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AHLFORS Q-REGULAR SPACES WITH ARBITRARY $Q > 1$ ADMITTING WEAK POINCARÉ INEQUALITY

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Abstract

Given $Q > 1$, we construct an Ahlfors Q-regular space that admits a weak $(1,1)$-Poincaré inequality.

0 Introduction

Recently a lot of analysis has been done in metric measure spaces with controlled geometry. In particular, Ahlfors-regular spaces admitting an appropriate Poincaré inequality have been shown to carry a very Euclidean-like theory. See for example [HK] and [C]. Recall that a metric measure space (X,μ) admits a weak $(1,1)$-Poincaré inequality if for any ball B

$$\int_B |u-u_B|d\mu \le C(\mathrm{diam}(B))\left(\int_{CB}\rho d\mu\right) \tag{0.1}$$

whenever u is a bounded continuous function in a ball CB and ρ its upper gradient there. The constant C should be independent of B and u. Ahlfors Q-regularity means that $r^Q/C \le \mu(B(x,r)) \le Cr^Q$ for some C independent of x and r, whenever $r \le \mathrm{diam}(X)$.

It has nevertheless not been clear, exactly how strong these assumptions are. In [HS] Heinonen and Semmes asked whether there are fractional dimensional Ahlfors regular spaces admitting a weak $(1,1)$-Poincaré inequality. Recently Bourdon and Pajot showed the existence of such spaces for a discrete countable set of non-integer dimensions, see [BP]. Thus it is natural to ask, whether there are gaps in the set of possible dimensions. The purpose of this paper is to show that there are no restrictions: For any $Q \ge 1$ there exists an Ahlfors Q-regular space admitting a weak (1,1)-Poincaré inequality. Note that an Ahlfors Q-reqular space has always Hausdorff-dimension Q.

Our method is based on ideas of Semmes in [S]. Semmes proved the Poincaré and Sobolev inequalities from the existence of curve-families that are controlled by a suitable measure and connect given points. It is easy to find arbitrary-dimensional spaces with curve families that are controlled by the measure of the space. Take for example $I \times K$, where I is an interval and K is a Cantor set. In such spaces the suitable curve families do not usually connect points. In a suspension of K two points can be connected by a suitable family, but others cannot. Our space is a factor space of $I \times K$ over a much more complex, nearly self-similar relation. It is almost like an iterated suspension, but in a way that equivalence classes consist only of pairs of points. Such equivalence classes resemble the notion of wormholes in physics.

1 Construction

Restrict the dimension Q first to the interval $(1, 2)$. Let t satisfy $1 + \ln(2)/\ln(1/t) = Q$. We have thus $t \in (0, 1/2)$. Denote by $K = K(t)$ the closed Cantor set in $I = [0, 1]$ satisfying $K = tK \cup (tK + 1 - t)$. The Hausdorff dimension of K is thus $Q - 1$. Set $K_0 = tK$ and $K_1 = tK + 1 - t$. Denote by K_{00} and K_{01} the similar parts of K_0, i.e. t^2K and $t\ (tK + 1 - t)$ respectively. Continue this naming inductively. For K_a, $|a|$ denotes the length of the binary string a and $a1$ denotes concatenation of the strings a and "1".

We construct our space F from $I \times K$ as follows. In $I \times K$ we take the metric induced from R^2. Let n be the integer such that $1/(n+1) < t \leq 1/n$. Let j be an infinite sequence of integers n and $n + 1$ such that

$$\frac{n}{n+1} \prod_{i=1}^{m} j_i^{-1} \leq t^m \leq \frac{n+1}{n} \prod_{i=1}^{m} j_i^{-1} \tag{1.0}$$

for every m.

Set $w(i) = i\ j_1^{-1}$, where i ranges from 0 to $j_1 - 1$. Similarly, $w(h, i) = j_1^{-1}(h + i\ j_2^{-1})$, with integers h and i in ranges $0 \leq h < j_1$ and $0 \leq i < j_2$. In general,

$$w(m_1, \dots, m_k) = \sum_{i=1}^{k} m_i \prod_{h=1}^{i} j_h^{-1} ,$$

where $0 \leq m_i < j_i$. Call sets $w(m_1, \dots, m_k) \times K$ in $I \times K$ *wormhole levels* of order k provided that $m_k > 0$. The requirement $m_k > 0$ is needed to exclude the overlapping of several levels. In a wormhole level of order k identify $w(m_1, \dots, m_k) \times K_{a0}$ with $w(m_1, \dots, m_k) \times K_{a1}$ for each binary string a of length $k - 1$. A point (x_1, x_2) in $w(m_1, \dots, m_k) \times K_{a0}$ is thus identified with $(x_1, x_2 + t^{k-1}(1-t))$. Such a point located in two places in the original space $I \times K$ is called a *wormhole*. Call the resulting space F. Denote the natural projection from $I \times K \to F$ by s.

In the space F we define the metric as follows. Set

$$|x - y| = \inf \left\{ H^1(p) \;\middle|\; s(p) \text{ is a path joining } x \text{ and } y \right\},$$

where naturally $p \subset I \times K$. We could as well measure the preimage of a given path in F, and that would give the same metric, but it would complicate the explanation.

We call the I-direction of $I \times K$ vertical, and the other horizontal. Note that s does not change vertical distance. It therefore makes sense to speak about the height $h(x)$ of a point $x \in F$, i.e. the I-component of its preimage in $I \times K$. By height of a set $X \in F$ we mean the measure of the I-projection of $s^{-1}(X)$. Similarly concepts like upwards and downwards remain meaningful. We say that a path p jumps via wormhole level L, if p can be divided to two connected parts $p_1 \in s(I \times K_{a0})$ and $p_2 \in s(I \times K_{a1})$ (both containing at least two points), in every sufficiently small neighbourhood of a point $x \in L \cap p$.

If p is an s-image of countable number of line segments , then the length of p is the sum of the *heights* of parts going only upwards or only downwards, i.e. lengths of corresponding line segments in $I \times K$. More generally, if p is any path going only upwards (or only downwards), we see that its length is equal to its height. Namely, if we have two points x and y that can be joined by an upwards-going path, then all the wormhole levels needed to join them can naturally be found in heights between them. Points can thus be connected by an upwards-going path that is an image of countable number of line segments. Thus $|x - y| = |h(x) - h(y)|$. This is true for all $x, y \in p$ in particular, whence the length of p is equal to its height. On the other hand, no path going from height a to height b can be shorter than $|a - b|$. From these we see the following elementary but essential propositions about geodesics.

PROPOSITION 1.1. *Let $[a, b] \subset I$ be a smallest possible interval that contains heights of x and y and all the wormhole levels that are needed to join x to y by a path. (There may be several such intervals.) Assume $h(x) \le h(y)$. Let p be any path starting from x, going downwards to height a, then upwards to height b and again downwards to y. Then p is a geodesic between x and y. All geodesics are of that form for some $[a', b']$ with $b - a = b' - a'$.*

Remark. There are indeed paths and even geodesics γ such that $s^{-1}(\gamma)$ is totally disconnected. Such geodesics are not needed in this paper and are therefore not considered here.

Proposition 1.2. *Let $[a,b]$ be as in 1.1. Then $|x-y| = 2b-2a+h(x)-h(y)$ and because $b-a$ is always at least $h(y)-h(x)$ by definition, we have $|a-b| \leq |x-y| \leq 2|a-b|$.*

We shall see that F has Hausdorff dimension Q. So, it is natural to use the Hausdorff Q-measure in F. To increase the dimension to values greater than or equal to 2 consider cartesian products $I \times K(t)^M$ with $t \in (0,1/2)$ equipped with similar sets of identifications like before in each $K(t)$. Thus 2^M points are identified with each other in turn.

2 Properties of the Space *F*

For the terminology we refer in general to [HK]. However, we begin this section with some definitions. In some places our terminology differs from that used in [HK] or [S]. All rectifiable paths are assumed to be parametrized by the arc-length.

Definition 2.1. Let X be a metric space, and let f and ρ be two Borel measurable functions in X, with f real-valued and ρ taking values in $[0,\infty]$. We say that ρ is an *upper gradient* of f if

$$\big|f(x) - f(y)\big| \leq \int_p \rho ds \tag{2.2}$$

whenever p is a rectifiable path connecting x and y.

Definition 2.3. A family G of curves in F with endpoints x and y for each $g \in G$, is called a *pencil of curves joining x and y* if there is a constant C and a probability measure dg on G such that G is contained in $B(x, C|x-y|)$ and

$$\int_G \left(\int_g \chi_A ds \right) dg < Cm'(A)\,, \qquad A \subset F \;\; \text{Borel}\,, \tag{2.4}$$

where $dm' = \delta dH^Q$ and $\delta(z) = |x-z|^{1-Q} + |y-z|^{1-Q}$. Here we need the assumption that the inner integral is dg-measurable for each Borel set A.

Combining Definitions 2.1 and 2.3 leads to a pointwise potential estimate, provided that an ample pencil joining x and y exists.

Namely, let ρ be an upper gradient of f. Then we have

$$\big|f(x)-f(y)\big| \le \int_G \bigg(\int_g \rho ds \bigg) dg \le C \int_{B(x,C|x-y|)} \rho(z) \big(|z-x|^{1-Q} + |z-y|^{1-Q}\big) dm, \tag{2.5}$$

where the first inequality is due to (2.2) and the fact that dg is a probability measure, and the second follows from (2.4) by approximating ρ by simple functions. If we can find an ample pencil between every pair $\{x, y\}$ of points with constant independent of x and y, then also in (2.5) constants are independent of x and y. Then a lot of analysis can be done using only (2.5) and the Q-regularity of F. This includes the Poincaré inequality. See the remark made by Semmes in [S, p. 277] concerning his Theorem B.15.

Theorem 2.6. *The space F is compact, Ahlfors Q-regular and every pair of points can be connected by pencils with constants independent of points. Moreover, pencils can be chosen to contain only geodesics.*

The space F can be extended to unbounded space with similar properties by extending both I and K and using a sequence with indices in Z such that an analog of (1.0) holds. The proof of 2.6 extends easily to that case. Using also the theory of Semmes we have another formulation of Theorem 2.6.

Theorem 2.7. *For any real $Q \ge 1$ there exists an Ahlfors Q-regular unbounded, proper and geodesic space admitting a weak $(1,1)$-Poincaré inequality.*

Proof of Theorem 2.6. Compactness of F is evident. For simplicity of notation, we prove the case $F = s(I \times K)$, where K is a two-part Cantor set and only two-point identifications are used. The general case is similar. To show the Ahlfors-regularity we prove that the natural projection s is David–Semmes regular. The claim therefore follows from the Ahlfors-regularity of $I \times K$ by the definition of David–Semmes regularity and the fact that such maps keep Hausdorff-measure comparable to what it was.

Definition 2.8. A mapping $f : X \to Y$ between metric spaces is called *David–Semmes regular with data* (L, M, N) if f is L-Lipschitz and if for each closed ball B in Y the preimage $f^{-1}(B)$ can be covered by M closed balls in X each of diameter at most N times the diameter of B.

To prove the Lipschitz condition, let x and y be two points in $I \times K$. There exists K_a such that $\{x, y\} \subset I \times K_a$ and $\mathrm{diam}(K_a) \le C|x - y|$. Now all the wormhole levels needed to pass from any point of K_a to any other are within the distance $2t^{|a|-1}/n$ which is proportional to $\operatorname{diam}(K_a)$. This together with Proposition 1.2. proves that $|s(x)-s(y)| \le C|x-y|$.

Next consider $s^{-1}(B(x,r))$. Let m be such that $t^m > (3n+3)\, r \ge t^{m+1}$. The minimal distance between wormhole levels of orders from 1 to $m+1$ is at least $t^m/(n+1)$, so $B(x,r)$ can reach only one wormhole level of those orders. On the other hand, wormhole levels of greater orders can only decrease distances of amount $\sum_{k=m+1}^{\infty} t^k$, because passing twice via wormholes of the same order does not help at all. The preimage $s^{-1}(B(x,r))$ is therefore contained in at most two balls of radius $r + t^{m+1}/(1-t) \le Cr$.

We now turn into the more interesting issue of constructing pencils connecting given x and y. Because our pencils are made of geodesics, they are automatically contained in $B(x,|x-y|)$. The key fact is that the height of every geodesic is proportional to its length. Hence a geodesic meets enough wormhole levels needed to reach any point of some K_w whose width is proportional to its length.

We know that the Fubini theorem holds in $I \times K$. This can be expressed as

$$\int_P \left(\int_p \chi_A ds \right) dp = H^Q(A)\,,$$

where the path family P consists of all vertical paths $I \times \{w\} \subset I \times K$ and dp is a measure in the path-space P that is originally the $(Q-1)$-dimensional Hausdorff-measure in K with the obvious correspondence between points and paths. Note that for the existence of pencils that is our goal, it does not matter where the measure that we use in the path-family comes from. We only need the existence of a suitable measure. We can thus think the path family P transported to F by s and use the same measure dp we used in $I \times K$. Because the vertical distance is the same in $I \times K$ and F, we have $\int_p \chi_{s^{-1}(A)} ds = \int_{s(p)} \chi_A ds$. Therefore

$$\int_P \left(\int_p \chi_{s^{-1}(A)} ds \right) dp = \int_{s(P)} \left(\int_{s(p)} \chi_A ds \right) dp\,.$$

Because s is David–Semmes regular, we have $H^Q(s^{-1}(A)) < CH^Q(A)$. Therefore

$$\int_{s(P)} \left(\int_{s(p)} \chi_A ds \right) dp < C_Q H^Q(A) \tag{2.9}$$

for every Borel set $A \subset F$.

Let g be a geodesic of the form of Proposition 1.1 joining x and y. Suppose in addition that g takes only necessary jumps via wormholes. Let $[a,b]$

be as in Proposition 1.1. Let $\beta(u)$ be the smallest k such that $2t^{k-1}/n \leq |u|$. Then between heights h and i there are wormhole levels of any order greater than or equal to $\beta(h-i)$ and, in particular, g meets a wormhole level of any order $k \geq \beta((a-b)/2)$ both in height-intervals $[a,(a+b)/2]$ and $[(a+b)/2,b]$. Let K_w be such that $s^{-1}(g)$ meets $\{(a+b)/2\} \times K_w$ and $|w| = \beta((a-b)/2)$. Varying our geodesic such that it does additional jumps via levels of orders greater than or equal to $\beta((a-b)/2)$, we can reach every point of $s(\{(a+b)/2\} \times K_w)$. To construct a pencil of these geodesics we require that all jumps via wormhole levels are done as close to $\{x,y\}$ as possible. Call such a family $G_{x,y}$ and its defining requirement *the closeness requirement.* Note that it implies also that no unnecessary jumps are done. For the probability measure dg we take the Hausdorff-measure in K_w weighted by factor $H^{Q-1}(K_w)^{-1}$. We show that $G_{x,y}$ satisfies the inequality (2.4) with constants independent of x and y. It is obviously enough to show this for the other end, i.e. between $s(\{(a+b)/2\} \times K_w)$ and x.

From the closeness requirement it follows immediately that if two geodesics need a wormhole level of order k to reach x, then they use the same level. (Each level is obviously needed only once.) Jumps of orders less than $\beta((a-b)/2)$ that are needed, are common to all geodesics in the family. They do not affect (2.4) and can be ignored. Consider next the other jumps. If the order of the jump is k, then for every binary string u with $|u| = k-1$ all paths in $s(I \times K_{u0})$ continue in $s(I \times K_{u1})$ (or vice versa depending on the location of $s^{-1}(x)$) . If x is at the wormhole level of order k, no jumps via that order level are needed, so we can assume the location of $s^{-1}(x)$ is unique. There can be no difference between subfamilies of paths in $s(I \times K_{u0})$ and $s(I \times K_{u1})$, because the jump is the only one of order k and all the other jumps affect the subfamilies similarly. Namely, in the height of an earlier jump, either s identifies a part containing both K_{u0} and K_{u1} to the adjacent part, or smaller parts to each other similarly inside both K_{u0} and K_{u1}. Hence the following paragraph remains true after every jump.

Between two successive jumps the pencil consists of a finite number of s-images of restrictions of P into a 2-interval $I_1 \times K_v$, where $I_1 \subset I$ is a 1-interval between heights of succesive jumps in question. The assumption of measurability of the inner integral in (2.4) is obviously satisfied. There are m pieces with the same K_v, with m depending only on I_1. There is a direct correspondence between paths in a restriction and points in a part of K_w of the same size. Inequality (2.9) holds thus in $s(I_1 \times K_v)$ with a constant $mC_Q H^{Q-1}(K_w)^{-1}$. After jumping via the next wormhole level m

is replaced by $2m$.

Take two parts γ_1 and γ_2 of any $\gamma \in G_{x,y}$ such that γ_1 goes from x to height a and γ_2 goes from height a to $(a+b)/2$. Note that γ_1 may be empty. If not, then by the closeness requirement at least jumps of orders $k \geq \beta(h(x)-a)$ are done in the part γ_1, at most at the distance $2t^{k-1}/n$ from x. For $k < \beta(h(x)-a)$ jumps may be done in γ_2, but now $2t^{k-1}/n$ is greater than the length of γ_1, and therefore the distance from x to a jump via that kind of wormhole is at most $4t^{k-1}/n$.

So, when the distance from x is more than Ct^k, only $k - \beta((a-b)/2)$ jumps may have happened, and thus inequality (2.9) holds with a constant $2^{k-\beta((a-b)/2)} C_Q H^{Q-1}(K_w)^{-1}$. The diameter of K_w is $t^{\beta((a-b)/2)}$. By the Ahlfors-regularity of K, $H^{Q-1}(K_w)^{-1}$ is proportional to $t^{(1-Q)\beta((a-b)/2)}$. Because $t^{1-Q} = 2$, the inequality (2.9) holds with a bound proportional to $t^{k(1-Q)}$. Thus $G_{x,y}$ satisfies (2.4).

The general case $F = s(I \times K(t)^M)$ differs in that after each jump the amount of overlapping restrictions of P is multiplied by 2^M, and, on the other hand, $t^{1-Q} = 2^M$. These two things balance each other. Instead of K_w we use the product of M K_{w_i}'s, with $|w_i| = \beta((a-b)/2)$. This product has $(Q-1)$-measure proportional to $t^{(1-Q)\beta((a-b)/2)} = 2^{M\beta((a-b)/2)}$ like before. The proof of Theorem 2.6 is complete.

3 Remarks

3.1 From the inequality (1.0) we needed only the fact that $\prod_{i=1}^{k}(j_i)^{-1}$ and t^k are comparable. So there is no need to assume that j consists of numbers n and $n+1$. Any finite amount of integers will do. Therefore we can make j periodic in a dense set of dimensions. Namely, if for some integer m we can represent t^{-m} as a product of m integers, then j can be made periodic with a period m.

3.2 There is no need to restrict ourselves to two-part Cantor sets or their products as a "base". One could take any self-similar set with $m \geq 2$ parts such that every part is isometric to the whole after scaling of a factor t, and parts have positive distance from each other. The dimension of such a space is $\ln(m)/\ln(1/t)$. Instead of I in $I \times K$ one could also use I^n, giving dimension $Q = n + \ln(m)/\ln(1/t)$ for the entire space. For more general constructions with similar properties see [L1].

3.3 If the sequence j associated to given t as in (1.0) is periodic, then the space is self-similar, i.e. it can be divided into finitely many parts that are isometric to the whole space after a scaling of the metric. Note that those parts overlap at certain wormhole levels. By 3.1 we can give a periodic j for any $0 < t < 1/2$ that satisfies $t^{-p} = \prod_{i=1}^{p} k_i$ for some integers $p > 0$ and $k_i > 1$. Remembering also 3.2 we see that there exists a Q-dimensional self-similar space with properties listed in Theorem 2.6 if $Q = n + \ln(m)/\ln(1/t)$ for some integers $n > 0$, $m > 1$ and real t as above.

Note that this set of dimensions is closed under finite sums. Therefore taking cartesian products does not give self-similarity in any new dimensions. One can obtain BPI self-similarity with Poincaré inequality with any dimension $Q > 1$ by modifying the construction presented here. See [L2].

4 Nonembeddability to $\boldsymbol{R}^n$

Theorem 4.1. *There exists no bilipschitz embedding $f : F \to R^n$ for any n.*

Proof. Assume that $f : F \to R^n$ is L-bilipschitz. Take from F two images of vertical lines, V_1 and V_2 that are connected by wormholes. Let x_1 and x_2 be two successive common points of these two lines. Take two points $p_1 \in V_1$ and $p_2 \in V_2$ such that $|p_i - x_1| = |p_i - x_2|$ for $i = 1, 2$. Their distance in F is $|x_1 - x_2|$. Set $M = \max(|f(p_i) - f(x_j)|)$. We claim that $2M/|f(x_1) - f(x_2)|$ has a definite lower bound $C_L > 1$ depending only on the bilipschitz constant of f. This says that f increases more (or decreases less) the distance between some p_i and x_j, than between x_1 and x_2. Moreover, the difference is of some definite factor of distances, independent of the points. With suitable iteration, this will lead to contradiction to Lipschitz condition. Now, by the definition of M, $f(p_1)$ and $f(p_2)$ are contained in $B(f(x_1), M) \cap B(f(x_2), M)$. Therefore

$$\begin{aligned} \operatorname{diam}\big(B(f(x_1), M) \cap B(f(x_2), M)\big) &\geq \big|f(p_1) - f(p_2)\big| \\ &\geq |p_1 - p_2|/L = |x_1 - x_2|/L \geq \big|f(x_1) - f(x_2)\big|/L^2 \,. \end{aligned}$$

This implies $M \geq \sqrt{|f(x_1) - f(x_2)|^2 + |f(x_1) - f(x_2)|^2/L^4}/2$ by the Pythagorean theorem, which proves the claim.

Take a part V_3 of V_1 or V_2 that joins p_i and x_j with $|f(p_i) - f(x_j)| = M$. Because of the claim we can find two succesive wormholes x_1' and x_2' of the same order from V_3 such that $|f(x_1') - f(x_2')|/|x_1' - x_2'| \geq C_L|f(x_1) - f(x_2)|/|x_1 - x_2|$. (Indeed, intervals between successive wormholes can be used to make a disjoint covering of V_3, with full 1-measure. If aforementioned x_1' and x_2' do not exist, the image of V_3 is therefore too short for the claim to be true.) Now we begin the process again with x_1' and x_2'. Continuing this way will eventually contradict the bilipschitz condition.

5 Gromov–Hausdorff Limits

5.1 Non-pointed limits. Let F^1 and F^2 be two spaces constructed like in the section 1 using j^1 and j^2 with first m numbers equal. The Gromov–Hausdorff distance of F^1 and F^2 is less than or equal to $2\prod_{i=1}^{m}(j_i)^{-1}$, where j denotes the common part of j_1 and j_2. See [GLP] for definition. To see the claim, look at the set A appearing in both F^1 and F^2 and consisting of points that can be joined to $s(I \times \{0\})$ using only wormholes of orders from 1 to m. The mutual distance of points in A is not affected by wormholes of orders greater than m. Therefore we can glue F^1 and F^2 together along A without changing the metric. The distance of successive wormhole levels of order m is at most $2\prod_{i=1}^{m}(j_i)^{-1}$. In that distance all the greater order wormholes are found, and any other point of F^1 or F^2 can be connected to A in that length.

Remark 5.1.1. Note that in the previous argument we do *not* need the assumption that F^1 and F^2 have the same dimension. In addition if we fix first m elements of j, then for any given t we can continue j such that it satisfies (1.0) for some, possibly very big, constant. Given Q_1 and F such that $\dim(F) = Q_2$ we can thus easily make a sequence F_k such that $\dim(F_k) = Q_1$ for any k and the Gromov–Hausdorff limit of F_k is F. Because the constant in inequality (2.4) depends on the constant in (1.0) that goes obviously to infinity with k in our case, we see that F_k's do not necessarily satisfy Poincaré inequality with common bound. The same is true for the Ahlfors-regularity.

Remark 5.1.2. There is a height-preserving isometry that maps $s(I \times K_{a0b})$ to $s(I \times K_{a1b})$ and vice versa simultaneously for all a of the same length and for all b. Combining such isometries leads to isometry that maps a given point to $s(I \times \{0\})$ and preserves height.

5.2 Pointed limits. The method of 5.1 also shows that tangent cones of the space F are spaces of the similar type. See [GLP] for the definition. Consider a tangent cone at $p \in F$. By 5.1.2 only the height of p matters.

Let j be the sequence associated to F like before satisfying (1.0) with some constant in the place of $(n+1)/n$. Use the following representation of $h(p)$. Let b be an infinite sequence of integers such that $0 \le b_i < j_i$ for each i and

$$h(p) = \sum_{i=1}^{\infty} b_i \prod_{m=1}^{i} j_m^{-1} .$$

So if $j_k = c$ for all k, then b is the ordinary c-ary representation of $h(p)$. Just as in the ordinary representation we make the sequence b unique by picking, if possible, the alternative with $b_k = 0$ when $k > M_b$. So, when the first order wormhole levels divide I to j_1 pieces, b_1 expresses, in which one of the pieces the point is (counted from the bottom). Wormhole levels themselves are counted to the lower piece they touch by the uniqueness requirement. Similarly, when each of these pieces are divided into j_2 pieces, b_2 says in which piece the point is.

We can naturally reconstruct the entire space around p from j and b. Correspondent partial sum

$$\sum_{i=k}^{\infty} b_i \prod_{m=1}^{i} j_m^{-1}$$

gives the distance from p to the closest wormhole level below of order at most k. Because we know this also for $k-1$, we know exactly, where to put kth order wormhole levels.

The family of scalings of the metric by factors $r_k = \prod_{i=1}^{k} j_i$ leads into spaces F_k with $j^{(k)}$ that are left-shiftings of the original j, i.e. $j_m^{(k)} = j_{k+m}$. By this equation $j_m^{(k)}$ is defined also for all $m > -k$. Along negative order wormhole levels the space extends thus sidewise. Because of inequality (1.0), j has a bounded range. By compactness of the space of such sequences, we can take a subsequence $j^{(i_k)}$ converging to a sequence $\bar{j}$ with indices in Z. Similarly, when scaled as before, the representation b of $h(p)$ is replaced by $b^{(k)}$ that are left-shiftings of the original b. Also b has a bounded range, so passing to another subsequence makes $b^{(k)}$ to converge to some $\bar{b}$ with indices in Z. Let $\bar{F}$ be the unbounded space reconstructed from $\bar{j}$ and $\bar{b}$ as before, with the following modifications: If $h(p) = 0$, let $\bar{F}$ continue from height 0 upwards. Analogously for $h(p) = 1$. (Otherwise let $\bar{F}$ continue to arbitrary great and small heights.) If p is a wormhole, we need two copies of the reconstructed space that are glued at the height $h(p)$. Denote also by (F_k, p_k) the suitable subsequence such that $j_i^{(k)} = \bar{j}_i$ and $b_i^{(k)} = \bar{b}_i$, when $|i| < k$, (and distance from p_k to the bottom and top of the space is more than Ct^k unless p is in either place).

We show as in 5.1 that $(\bar{F}, \bar{p})$ is a pointed Gromov–Hausdorff limit of (F_k, p_k). Let A_k be the set appearing in both (F_k, p_k) and $(\bar{F}, \bar{p})$ that consists of lines that can be connected to $I \times \bar{p}$ or $I \times p_k$ by wormhole levels of orders $-k \leq n \leq k$. Because all $j^{(k)}$ satisfy (1.0) by uniform constant, the distance of successive wormhole levels of order k is at most Ct^k. The convergence of the representation of p_k says thus that $|p_k - \bar{p}| < Ct^k$ in A_k. Glue F_k to one end of $[0, Ct^k] \times A_k$ and $\bar{F}$ to the other. In the glued space we can identify $\bar{p}$ and p_k with changing the metric neither in F_k nor in $\bar{F}$. Now $B(\bar{p}, Ct^{-k}/6)$ can reach only one wormhole level of order less than $-k$. We can thus extend A_k to the other side of that level similarly in F_k and $\bar{F}$ without knowing what order the level has. (It is the wormhole level at $h(p)$, if such exists.) When restricted to $B(\bar{p}, Ct^{-k}/6)$, all points of F_k and $\bar{F}$ have distance at most Ct^k to A_k as in 5.1, and the Hausdorff-distance of F_k and $\bar{F}$ is thus less than $3Ct^k$ as required.

We showed so far only that F has some tangent cones like itself, but it is easy to see that actually all the tangent cones are of the same type. From any sequence r_k of scaling-factors we can take a subsequence that leads to a previous kind of convergence. In particular, any convergent sequence has to converge to a previous kind of space. Namely, factors r_k can be represented as $u_k v_k$, where u_k is the distance between wormhole levels of order $i(k)$ and $1 \leq v_k < M$. Now taking a suitable subsequence makes v_k converge, and u_k leads to spaces $F_{i(k)}$ with left-shifted $j^{(i(k))}$. Thus we can continue as before. This remains true even if we allow the centers to vary when scaling, because the representations $b^{(k)}$ of the centers still vary in a compact space and converge for a suitable subsequence.

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References

[BP] M. Bourdon, H. Pajot, Poincaré inequalities and quasiconformal structure on the boundaries of some hyperbolic buildings, Proc. Amer. Math. Soc. 127:8 (1999), 2315–2324.

[C] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, GAFA 9:3 (1999), 413–640.

[GLP] M. Gromov, J. Lafontaine, P. Pansu, Structures métriques pour les variétiés Riemanniennes, Cedic-Fernand Nathan, Paris, 1981.

[HK] J. Heinonen, P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, Acta Math. 181 (1998), 1–61.

[HS] J. Heinonen, S. Semmes, Thirty-three YES or NO questions about mappings, measures and metrics, Conformal Geometry and Dynamics (Electronic Journal of the American Mathematical Society) 1 (1997), 1–12.

[L1] T. Laakso, Analysis in one-directional spaces, in preparation.

[L2] T. Laakso, Fractals with decent calcunus, in preparation.

[M] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, 1995.

[S] S. Semmes, Finding curves on general spaces through quantitative topology with application for Sobolev and Poincaré inequalities, Selecta Math. (N.S.) 2 (1996), 155–295.

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