $\pi_{k+n-1}(S^{n-1})$ is again finite, making this case the same as the previous one.

In addition, the case $k = 0$ must be coded separately: G_n always has two components.

Now we use the fact that $\pi_k(G_n) = \pi_{k+1}(BG_n)$ to build successive approximations $B_{i,n}$ of BG_n, together with the pullbacks $p_i : E_{i,n} \to B_{i,n}$ of the tautological S^{n-1}-fibration.

Given a map $f : S^k \times S^{n-1} \to S^{n-1}$ representing an element $\alpha \in \pi_k(G_n)$ (in particular with $f|_{* \times S^{n-1}} = \mathrm{id}$) we define the space

$$E_f = D^{k+1} \times S^{n-1} / \{(x, y) \sim (*, f(x, y)) : x \in \partial D^{k+1}, y \in S^{n-1}\};$$

then the projection $E_f \to S^{k+1}$ onto the first factor has homotopy fiber S^{n-1}, and α is the obstruction to constructing a fiberwise homotopy equivalence $S^{n-1} \times S^{k+1} \to E_f$. We use the E_f's as building blocks for our construction.

We set $p_1 : E_{1,n} \to B_{1,n}$ to be the map

$$\bigvee_{[f] \text{ generating } \pi_0(G_n)} E_f \to \bigvee S^1.$$

Now suppose we have constructed $p_i : E_{i,n} \to B_{i,n}$ which is the homotopy pullback of the tautological bundle over BG_n along an i-connected map. Then we construct p_{i+1} using the following algorithm. Here the CW structure can be given via simplicial maps from subdivided simplices corresponding to each cell.

(1) First, we compute the kernel of the map $\pi_i(B_{i,n}) \to \pi_i(BG_n)$; since $E_{i,n}$ is the pullback of the tautological bundle, this means determining which elements of $\pi_i(B_{i,n})$ pull $E_{i,n}$ back to a trivial fibration. We do this in two stages: first, compute the kernel of the obstruction map $\pi_i(B_{i,n}) \to \pi_{i-1}(S^{n-1})$ to lifting a given map $S^i \to B_{i,n}$ to $E_{i,n}$; then, compute the kernel of the obstruction in $\pi_{n+i-3}(S^{n-1})$ to extending this to a map $S^i \times S^{n-1} \to E_{i,n}$ whose restriction to $* \times S^{n-1}$ is the homotopy fiber.

(2) Now given a generating set for this kernel, we glue in an $(i+1)$-cell for each generator to $B_{i,n}$ and a corresponding copy of $D^{i+1} \times S^{n-1}$ to $E_{i,n}$. This ensures that the map $\pi_i(B_{i,n}) \to \pi_i(BG_n)$ is an isomorphism.

(3) Finally, we wedge on $E_f \to B_f$ for a set of functions f which generate $\pi_i(G_n)$. This ensures that $B_{i+1,n} \to BG_n$ is an $(i+1)$-connected map.

Finally, we construct the relative Postnikov tower of the map $p_m : E_{m,n} \to B_{m,n}$. This completes the proof of (i).

For both (ii) and (iii), we will need a subroutine which, given a map $f : E \to B$ whose homotopy fiber is S^{n-1} and such that B is m-dimensional, computes the classifying map to $B_{m,m+1}$. We note that since the homotopy groups of BG are finite, so are the homotopy groups of $B_{m,m+1}$ through dimension m. Therefore there are a finite number of homotopy classes of maps $B \to B_{m,m+1}$, which can be enumerated via obstruction theory; we choose the one for which f is the homotopy pullback of p_m, which can be verified by induction on the relative Postnikov tower.

For (ii), we can construct the map f by repeatedly taking the double mapping cone of p_m.

For (iii), given a PL embedding of M, we need to compute the Spivak normal S^{m+k-1}-fibration $E \to M$ before we compute its classifying map. It is enough to find the following data:

- a compact PL $(2m+k)$-manifold with boundary $N(M)$ embedded in $\mathbb{R}^{2m+k}$ which contains a subdivision of M in its interior;
- a strong deformation retraction of $N(M)$ to M.

Then the induced map $\partial N \to M$ is the Spivak normal fibration. Since these properties are checkable and we can iterate through all subdivisions of M, simplicial complexes in $\mathbb{R}^{2m+k}$ with rational vertices, and simplicial maps from a subdivision of $N(M) \times I$ to M, we can find this data via exhaustive search. □

3. Immersibility

Theorem 3.1. *Let $n \geq 4$ and $m < n$ be natural numbers.*

(i) Whenever $n - m$ is odd or $3m \leq 2n - 1$, the immersibility of a smooth m-manifold with boundary (given as a semialgebraic set in some $\mathbb{R}^N$) in $\mathbb{R}^n$ is algorithmically decidable.

(ii) Whenever $n - m$ is even and $5m \geq 4n$, the immersibility of a smooth m-manifold in $\mathbb{R}^n$ is undecidable (including if only closed manifolds are considered.)

(iii) Whenever $n - m \neq 2$, the immersibility of a PL m-manifold with boundary in $\mathbb{R}^n$ is decidable.

(iv) When $n - m = 2$, it is undecidable (at least for $n \geq 10$) whether a PL m-manifold has a locally flat immersion in $\mathbb{R}^n$, but there is an algorithm to decide whether it has a not necessarily locally flat immersion.

Moreover, over the cases for which an algorithm exists, it can be made uniform with respect to m and n.

Note that for certain pairs with $n - m$ even, we have not determined whether immersibility is decidable. We suspect that it is in fact undecidable in those cases, since the corresponding homotopy-theoretic problem is undecidable.

Proof. We assume at first that the manifold is oriented, to avoid fundamental group issues.

In each case, the problem of immersibility can be reduced to a homotopy lifting problem: these are the h-principles of Smale–Hirsch [Hir59] in the smooth case and Haefliger–Poenaru [HP64] in the PL case. Both of these results state that the space of immersions $M \to N$ in the appropriate category is homotopy equivalent to the space of tangent bundle monomorphisms $TM \to TN$, or simply $TM \to \mathbb{R}^n$ when $N = \mathbb{R}^n$. These in turn can be thought of as lifts of the classifying map of the tangent bundle to the Grassmannian of m-planes in $\mathbb{R}^n$.

The smooth case. We have reduced immersibility to the homotopy lifting problem

$$\begin{tikzcd} & \mathrm{Gr}_m(\mathbb{R}^n) \arrow[d] \\ M \arrow[r, "\kappa"] \arrow[ur, dashed] & BSO(m) \end{tikzcd}$$

where κ is the classifying map of the tangent bundle of M. Moreover, this lifting property is *stable*: such a lift exists if and only if the corresponding lift

$$\begin{tikzcd} & \mathrm{Gr}_m(\mathbb{R}^n) \arrow[r] \arrow[d] & \mathrm{Gr}_{m+1}(\mathbb{R}^{n+1}) \arrow[d] \\ M \arrow[r, "\kappa"'] \arrow[urr, dashed] & BSO(m) \arrow[r] & BSO(m+1) \end{tikzcd}$$

exists. This is because the map between the corresponding homotopy fibers

$$V_m(\mathbb{R}^n) \to V_{m+1}(\mathbb{R}^{n+1})$$

is $(n-1)$-connected. Therefore, when $m < n$, it suffices to resolve the lifting problem

$$\begin{tikzcd} & BSO(n-m) \arrow[d] \\ M \arrow[r, "\kappa"] \arrow[ur, dashed] & BSO. \end{tikzcd}$$

Since $BSO(n-m)$ and BSO are both formal spaces, to understand this map rationally it is enough to consider their rational cohomology. $H^*(BSO)$ is a free algebra generated by the Pontryagin classes in degree $4k$ for every k. $H^*(BSO(n-m))$ is generated by Pontryagin classes in degree $4k$ where $2k \le n-m$; when $n-m$ is even, there is also an Euler class in degree $n-m$ whose square is the top Pontryagin class.

The smooth case in odd codimension. In the case when $n-m$ is odd, the immersibility of M in $\mathbb{R}^n$ can be determined via the following algorithm. Let M be given to us by a C^1 triangulation. Then:

(1) Test whether the ith Pontryagin classes of M are zero, for $2(n-m) < 4i \le m$, by pulling back the relevant cohomology classes along f. If any of them are not, then M cannot be immersed in $\mathbb{R}^n$. Otherwise, we have a unique lift $\hat{f} : X \to \hat{B}$ to a space $\hat{B}$ in between $\mathrm{Gr}_m(\mathbb{R}^n)$ and B; moreover, the map $\mathrm{Gr}_m(\mathbb{R}^n) \to \hat{B}$ is rationally m-connected.

(2) Now we can determine the existence of a lift of $\hat{f}$ to $\mathrm{Gr}_m(\mathbb{R}^n)$ via obstruction theory, by testing all possibilities. Since the homotopy groups of the fiber are finite, so is the universe of potential lifts.

The smooth case in the metastable range. When $2n \ge 3m+1$, all Pontryagin classes in the relevant range are zero. However, when $n-m$ is even, there may be a nonzero Euler class in degree $n-m$, whose square is always zero. This is the only infinite-order homotopy group of the fiber of the map $BSO(n-m) \to BSO$ below dimension n. Moreover, this map is $(n-m)$-connected. To show that the resulting lifting problem is decidable, we can use the results of Vokřínek [Vok17], who shows that a lifting problem through a k-connected fiber is decidable if the only infinite-dimensional homotopy groups of this fiber are dimensions $< 2k$.

The smooth case in even codimension. Now suppose that $n-m$ is even. Then the lifting problem

$$\begin{tikzcd} & BSO(c) \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "f"] & BSO \end{tikzcd}$$

is undecidable for general $2c$-complexes X and maps $f : X \to BSO$. We prove this by a method used in [ČKM$^+$14a], reducing an undecidable algebraic problem to a question about lifts. This is a special case of Hilbert's 10th problem: determining the existence of an integer solution to a system of equations each of the form

$$\sum_{1 \le i < j \le r} a_{ij}^{(k)} x_i x_j = b_k,$$

where $x_1, \ldots, x_r$ are variables and b_k and $a_{ij}^{(k)}$ are coefficients. Given such a system of s equations, we can form a CW complex X as follows. We take the wedge of r copies of S^c and attach s $2c$-cells the kth one via an attaching map whose homotopy class is

$$\sum_{1 \le i < j \le r} a_{ij}^{(k)} [\mathrm{id}_i, \mathrm{id}_j],$$

where id_i is the inclusion map of the ith c-cell. We fix a map $f : X \to BSO$ by taking the c-cells to the basepoint and the kth $2c$-cell to b_k times the generator of $\pi_{2c}(BSO)$ dual to the Pontryagin class in that degree. Then choosing a lift to $BSO(c)$ means determining an Euler class in $H^c(X)$ whose square is the top Pontryagin class; that is, choosing an assignment of the variables x_i so that the system is satisfied.

It remains to show that one can construct a manifold whose homotopy type and Pontryagin classes determine any such system. This can be done, at the cost of some increase in dimension; our examples are of dimension $4c$, which is probably not optimal.

Such manifolds exist by an argument of Wall [Wal66, §5], who shows that for any $2c$-complex X and map $f : X \to BSO$, there is a corresponding $(4c+1)$-dimensional *thickening* of X, i.e. a manifold with boundary M homotopy equivalent to X such that the classifying map of its tangent bundle is homotopic to f. Moreover, the pair $(M, \partial M)$ is $2c$-connected and any extra topology of ∂M is sent to zero by the classifying map. Thus ∂M is a closed $4c$-manifold which immerses in $\mathbb{R}^{5c}$ if and only if the system of equations above has a solution.

Now suppose there were an algorithm to decide smooth immersibility of $4c$-manifolds in $\mathbb{R}^{5c}$ for some fixed even c. Then given a system of equations, we could iterate over smooth closed $4c$-manifolds M and bases for $H^c(M)$ until we find one with the right cohomology algebra and classifying map. This search terminates since Wall guarantees the existence of such a manifold. Then we could decide whether the system has a solution using our solution to the immersibility problem. Thus immersibility cannot be decidable.

The PL case in codimension ≥ 3. Here the unstable lifting problem reduces to the stable problem

$$\begin{tikzcd} & \widetilde{BPL}(n-m) \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "\kappa"] & \widetilde{BPL}. \end{tikzcd}$$

Moreover, when $n-m \geq 3$, the diagram

$$\begin{tikzcd} BPL(n-m) \arrow[r] \arrow[d] & BPL \arrow[d] \\ BG(n-m) \arrow[r] & BG, \end{tikzcd}$$

where BG is the classifying space of spherical fibrations, is a homotopy pullback square [Wal99, p. 123]. Thus, equivalently, we must solve the lifting problem

$$\begin{tikzcd} & \widetilde{BG}(n-m) \arrow[d] \\ M \arrow[r, "\kappa"] \arrow[ur, dashed] & \widetilde{BG}. \end{tikzcd}$$

The argument in §2 shows that the only infinite homotopy group of $G(n-m)$ is

$$\begin{cases} \pi_{n-m-1} & \text{when } n-m \text{ is even} \\ \pi_{2(n-m)-3} & \text{when } n-m \text{ is odd,} \end{cases}$$

and therefore $BG(n-m)$ only has an infinite homotopy group in dimension $n-m$ or $2(n-m)-2$. Moreover, in both cases the map $\widetilde{BG}(n-m) \to \widetilde{BG}$ is $(n-m-2)$-connected, by the stability of homotopy groups of spheres.

Thus to decide immersibility we can use the following algorithm. Suppose M is given to us as a simplicial complex.

(1) We embed the simplicial complex linearly in $\mathbb{R}^N$, for some large N.
(2) This gives us a map $M \to \widetilde{BG}(N)$ which can be computed by Lemma 2.1(iii).
(3) Decide whether the map lifts to a map $M \to \widetilde{BG}(n-m)$. In the even case, this can be done by the aforementioned work of Vokřínek [Vok17], since the only infinite obstruction is below twice the connectivity of the map $\widetilde{BG}(n-m) \to \widetilde{BG}$. In the odd-dimensional case, we can split the work into two steps:
   - Compute all possible lifts to the $(2(n-m)-3)$rd stage of the relative Postnikov tower of $\widetilde{BG}(n-m) \to \widetilde{BG}$. This can be done since all the obstructions are finite.
   - For each lift computed, use the algorithm of Vokřínek to decide whether it can be extended to $\widetilde{BG}(n-m)$.

PL immersions in codimension 2. In codimension 2, there are two somewhat different things we may mean by PL immersion: locally flat immersion, in which the link of every vertex is unknotted, and immersion which is not necessarily locally flat.

A PL manifold M has a locally flat immersion in codimension 2 if and only if it has a smoothing which immerses smoothly in codimension 2. This is because by the fundamental theorem of smoothing theory [HM74, Part II], M is smoothable if and only if the classifying map $M \to \widetilde{BPL}$ of the stable tangent bundle lifts to BSO; but immersibility is equivalent to the existence of a further lift

$$\begin{tikzcd} & BSO(2) \cong BPL(2) \arrow[d] \\ & BSO \arrow[d] \\ M \arrow[r] \arrow[uur, dashed] & \widetilde{BPL}. \end{tikzcd}$$

Moreover, the homotopy fiber PL/O has finite homotopy groups, so the rational obstructions discussed above are the same in the PL case as in the smooth case. Therefore, this problem is undecidable for $\dim M \geq 8$ by the same argument as above: the examples we produced are PL immersible if and only if they are smoothly immersible.

The case of immersions which are not necessarily locally flat was studied by Cappell and Shaneson [CS76] [CS73]. Such immersions are classified by maps to a space $BSRN_2$. Unlike in the higher codimension case, the diagram

$$\begin{tikzcd} BSRN_2 \arrow[r] \arrow[d] & \widetilde{BG_2} \arrow[d] \\ \widetilde{BPL} \arrow[r] & \widetilde{BG} \end{tikzcd}$$

is not a homotopy pullback square, but the map from $BSRN_2$ to the pullback splits up to homotopy. Therefore it is again sufficient to solve the lifting problem from $\widetilde{BG}$ to $\widetilde{BG_2}$.

Codimension 1. In codimension one, the lifting problem above, and therefore the question of smooth immersibility, boils down to whether the suspension of the tangent bundle is trivial, that is, whether the composition

$$M \to BSO(m) \to BSO(m+1)$$

is nullhomotopic. Once this composition is given as an explicit map, we use the fact that whether two explicit maps between finite complexes are homotopic is a decidable question, a theorem of [FV13].

The oriented PL case is formally identical: one needs to determine whether the map $M \to \widetilde{BPL}(m+1)$ induced by the tangent bundle, or equivalently the map $M \to \widetilde{BPL}$, is trivial. However, up until now we have gotten away with only studying maps to $\widetilde{BG}$, and we have neither an explicit finite-type model for $\widetilde{BPL}$ nor a way of constructing the map. One way of getting around this would be to first determine whether the map to $\widetilde{BG}$ is trivial; if it is, then there is an induced map to G/PL which must also be trivial. To determine its triviality, we would need to compute the Pontryagin and Kervaire classes of M from its combinatorial structure. Such an algorithm was produced for the rational Pontryagin classes by Gelfand and MacPherson [GM92] (see also the survey paper [Gai05]), but the integral version remains open.

The path we take uses smoothing theory. As in the codimension 2 case, if the classifying map $M \to \widetilde{BPL}$ is trivial, M admits a smoothing which immerses smoothly in $\mathbb{R}^{m+1}$. Thus it is enough to construct all possible smoothings of M (finitely many, and perhaps none); then we can use the smooth algorithm to determine whether one of them immerses. This construction is given in §5.

Non-orientable manifolds. In this case constructing an immersion is equivalent to constructing a $\mathbb{Z}/2\mathbb{Z}$-equivariant immersion of the oriented double cover. In other words, we must do what we did above but in a way that respects the natural free $\mathbb{Z}/2\mathbb{Z}$-action on each of the classifying spaces. This action is easy to encode computationally; moreover, as pointed out by Vokřínek [Vok17, §5] and elaborated in [ČKV17], the relevant homotopy theory can be done as easily as in the non-equivariant case. □

4. Applications to embeddings

4.1. **Immersions which extend to embeddings.** The following is a well-known fact, noted for example in [Mas59].

Lemma 4.1. *The normal bundle to an embedded smooth closed oriented submanifold $M^m \subseteq \mathbb{R}^n$ always has vanishing Euler class.*

Proof. Consider the diagram

$$\begin{array}{ccccc} H^{n-m}(\mathbb{R}^n, \mathbb{R}^n \setminus M) & \longrightarrow & H^{n-m}(\mathbb{R}^n) & & \\ \downarrow & & \downarrow & & \\ H^{n-m}(\nu_M, \nu_M \setminus M) & \xrightarrow{(*)} & H^{n-m}(\nu_M) & \longrightarrow & H^{n-m}(M). \end{array}$$

The Euler class is the image of the generator of $H^{n-m}(\nu_M, \nu_M \setminus M)$ along the bottom row. The left vertical arrow is an isomorphism by excision. Since $H^{n-m}(\mathbb{R}^n) = 0$, the arrow labeled $(*)$ is zero. □

This means that if M is closed and oriented, an immersion of M can only be regularly homotopic to an embedding if it has zero Euler class. Unlike the existence of an immersion in general, the existence of such an immersion is decidable via the same algorithm as in odd codimension: test whether all Pontryagin classes in degrees $2(n-m) \leq 4i \leq 2m$ are zero, and then resolve the remaining finite-order questions.

In other words, while it may well be that the embeddability of closed smooth manifolds in $\mathbb{R}^n$ is undecidable outside the metastable range, this cannot be a result of immersion theory.

4.2. **Embeddability is undecidable.**

Theorem 4.2. *Whenever $n - m$ is even and $11m \geq 10n + 1$, the embeddability of a smooth m-manifold with boundary in $\mathbb{R}^n$ is undecidable.*

We note that the method used here depends both on using the smooth category and on allowing the manifold to have boundary.

Proof. We reduce this statement to Theorem 3.1(ii). We note first that by the stability property discussed above, when $n \geq m+2$, an m-manifold M immerses smoothly in $\mathbb{R}^n$ if and only if $M \times D^k$ immerses smoothly in $\mathbb{R}^{n+k}$.

In general position, the self-intersection of an immersion $f : M \to N$ is a $(2m-n)$-dimensional CW complex. If we stabilize by crossing with $\mathbb{R}^k$ for $k \geq 4m - 2n + 1$, then this complex always has an embedding in $\mathbb{R}^k$; therefore the immersion

$$f \times \mathrm{id} : M \times D^k \to \mathbb{R}^{n+k}$$

can be deformed to an embedding, by pushing a neighborhood of the self-intersection off itself in the $\mathbb{R}^k$ direction. Conversely, if M does not immerse in $\mathbb{R}^n$, then $M \times D^k$ does not embed in $\mathbb{R}^{n+k}$.

If $m = 4c$ and $n = 5c$, then we can choose $k = 6c + 1$. In other words, it is undecidable whether a $(10c+1)$-manifold embeds into $\mathbb{R}^{11c+1}$ when c is even. □

5. Computing all smoothings of a PL manifold

In this section we sketch an algorithm which, given a triangulation of a PL manifold M^m, computes a set of smoothings which contains at least one (but usually many) representatives of each diffeomorphism type of smoothing. Of course, if M is not smoothable, the algorithm yields the empty set.

The basic algorithm is as follows. We start by finding a subdivision M' of M so that for each simplex σ of M, there is a subcomplex V_σ of M' which is a "thickened version" of σ, that is:

- $\bigcup_{\tau \in \sigma} V_\tau$ contains a neighborhood of σ.
- All the V_σ are homeomorphic to closed m-balls, have disjoint interiors, and deformation retract to $V_\sigma \cap \sigma$.

This can be done by applying the same symmetric subdivision to every k-simplex. Furthermore, we embed M' in some $\mathbb{R}^N$, $N \geq 2m+1$.

We then inductively construct all possible smoothings first away from the $(m-1)$-skeleton of M (that is, on those V_σ corresponding to m-simplices σ of M), then away from the $(m-2)$-skeleton, and so on. At each step each subsequent representative will be encoded via a smooth map from a further subdivision to $\mathbb{R}^N$. Once we have filled in the neighborhoods of vertices, we have generated representatives of all possible smoothings.

Write $V_k = \bigcup_{\dim \sigma > m-k} V_\sigma$ for the part of M' away from the $(m-k)$-skeleton of M. Suppose we have defined a particular smoothing on V_k. Then the kth step of the induction proceeds as follows, for every $(m-k)$-simplex σ of M:

(1) Determine whether the map on $V_k \cap V_\sigma$ is diffeomorphic to the standard cylinder $S^{k-1} \times D^{m-k}$. If it isn't, then the smoothing does not extend.
(2) If the smoothing extends, we extend it by iterating over all possible piecewise polynomial smooth maps until we find one that works.
(3) Finally, for every exotic k-sphere, we glue in a $D^k \times D^{m-k}$ which modifies the smoothing on V_σ by that k-sphere. This entails a further subdivision of V_σ.

It remains to describe algorithms for constructing and classifying exotic spheres. In every dimension k, the exotic spheres are classified by a finite abelian group Θ_k whose group operation is connect sum. To perform steps (1) and (3), it would be enough to have an algorithm which, given a smooth manifold PL homeomorphic to the sphere, computes the corresponding element of Θ_k. By the stability of smoothing theory, for step (1), we must simply test whether the element of Θ_k corresponding to some arbitrarily chosen S^{k-1} fiber is zero. For step (3), we can generate all the necessary exotic disks by iterating over all possible piecewise polynomial smooth maps from subdivisions of $\partial\Delta^{k+1}$ to $\mathbb{R}^{2k+1}$ until we find representatives for every element of Θ_k. Then we can get the desired disks by cutting out a k-simplex from each of these.

In fact, we find something slightly weaker. To analyze the group Θ_k, we look at the original paper of Kervaire and Milnor [KM63] where it is defined, as well as an expository paper of Levine [Lev85] which fills in certain details developed later. It turns out that Θ_k naturally fits into an exact sequence, whose terms we will define later:

$$0 \to bP_{k+1} \to \Theta_k \xrightarrow{\psi} \operatorname{coker}\big(\pi_k(SO_{k+1}) \xrightarrow{J_k} \pi_{2k+1}(S^{k+1})\big) \xrightarrow{\phi} P_k.$$

We sketch algorithms which, given a smooth manifold PL homeomorphic to the sphere,

(∗) compute the corresponding element of Θ_k / bP_{k+1};
(†) if this element is zero, compute the corresponding element of bP_{k+1}.

This is clearly enough for step (1); for step (3), if we generate representatives of all elements of Θ_k/bP_{k+1} and all elements of bP_{k+1}, we can generate representatives of all elements of Θ_k by taking connect sums.

We now discuss the terms of the exact sequence above:

- The group $P_k = \begin{cases} 0 & k \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & k \equiv 2 \mod 4 \\ \mathbb{Z} & k \equiv 0 \mod 4. \end{cases}$
- The map ϕ sends a smooth map $f : S^{2k+1} \to S^{k+1}$ to the Kervaire invariant (if $k \equiv 2 \mod 4$) or $1/8$ times the signature (if $k \equiv 0 \mod 4$) of the preimage of a regular point.
- The map J_k is the usual J-homomorphism, defined as follows. An element of $\pi_r(SO(q))$ can be interpreted as a map $S^r \times S^{q-1} \to S^{q-1}$. This in turn induces a map from the join $S^r * S^{q-1} \cong S^{r+q}$ to the suspension of S^{q-1}, that is, S^q.

- The group bP_{k+1} is a certain finite quotient of P_{k+1}. In the nontrivial case $k+1=2r$, this has order which divides
$$2^{2r-1} \cdot (2^{2r-1}-1) \cdot \text{numerator}(B_r/r),$$
where $2r = k+1$ and B_r is the rth Bernoulli number.
- The map ψ is constructed as follows. Every smooth homotopy sphere Σ is stably parallelizable. This means that given an embedding $\Sigma \hookrightarrow S^{2k+1}$, one can construct a trivialization of the normal bundle and use the Pontryagin–Thom construction to give a map $S^{2k+1} \to S^{k+1}$. This depends on the choice of trivialization, and the indeterminacy is exactly the image of the J-homomorphism.
- Finally, an isomorphism between $\ker\psi$ and bP_{k+1} is given as follows. If $\Sigma \in \ker\psi$, then Σ is framed nullcobordant. Then the corresponding element in P_{k+1} is given by the Kervaire invariant or 1/8 signature (rel boundary) of a nullcobordism with parallelizable normal bundle; this has an indeterminacy which induces the quotient map b.

It remains to show that all of these elements can be computed.

The signature and Kervaire invariant are cohomological notions and so are unproblematic to compute from a triangulation.

A generator for $\pi_k(SO_{k+1})$ can be constructed explicitly as a simplicial map by the main theorem of [FFWZ17]. Then a corresponding simplicial map $S^r \times S^{q-1} \to S^{q-1}$ can be constructed by induction on skeleta of S^r. Finally, the Hopf construction of a map from the join to the suspension is clearly algorithmic. This gives an algorithm for determining the image of the J-homomorphism.

By results of [ČKM+14b], $\pi_k(S^{k+1})$ is fully effective: that is, we can compute a set of generators and find the combination of generators corresponding to a given simplicial map. In particular, this allows us to compute the cokernel of the J-homomorphism.

The main remaining obstacle is constructing a framing of a manifold known to be stably parallelizable. That is, given an oriented smooth manifold M^m embedded in $\mathbb{R}^n$, perhaps with boundary, we would like to lift the classifying map $M \to \mathrm{Gr}(n-m,n)$ of the normal bundle to the Stiefel manifold $V(n-m,n)$.

There are many formalisms we could use to do this algorithmically. One is as follows. We first fix a simplicial structure on $\mathrm{Gr}(n-m,n)$ which is sufficiently fine that every point within the star of a vertex corresponds to a subspace whose angle is at most $\pi/8$ from that at the vertex. Then the simplicial approximation to the classifying map $M \to \mathrm{Gr}(n-m,n)$ is at most at an angle $\pi/8$ from the "true" normal bundle. Now we fix a simplicial structure on $V(n-m,n)$ with a similar property and construct (by exhaustion) a simplicial map which approximates a lift. This may require a further subdivision of M, beyond that required for the original simplicial approximation. Then at every point of M the corresponding frame spans a subspace at an angle at most $\pi/4$ from the normal space. In particular, none of the vectors are tangent to M.

By minimizing over all simplices and pairs of simplices of the subdivision of M, we obtain a lower bound on the thickness of the resulting tubular neighborhood. Then given a framing, the Pontryagin–Thom construction can be implemented, and thus we can compute the image of a homotopy sphere under ψ. Moreover, given a map in $\ker\psi$, we can find a framed nullcobordism by iterating over all candidate nullcobordisms and framings. Thus, given a homotopy sphere, we can assign it either to a nonzero element of Θ_k/bP_{k+1} or an element of bP_{k+1}. By iterating over all smooth triangulations corresponding to barycentric subdivisions of $\partial\Delta^{k+1}$, we eventually generate representatives for all the elements of both the subgroup and the quotient group.

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