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Weak curvature conditions and functional inequalities

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Abstract

We give sufficient conditions for a measured length space (X, d, ν) to admit local and global Poincaré inequalities, along with a Sobolev inequality. We first introduce a condition DM on (X, d, ν), defined in terms of transport of measures. We show that DM, together with a doubling condition on ν, implies a scale-invariant local Poincaré inequality. We show that if (X, d, ν) has nonnegative N-Ricci curvature and has unique minimizing geodesics between almost all pairs of points then it satisfies DM, with constant 2^N. The condition DM is preserved by measured Gromov–Hausdorff limits. We then prove a Sobolev inequality for measured length spaces with N-Ricci curvature bounded below by $K > 0$. Finally we derive a sharp global Poincaré inequality.

Keywords: Poincaré inequality; Ricci curvature; Metric-measure spaces; Sobolev inequality

There has been recent work on giving a good notion for a compact measured length space (X, d, ν) to have a "lower Ricci curvature bound." In our previous work [10] we gave a notion of (X, d, ν) having nonnegative N-Ricci curvature, where $N \in [1, \infty)$ is an effective dimension. The definition was in terms of the optimal transport of measures on X. A notion was also given of (X, d, ν) having ∞-Ricci curvature bounded below by $K \in \mathbb{R}$; a closely related definition in this case was given independently by Sturm [13]. In a recent contribution, Sturm has suggested a notion of (X, d, ν) having N-Ricci curvature bounded below by $K \in \mathbb{R}$ [14]. These notions are

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preserved by measured Gromov–Hausdorff limits; when specialized to Riemannian manifolds, they coincide with classical Ricci curvature bounds.

Several results in Riemannian geometry have been extended to these generalized settings. For example, the Lichnerowicz inequality of Riemannian geometry implies that for a compact Riemannian manifold with Ricci curvature bounded below by $K > 0$, the lowest positive eigenvalue of the Laplacian is bounded below by K. In [10] we showed that this inequality extends to measured length spaces with ∞-Ricci curvature bounded below by K in the form of a global Poincaré inequality.

When doing analysis on metric-measure spaces, a useful analytic property is a "local" Poincaré inequality. A metric-measure space (X, d, ν) admits a local Poincaré inequality if, roughly speaking, for each function f and each ball B in X, the mean deviation (on B) of f from its average value on B is quantitatively controlled by the gradient of f on a larger ball; see Definition 2.3 of Section 2 for a precise formulation. Cheeger showed that if a metric-measure space has a doubling measure and admits a local Poincaré inequality then it has remarkable extra local structure [2].

Cheeger and Colding showed that local Poincaré inequalities exist for measured Gromov–Hausdorff limits of Riemannian manifolds with lower Ricci curvature bounds [4]. The method of proof was to show that such Riemannian manifolds satisfy a certain "segment inequality" [3, Theorem 2.11] and then to show that the property of satisfying the segment inequality is preserved under measured Gromov–Hausdorff limits [4, Theorem 2.6]. This then implies the local Poincaré inequality.

Following on the work of Cheeger and Colding, in the present paper we introduce a certain condition DM on a measured length space, with DM being short for "democratic." The condition DM is defined in terms of what we call "dynamical democratic transference plans." A dynamical democratic transference plan is a measure on the space of all geodesics with both endpoints in a given ball. The "democratic" condition is that the geodesics with a fixed initial point must have their endpoints sweeping out the ball uniformly, and similarly for the geodesics with a fixed endpoint. Roughly speaking, the condition DM says that there is a dynamical democratic transference plan so that a given point is not hit too often by the geodesics.

We show that the condition DM is preserved by measured Gromov–Hausdorff limits. We show that DM, together with a doubling condition on the measure, implies a scale-invariant local Poincaré inequality. We then show that if (X, d, ν) has nonnegative N-Ricci curvature in the sense of [10], and in addition for almost all $(x_0, x_1) \in X \times X$ there is a unique minimal geodesic joining x_0 and x_1, then (X, d, ν) satisfies DM. Since nonnegative N-Ricci curvature implies a doubling condition, it follows that (X, d, ν) admits a local Poincaré inequality. We do not know whether the condition of nonnegative N-Ricci curvature is sufficient in itself to imply a local Poincaré inequality.

In the last section of the paper we prove a Sobolev inequality for compact measured length spaces with N-Ricci curvature bounded below by $K > 0$. Our definition of N-Ricci curvature bounded below by K is a variation on Sturm's $\mathrm{CD}(K, N)$ condition [14]. We use the Sobolev inequality to derive a global Poincaré inequality. In the case $N = \infty$, a global Poincaré inequality with constant K was proven in [10]; we show that when $N < \infty$, one can improve this by a factor of $\frac{N}{N-1}$. In the Riemannian case, this is the sharp Lichnerowicz inequality for the lowest positive eigenvalue of the Laplacian [9].

Appendix A contains a compactness theorem for probability measures on spaces of geodesics.

After our research concerning DM was completed, we learned of preprints by Ohta [11], von Renesse [12] and Sturm [14] that consider somewhat related conditions. In [12] a local

Poincaré inequality is proved, also along the Cheeger–Colding lines, based on a "measure contraction property" and almost-everywhere unique geodesics. The measure contraction property is also considered in [11,14]; compare with the proof of Lemma 3.4.

1. Democratic couplings

We recall some notation from [10, Section 2]. Let (X, d) be a compact length space; see [1] for background material on such spaces. (Many results of the paper extend to the locally compact case, but for simplicity we will assume compactness.) Let Γ denote the set of minimizing constant-speed geodesics $\gamma : [0, 1] \to X$, with the time-t evaluation map denoted by $e_t : \Gamma \to X$. The endpoint map $E : \Gamma \to X \times X$ is $E = (e_0, e_1)$.

We let $P(X)$ denote the set of Borel probability measures on X. A *transference plan* $\pi \in P(X \times X)$ between $\mu_0, \mu_1 \in P(X)$ is a probability measure whose marginals are μ_0 and μ_1. The 2-Wasserstein space $P_2(X)$ is $P(X)$ equipped with the metric of optimal transport, $W_2(\mu_0, \mu_1) = [\inf \int_{X\times X} d(x_0, x_1)^2 \, d\pi(x_0, x_1)]^{1/2}$. Here the infimum is over transference plans between μ_0 and μ_1. A transference plan is said to be *optimal* if it achieves the infimum in the above variational problem. (See [15] for background on the theory of optimal transport.) When such a π is given, we can disintegrate it with respect to its first marginal μ_0 or its second marginal μ_1. We write this in a slightly informal way:

$$d\pi(x_0, x_1) = d\pi(x_1|x_0)\, d\mu_0(x_0) = d\pi(x_0|x_1)\, d\mu_1(x_1). \tag{1.1}$$

A *dynamical transference plan* consists of a transference plan π and a Borel measure Π on Γ such that $E_*\Pi = \pi$; it is said to be optimal if π itself is. If Π is a dynamical transference plan then for $t \in [0, 1]$, we put $\mu_t = (e_t)_*\Pi$. Then Π is optimal if and only if $\{\mu_t\}_{t\in[0,1]}$ is a Wasserstein geodesic [10, Lemma 2.4]. Any Wasserstein geodesic arises from some optimal dynamical transference plan in this way [10, Proposition 2.10].

Definition 1.2 *(Democratic coupling).* Given $\mu \in P(X)$, the democratic transference plan between μ and itself is the tensor product $\mu \otimes \mu \in P(X \times X)$. A probability measure $\Pi \in P(\Gamma)$ is said to be a *dynamical democratic transference plan* between μ and itself if $E_*\Pi = \mu \otimes \mu$.

Example 1.3. Let (X, d) be equipped with a reference measure $\nu \in P(X)$. Suppose that one has almost-everywhere uniqueness of geodesics in the following sense:

$$\begin{cases} \text{For } \nu \otimes \nu\text{-almost all } (x_0, x_1) \in X \times X, \text{ there is a unique geodesic } \gamma = \gamma_{x_0,x_1} \in \Gamma \\ \text{with } \gamma(0) = x_0 \text{ and } \gamma(1) = x_1. \end{cases} \tag{1.4}$$

Define $S : X \times X \to \Gamma$ measurably by $S(x_0, x_1) = \gamma_{x_0,x_1}$. If μ is absolutely continuous with respect to ν then there is a unique dynamical democratic transference plan between μ and itself given by

$$\Pi = S_*(\mu \otimes \mu). \tag{1.5}$$

Definition 1.6. A compact measured length space (X, d, ν) is a compact length space (X, d) equipped with a Borel probability measure $\nu \in P(X)$. Given $C > 0$, the triple (X, d, ν) is said to satisfy DM(C) if for each ball $B = B_r(x) \subset X$ with $\nu[B] > 0$, there is a dynamical democratic transference plan Π from $\mu = \frac{1_B}{\nu[B]}\nu$ to itself with the property that if we put $\mu_t = (e_t)_*\Pi$ then

$$\int_0^1 \mu_t \, dt \leqslant \frac{C}{\nu[B]}\nu. \tag{1.7}$$

We recall that a sequence $\{(X_i, d_i)\}_{i=1}^{\infty}$ of compact metric spaces converges to a compact metric space (X, d) in the Gromov–Hausdorff topology if there is a sequence of Borel maps $f_i : X_i \to X$ and a sequence of positive numbers $\epsilon_i \to 0$ so that:

1. For all $x_i, x_i' \in X_i$, $|d_X(f_i(x_i), f_i(x_i')) - d_{X_i}(x_i, x_i')| \leqslant \epsilon_i$.
2. For all $x \in X$ and all i, there is some $x_i \in X_i$ such that $d_X(f_i(x_i), x) \leqslant \epsilon_i$.

The maps f_i are called ϵ_i-approximations. If each (X_i, d_i) is a length space then so is (X, d). A sequence $\{(X_i, d_i, \nu_i)\}_{i=1}^{\infty}$ of compact measured length spaces converges to (X, d, ν) in the measured Gromov–Hausdorff topology if in addition, one can choose the f_i's so that $\lim_{i\to\infty}(f_i)_*\nu_i = \nu$ in the weak-$*$ topology on $P(X)$.

Theorem 1.8. *Suppose that $\{(X_i, d_i, \nu_i)\}_{i=1}^{\infty}$ is a sequence of compact measured length spaces that converges to (X, d, ν) in the measured Gromov–Hausdorff topology. Suppose that each ball B in X has $\nu[B] = \nu[\overline{B}]$. If each (X_i, d_i, ν_i) satisfies* DM(C) *then so does (X, d, ν).*

Proof. Let $f_i : X_i \to X$ be a sequence of ε_i-approximations, with $\varepsilon_i \to 0$, that realizes the Gromov–Hausdorff convergence. Let $B = B_r(x)$ be a ball in X with $\nu[B] > 0$. For each i, choose a point $x_i \in X_i$ so that $d_X(f_i(x_i), x) \leqslant \varepsilon_i$ and put $B_i = B_r(x_i)$. By elementary estimates,

$$f_i^{-1}\big(B_{r-2\varepsilon_i}(x)\big) \subset B_i \subset f_i^{-1}\big(f_i(B_i)\big) \subset f_i^{-1}\big(B_{r+2\varepsilon_i}(x)\big). \tag{1.9}$$

Combining this with the convergence of $(f_i)_*\nu_i$ to ν, and the fact that $\nu[B] = \nu[\overline{B}]$, it is easy to deduce that $\nu_i[B_i] \to \nu[B]$ (and in particular $\nu_i[B_i] > 0$ for i large enough). A similar "squeezing" argument shows that $\int_{B_i} \varphi \circ f_i \, d\nu_i$ converges to $\int_B \varphi \, d\nu$ for all nonnegative continuous functions φ. As a consequence, if we put $\mu = \frac{1_B}{\nu[B]}\nu$ and (for i large enough) $\mu_i = \frac{1_{B_i}}{\nu_i[B_i]}\nu_i$ then $\lim_{i\to\infty}(f_i)_*\mu_i = \mu$ in the weak-$*$ topology.

For each i, we can introduce a dynamical democratic transference plan Π_i as in Definition 1.6, relative to the ball B_i. We write $\mu_{i,t} = (e_t)_*\Pi_i$. From Theorem A.45, there is a dynamical transference plan $\Pi \in P(\Gamma(X))$, with associated transference plan $\pi = E_*\Gamma$ and measures $\mu_t = (e_t)_*\Pi$, so that (up to extraction of a subsequence) $\lim_{i\to\infty}(f_i, f_i)_*\pi_i = \pi$ and $\lim_{i\to\infty}(f_i)_*\mu_{i,t} = \mu_t$.

For any $F_1, F_2 \in C(X)$, we have

$$\begin{aligned}\int_{X\times X} F_1(x)F_2(y)\, d\pi(x,y) &= \lim_{i\to\infty} \int_{X\times X} F_1(x)F_2(y)\, d\big((f_i, f_i)_*\pi_i\big)(x,y) \\ &= \lim_{i\to\infty} \int_{X_i\times X_i} \big(f_i^*F_1\big)(x_i)\big(f_i^*F_2\big)(y_i)\, d\pi_i(x_i, y_i)\end{aligned}$$

$$= \lim_{i\to\infty} \int_{X_i} \left(f_i^* F_1\right) d\mu_i \int_{X_i} \left(f_i^* F_2\right) d\mu_i$$

$$= \lim_{i\to\infty} \int_{X} F_1\, d(f_i)_*\mu_i \int_{X} F_2\, d(f_i)_*\mu_i$$

$$= \int_X F_1\, d\mu \int_X F_2\, d\mu. \tag{1.10}$$

Thus $E_*\Pi = \pi = \mu \otimes \mu$, so Π is still a dynamical democratic transference plan.

It remains to check (1.7). Let φ be a nonnegative continuous function on X. For large i, we can write

$$\int_0^1 \int_{X_i} (f_i)^*\varphi\, d\mu_{i,t} \leqslant \frac{C}{\nu_i[B_i]} \int_{X_i} (f_i)^*\varphi\, d\nu_i. \tag{1.11}$$

In other words,

$$\int_0^1 \int_X \varphi\, d(f_i)_*\mu_{i,t} \leqslant \frac{C}{\nu_i[B_i]} \int_X \varphi\, d(f_i)_*\nu_i. \tag{1.12}$$

Passing to the limit as $i \to \infty$ gives

$$\int_0^1 \int_X \varphi\, d\mu_t \leqslant \frac{C}{\nu[B]} \int_X \varphi\, d\nu. \tag{1.13}$$

Since φ is arbitrary, this proves (1.7). □

2. From DM to a scale-invariant local Poincaré inequality

We first recall some notation and definitions about metric-measure spaces (X, d, ν). If $B = B_r(x)$ is a ball in X then we write λB for $B_{\lambda r}(x)$. The measure ν is said to be *doubling* if there is some $D > 0$ so that for all balls B, $\nu[2B] \leqslant D\nu[B]$. An admissible constant D is called a *doubling constant*. An *upper gradient* for a function $u \in C(X)$ is a Borel function $g : X \to [0, \infty]$ such that for each curve $\gamma : [0, 1] \to X$ with finite length $L(\gamma)$ and constant speed,

$$\left|u\big(\gamma(1)\big) - u\big(\gamma(0)\big)\right| \leqslant L(\gamma) \int_0^1 g\big(\gamma(t)\big)\, dt. \tag{2.1}$$

If u is Lipschitz then an example of an upper gradient is obtained by defining

$$g(x) = \begin{cases} \limsup_{y\to x} \frac{|u(y)-u(x)|}{d(x,y)} & \text{if } x \text{ is not isolated}, \\ 0 & \text{if } x \text{ is isolated}. \end{cases} \tag{2.2}$$

There are many forms of local Poincaré inequalities. The strongest one, in a certain sense, is as follows.

Definition 2.3. A metric-measure space (X, d, ν) admits a local Poincaré inequality if there are constants $\lambda \geqslant 1$ and $P < \infty$ such that for all $u \in C(X)$ and $B = B_r(x)$ with $\nu[B] > 0$, each upper gradient g of u satisfies

$$\fint_B \left|u - \langle u\rangle_B\right| d\nu \leqslant Pr \fint_{\lambda B} g\, d\nu. \tag{2.4}$$

Here the barred integral is the average (with respect to ν) and $\langle u\rangle_B$ is the average of u over the ball B. In the case of a length space, the local Poincaré inequality as formulated in Definition 2.3 actually implies stronger inequalities, for which we refer to [6, Chapters 4 and 9]. It is known that the property of admitting a local Poincaré inequality is preserved under measured Gromov–Hausdorff limits [7,8]. (This is an extension of the earlier result [2, Theorem 9.6]; Cheeger informs us that in unpublished work he also proved the extension.)

The use of a condition like DM to prove a local Poincaré inequality is implicit in the work of Cheeger and Colding [3, proof of Theorem 2.11]. The next theorem makes the link explicit.

Theorem 2.5. *If the compact measured length space (X, d, ν) satisfies* DM(C) *and ν is doubling, with doubling constant D, then (X, d, ν) admits a local Poincaré inequality* (2.4) *with $\lambda = 2$ and $P = 2CD$.*

Proof. Let x_0 be a given point in X. Given $r > 0$, write $B = B_r(x_0)$. Note that from the doubling condition, $\nu[B] > 0$. Put $\mu = \frac{1_B}{\nu[B]}\nu$.

For $y_0 \in X$, we have

$$u(y_0) - \langle u\rangle_B = \int_X \big(u(y_0) - u(y_1)\big)\, d\mu(y_1). \tag{2.6}$$

Then

$$\fint_B \left|u - \langle u\rangle_B\right| d\nu = \int_X \left|u(y_0) - \langle u\rangle_B\right| d\mu(y_0) \leqslant \int_{X\times X} \left|u(y_0) - u(y_1)\right| d\mu(y_0)\, d\mu(y_1). \tag{2.7}$$

Next, we estimate $|u(y_0) - u(y_1)|$ in terms of a geodesic path γ joining y_0 to y_1, where $y_0, y_1 \in B$. The length of such a geodesic path is clearly less than $2r$. Then, from the definition of an upper gradient,

$$\left|u(y_0) - u(y_1)\right| \leqslant 2r \int_0^1 g\big(\gamma(t)\big)\, dt. \tag{2.8}$$

Now let Π be a dynamical democratic transference plan between μ and itself satisfying (1.7). Integrating (2.8) against Π gives, with $\mu_t = (e_t)_* \Pi$,

$$\begin{aligned}
\int_{X\times X} \bigl|u(y_0)-u(y_1)\bigr|\,d\mu(y_0)\,d\mu(y_1) &\leqslant \int_{\Gamma}\left(2r\int_0^1 g\bigl(\gamma(t)\bigr)\,dt\right)d\Pi(\gamma)\\
&= 2r\int_0^1\left(\int_{\Gamma} g\bigl(\gamma(t)\bigr)\,d\Pi(\gamma)\right)dt\\
&= 2r\int_0^1\left(\int_{\Gamma}(g\circ e_t)\,d\Pi\right)dt\\
&= 2r\int_0^1\left(\int_X g\,d(e_t)_*\Pi\right)dt\\
&= 2r\int_0^1\int_X g\,d\mu_t\,dt. \qquad (2.9)
\end{aligned}$$

Combining this with (2.7), we conclude that

$$⨍_B \bigl|u-\langle u\rangle_B\bigr|\,d\nu \leqslant 2r\int_0^1\int_X g\,d\mu_t\,dt. \tag{2.10}$$

However, a geodesic joining two points in B cannot leave $2B$, so (2.10) and DM(C) together imply that

$$⨍_B \bigl|u-\langle u\rangle_B\bigr|\,d\nu \leqslant \frac{2Cr}{\nu[B]}\int_{2B} g\,d\nu. \tag{2.11}$$

By the doubling property, $\frac{1}{\nu[B]} \leqslant \frac{D}{\nu[2B]}$. The conclusion is that

$$⨍_B \bigl|u-\langle u\rangle_B\bigr|\,d\nu \leqslant 2CDr ⨍_{2B} g\,d\nu. \tag{2.12}$$

This proves the theorem. □

3. Nonnegative N-Ricci curvature and DM

In this section we show that a measured length space with nonnegative N-Ricci curvature satisfies the condition DM(2^N) as soon as geodesics are almost-everywhere unique.

We use the notion of nonnegative N-Ricci curvature from [10, Definition 5.12]. This is the same as the case $K=0$ of Section 4. We will be concerned here with the case $N<\infty$.

Theorem 3.1. *Suppose that a compact measured length space (X, d, ν) has nonnegative N-Ricci curvature, and that minimizing geodesics in X are almost-everywhere unique in the sense of* (1.4)*. Then (X, d, ν) satisfies* $\mathrm{DM}(2^N)$.

Remark 3.2. If (X, d, ν) has nonnegative N-Ricci curvature and (X, d) is nonbranching then minimizing geodesics in X are almost-everywhere unique [13].

Before proving Theorem 3.1, we state a corollary.

Corollary 3.3. *If a compact measured length space (X, d, ν) has nonnegative N-Ricci curvature and almost-everywhere unique geodesics then it satisfies the local Poincaré inequality of Definition* 2.3 *with $\lambda = 2$ and $P = 2^{2N+1}$. More generally, if $\{(X_i, d_i, \nu_i)\}_{i=1}^{\infty}$ is a sequence of compact measured length spaces with nonnegative N-Ricci curvature and almost-everywhere unique geodesics, and it converges in the measured Gromov–Hausdorff topology to (X, d, ν), then (X, d, ν) satisfies the local Poincaré inequality of Definition* 2.3 *with $\lambda = 2$ and $P = 2^{2N+1}$.*

Proof. First, [10, Theorem 5.19] implies that (X, d, ν) has nonnegative N-Ricci curvature. Then the generalized Bishop–Gromov inequality of [10, Theorem 5.31] implies that $\nu[B] = \nu[\overline{B}]$ for each ball B whose center belongs to the support of ν. It also implies that ν is doubling with constant $D = 2^N$. Then the conclusion follows from Theorems 1.8, 2.5 and 3.1. □

As preparation for the proof of Theorem 3.1, we first prove a lemma concerning optimal transport to delta functions.

Lemma 3.4. *Under the hypotheses of Theorem* 3.1*, let B be an open ball in X. Then for almost all $x_0 \in \mathrm{supp}(\nu)$, the* (*unique*) *Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ joining $\mu_0 = \delta_{x_0}$ to $\mu_1 = \frac{1_B}{\nu[B]}\nu$ can be written as $\mu_t = \rho_t \nu$ with*

$$\rho_t(x) \leqslant \frac{1}{t^N \nu[B]}. \tag{3.5}$$

Proof. Let Π be the (unique) optimal dynamical transference plan giving rise to $\{\mu_t\}_{t\in[0,1]}$. Let Y_0 be the set of $x_0 \in \mathrm{supp}(\nu)$ such that for ν-almost every $x \in X$ there is a unique geodesic joining x_0 to x. By assumption, Y_0 has full ν-measure. Consider $x_0 \in Y_0$. Given $t \in (0, 1)$, $y \in X$ and $r > 0$, let Z be the set of endpoints $\gamma(1)$ of geodesics γ with $\gamma(0) = x_0$, $\gamma(t) \in B_r(y)$ and $\gamma(1) \in B$. Then

$$\begin{aligned}\mu_t\big[B_r(y)\big] &= \big((e_t)_*\Pi\big)\big[B_r(y)\big] = \Pi\big[e_t^{-1}\big(B_r(y)\big)\big] = \Pi\big[e_1^{-1}(Z)\big] = \big((e_1)_*\Pi\big)[Z] \\ &= \mu_1[Z] = \frac{\nu[Z]}{\nu[B]}.\end{aligned} \tag{3.6}$$

If $\nu[Z] = 0$ then $\mu_t[B_r(y)] = 0$. Otherwise, put $\mu_1' = \frac{1_Z}{\nu[Z]}\nu$ and let $\{\mu_t'\}_{t\in[0,1]}$ be the (unique) Wasserstein geodesic joining $\mu_0' = \delta_{x_0}$ to μ_1'. By the construction of Z, $\mu_t'[B_r(y)] = 1$.

Put

$$\phi(s) = \int_X \big(\rho_s'\big)^{1-1/N}\, d\nu, \tag{3.7}$$

where ρ'_s is the density in the absolutely continuous part of the Lebesgue decomposition of μ'_s with respect to ν. As (X, d, ν) has nonnegative N-Ricci curvature, $-\phi$ satisfies a convexity inequality on [0, 1]. (We use here the uniqueness of the Wasserstein geodesic $\{\mu'_t\}_{t\in[0,1]}$. From the definition of nonnegative N-Ricci curvature, *a priori* one only has convexity along *some* Wasserstein geodesic from μ'_0 to μ'_1.)

As $\phi(0) = 0$ and $\phi(1) = \nu[Z]^{\frac{1}{N}}$, we obtain

$$\phi(t) \geqslant t\nu[Z]^{\frac{1}{N}}. \tag{3.8}$$

On the other hand, by Jensen's inequality

$$\begin{aligned}\phi(t) &= \nu\big[B_r(y)\big]\bigg(\frac{1}{\nu[B_r(y)]}\int\limits_{B_r(y)} \big(\rho'_t\big)^{1-\frac{1}{N}}\,d\nu\bigg) \leqslant \nu\big[B_r(y)\big]\bigg(\frac{1}{\nu[B_r(y)]}\int\limits_{B_r(y)} \rho'_t\,d\nu\bigg)^{1-\frac{1}{N}} \\ &\leqslant \nu\big[B_r(y)\big]^{\frac{1}{N}}\mu'_t\big[B_r(y)\big]^{1-\frac{1}{N}} = \nu\big[B_r(y)\big]^{\frac{1}{N}}. \end{aligned}\tag{3.9}$$

This, combined with (3.8), gives

$$t\nu[Z]^{\frac{1}{N}} \leqslant \nu\big[B_r(y)\big]^{\frac{1}{N}}. \tag{3.10}$$

Then by (3.6),

$$\frac{\mu_t[B_r(y)]}{\nu[B_r(y)]} = \frac{\frac{\nu[Z]}{\nu[B_r(y)]}}{\nu[B]} \leqslant \frac{1}{t^N\nu[B]}. \tag{3.11}$$

Since this is true for any ball centered at any y and since balls generate the Borel σ-algebra, we deduce that $\mu_t \leqslant \frac{\nu}{t^N\nu[B]}$; so μ_t is absolutely continuous with respect to ν and its density is bounded above by $\frac{1}{t^N\nu[B]}$. □

Proof of Theorem 3.1. As in Definition 1.6, let B be a ball in X with $\nu[B] > 0$ and put $\mu = \frac{1_B}{\nu[B]}\nu$. As in Example 1.3, there is a unique dynamical democratic transference plan from μ to itself. We want to show that the condition of Definition 1.6 is satisfied.

Define $\mu_t^{x_0}$ as in Lemma 3.4, with density $\rho_t^{x_0}$. From Lemma 3.4, $\rho_t^{x_0} \leqslant \frac{1}{t^N\nu[B]}$. The key point is that this is independent of x_0.

We now want to integrate with respect to x_0. With μ_t as in Definition 1.6 and $\varphi \in C(X)$, we have

$$\begin{aligned}\int\limits_X \varphi\,d\mu_t &= \int\limits_X \varphi\,d(e_t)_*\Pi = \int\limits_X (\varphi\circ e_t)\,d\Pi = \int\limits_\Gamma \varphi\big(\gamma(t)\big)\,d\Pi(\gamma) \\ &= \int\limits_{X\times X} \varphi\big(\gamma_{x_0,x_1}(t)\big)\,d\mu(x_0)\,d\mu(x_1)\end{aligned}\tag{3.12}$$

and

$$\int_X \varphi\, d\mu_t^{x_0} = \int_X \varphi\big(\gamma_{x_0,x_1}(t)\big)\, d\mu(x_1). \tag{3.13}$$

These equations show that

$$\mu_t = \int_X \mu_t^{x_0}\, d\mu(x_0). \tag{3.14}$$

In particular, μ_t admits a density ρ_t which satisfies the equation

$$\rho_t(x) = \int_X \rho_t^{x_0}(x)\, d\mu(x_0). \tag{3.15}$$

It follows immediately that

$$\rho_t(x) \leqslant \frac{1}{t^N \nu[B]}. \tag{3.16}$$

As geodesics are almost-everywhere unique, we can apply the preceding arguments symmetrically with respect to the change $t \to 1-t$. This gives

$$\rho_t(x) \leqslant \frac{1}{(1-t)^N \nu[B]}. \tag{3.17}$$

Then

$$\rho_t(x) \leqslant \min\left(\frac{1}{t^N}, \frac{1}{(1-t)^N}\right)\frac{1}{\nu[B]} \leqslant \frac{2^N}{\nu[B]}. \tag{3.18}$$

The theorem follows. □

Remark 3.19. The above bounds (3.18) can be improved as follows. Let $\mu = \rho\nu$ be a measure that is absolutely continuous with respect to ν, and arbitrary, otherwise. Then there exists a probability measure $\Pi \in P(\Gamma)$, with $E_*\Pi = \mu \otimes \mu$, such that $\mu_t = (e_t)_*\Pi$ admits a density ρ_t with respect to ν, and

$$\|\rho_t\|_{L^p} \leqslant \min\left(\frac{1}{t^{N/p'}}, \frac{1}{(1-t)^{N/p'}}\right)\|\rho\|_{L^p} \tag{3.20}$$

for all $p \in (1,\infty)$, where $p' = p/(p-1)$ is the conjugate exponent to p and $\|\rho\|_{L^p} = (\int_X \rho^p d\nu)^{1/p}$. Condition (3.20) is also stable by measured Gromov–Hausdorff limits. Yet we prefer to focus on condition DM because it is a priori weaker, and still implies the local Poincaré inequality.

4. Definition of N-Ricci curvature bounded below by K

We recall some more notation.

For $N \in [1, \infty)$, the class $\mathcal{DC}_N$ is the set of continuous convex functions $U : [0, \infty) \to \mathbb{R}$, with $U(0) = 0$, such that the function

$$\psi(\lambda) = \lambda^N U\big(\lambda^{-N}\big) \tag{4.1}$$

is convex on $(0, \infty)$. For $N = \infty$, the class $\mathcal{DC}_\infty$ is the set of continuous convex functions $U : [0, \infty) \to \mathbb{R}$, with $U(0) = 0$, such that the function

$$\psi(\lambda) = e^\lambda U\big(e^{-\lambda}\big) \tag{4.2}$$

is convex on $(-\infty, \infty)$. In both cases, such a ψ is automatically nonincreasing by the convexity of U. We write $U'(\infty) = \lim_{r\to\infty} \frac{U(r)}{r}$. If a reference probability measure $\nu \in P(X)$ is given, we define a function $U_\nu : P(X) \to \mathbb{R} \cup \{\infty\}$ by

$$U_\nu(\mu) = \int_X U(\rho)\, d\nu + U'(\infty)\mu_s(X), \tag{4.3}$$

where $\mu = \rho\nu + \mu_s$ is the Lebesgue decomposition of μ with respect to ν.

We now introduce some expressions that played a prominent role in [5] and [14]. Given $K \in \mathbb{R}$ and $N \in (1, \infty]$, define

$$\beta_t(x_0, x_1) = \begin{cases} e^{\frac{1}{6}K(1-t^2)d(x_0,x_1)^2} & \text{if } N = \infty, \\ \infty & \text{if } N < \infty,\ K > 0 \text{ and } \alpha > \pi, \\ \left(\frac{\sin(t\alpha)}{t\sin\alpha}\right)^{N-1} & \text{if } N < \infty,\ K > 0 \text{ and } \alpha \in [0, \pi], \\ 1 & \text{if } N < \infty \text{ and } K = 0, \\ \left(\frac{\sinh(t\alpha)}{t\sinh\alpha}\right)^{N-1} & \text{if } N < \infty \text{ and } K < 0, \end{cases} \tag{4.4}$$

where

$$\alpha = \sqrt{\frac{|K|}{N-1}}\, d(x_0, x_1). \tag{4.5}$$

When $N = 1$, define

$$\beta_t(x_0, x_1) = \begin{cases} \infty & \text{if } K > 0, \\ 1 & \text{if } K \leqslant 0. \end{cases} \tag{4.6}$$

Although we may not write it explicitly, α and β depend on K and N.

Definition 4.7. We say that (X, d, ν) has N-Ricci curvature bounded below by K if the following condition is satisfied. Given $\mu_0, \mu_1 \in P(X)$ with support in $\operatorname{supp}(\nu)$, write their Lebesgue decompositions with respect to ν as $\mu_0 = \rho_0\nu + \mu_{0,s}$ and $\mu_1 = \rho_1\nu + \mu_{1,s}$, respectively. Then there is *some* optimal dynamical transference plan Π from μ_0 to μ_1, with corresponding Wasserstein geodesic $\mu_t = (e_t)_*\Pi$, so that for all $U \in \mathcal{DC}_N$ and all $t \in [0, 1]$, we have

$$U_\nu(\mu_t) \leqslant (1-t) \int\limits_{X\times X} \beta_{1-t}(x_0,x_1)\, U\left(\frac{\rho_0(x_0)}{\beta_{1-t}(x_0,x_1)}\right) d\pi(x_1|x_0)\, d\nu(x_0)$$
$$+\, t \int\limits_{X\times X} \beta_t(x_0,x_1)\, U\left(\frac{\rho_1(x_1)}{\beta_t(x_0,x_1)}\right) d\pi(x_0|x_1)\, d\nu(x_1)$$
$$+\, U'(\infty)\big((1-t)\mu_{0,s}(X) + t\mu_{1,s}(X)\big). \tag{4.8}$$

Here if $\beta_t(x_0,x_1)=\infty$ then we interpret $\beta_t(x_0,x_1)U(\frac{\rho_1(x_1)}{\beta_t(x_0,x_1)})$ as $U'(0)\rho_1(x_1)$, and similarly $\beta_{1-t}(x_0,x_1)U(\frac{\rho_0(x_0)}{\beta_{1-t}(x_0,x_1)})$ as $U'(0)\rho_0(x_0)$. It is not difficult to show that if $N<\infty$ and (X,d,ν) has N-Ricci curvature bounded below by $K>0$, then the diameter of the support of ν is bounded above by $\pi\sqrt{(N-1)/K}$. In that case, the quantity α defined in (4.5) will vary only in $[0,\pi]$ as x_0, x_1 vary in the support of ν.

Remark 4.9. If μ_0 and μ_1 are absolutely continuous with respect to ν then the inequality can be rewritten in the more symmetric form

$$U_\nu(\mu_t) \leqslant (1-t) \int\limits_{X\times X} \frac{\beta_{1-t}(x_0,x_1)}{\rho_0(x_0)} U\left(\frac{\rho_0(x_0)}{\beta_{1-t}(x_0,x_1)}\right) d\pi(x_0,x_1)$$
$$+\, t \int\limits_{X\times X} \frac{\beta_t(x_0,x_1)}{\rho_1(x_1)} U\left(\frac{\rho_1(x_1)}{\beta_t(x_0,x_1)}\right) d\pi(x_0,x_1). \tag{4.10}$$

Remark 4.11. Note that (4.8) is unchanged by the addition of a linear function $r \to cr$ to U. Of course, the validity of (4.8) depends on the values of K and N. The parameter β_t is monotonically nondecreasing in K and the function $\beta \mapsto \beta U(\rho/\beta)$ is monotonically nonincreasing in β (because of the convexity of U). It follows that if $K \leqslant K'$ and (X,d,ν) has N-Ricci curvature bounded below by K' then it also has N-Ricci curvature bounded below by K, as one would expect. One can also show that if $N \leqslant N'$ and (X,d,ν) has N-Ricci curvature bounded below by K then it has N'-Ricci curvature bounded below by K.

We now compare Definition 4.8 with earlier definitions in the literature, starting with the case $N<\infty$. If $N<\infty$ and $K=0$ then one recovers the $N<\infty$, $K=0$ definition of [10]. If $N<\infty$ and one specializes to $U(r)$ being

$$U_N(r) = Nr\big(1-r^{-1/N}\big), \tag{4.12}$$

with corresponding entropy function

$$H_{N,\nu}(\mu) = N - N\int\limits_X \rho^{1-\frac{1}{N}}\, d\nu, \tag{4.13}$$

then one recovers the $N<\infty$ definition of Sturm [14]. (In [14] it was not required that π and $\{\mu_t\}_{t\in[0,1]}$ be related in the sense that they both arise from an optimal dynamical transference plan Π. We can make that requirement without loss of consistency.)

To deal with the $N=\infty$ case, we use the following lemma.

Lemma 4.14. *If $N = \infty$, with ψ defined as in* (4.2), *put*

$$\lambda(U) = \begin{cases} -K\psi'(\infty) & \text{if } K > 0, \\ 0 & \text{if } K = 0, \\ -K\psi'(-\infty) & \text{if } K > 0. \end{cases} \tag{4.15}$$

If μ_0 and μ_1 are absolutely continuous with respect to ν then

$$\int\limits_{X \times X} \frac{\beta_t(x_0, x_1)}{\rho_1(x_1)} U\left(\frac{\rho_1(x_1)}{\beta_t(x_0, x_1)}\right) d\pi(x_0, x_1) \leqslant \int\limits_X U(\rho_1)\, d\nu - \frac{1}{6}\big(1 - t^2\big)\lambda(U) W_2(\mu_0, \mu_1)^2. \tag{4.16}$$

Proof. Suppose first that $K > 0$. From the convexity of ψ, if $\rho_1(x_1) > 0$ then

$$\frac{\psi(-\ln \rho_1(x_1) + \frac{1}{6}K(1 - t^2)d(x_0, x_1)^2) - \psi(-\ln \rho_1(x_1))}{\frac{1}{6}K(1 - t^2)d(x_0, x_1)^2} \leqslant \psi'(\infty). \tag{4.17}$$

Then

$$\frac{\beta_t(x_0, x_1)}{\rho_1(x_1)} U\left(\frac{\rho_1(x_1)}{\beta_t(x_0, x_1)}\right) \leqslant \frac{1}{\rho_1(x_1)} U\big(\rho_1(x_1)\big) + \frac{1}{6} K\psi'(\infty)\big(1 - t^2\big)\, d(x_0, x_1)^2. \tag{4.18}$$

The lemma follows upon integration with respect to $d\pi(x_0, x_1)$. The cases $K = 0$ and $K < 0$ are similar. □

Using Lemma 4.14, and the analogous inequality for $\int_{X \times X} \frac{\beta_{1-t}(x_0,x_1)}{\rho_0(x_0)} U(\frac{\rho_0(x_0)}{\beta_{1-t}(x_0,x_1)})\, d\pi(x_0, x_1)$, one finds that (4.10) implies

$$U_\nu(\mu_t) \leqslant tU_\nu(\mu_1) + (1 - t)U_\nu(\mu_0) - \frac{1}{2}\lambda(U)t(1 - t)W_2(\mu_0, \mu_1)^2, \tag{4.19}$$

which is exactly the inequality used in [10] to define what it means for (X, d, ν) to have ∞-Ricci curvature bounded below by K, in the sense of [10]. We have only shown that (4.19) holds when μ_0 and μ_1 are absolutely continuous with respect to ν, but [10, Proposition 3.21] then implies that it holds for all μ_0, μ_1 with support in $\mathrm{supp}(\nu)$.

A consequence is that any result of [10] concerning measured length spaces with ∞-Ricci curvature bounded below by K, in the sense of [10], also holds for measured length spaces with ∞-Ricci curvature bounded below by K in the sense of Definition 4.7.

Finally, Sturm's notion of having ∞-Ricci curvature bounded below by K [13] is the specialization of the definition of [10] to the case $U(r) = U_\infty(r) = r\ln(r)$.

The notion of having N-Ricci curvature bounded below by K, in the sense of Definition 4.7, is preserved by measured Gromov–Hausdorff limits. We will present the proof, which is more complicated than that of the analogous statement in [10], elsewhere.

We now show that in the case of a Riemannian manifold equipped with a smooth measure, a lower Ricci curvature bound in the sense of Definition 4.7 is equivalent to a tensor inequality.

Let M be a smooth compact connected n-dimensional manifold with Riemannian metric g. We let (M, g) denote the corresponding metric space. Given $\Psi \in C^\infty(M)$ with $\int_M e^{-\Psi} \operatorname{dvol}_M = 1$, put $d\nu = e^{-\Psi} \operatorname{dvol}_M$.

Definition 4.20. For $N \in [1, \infty]$, let the N-Ricci tensor Ric_N of (M, g, ν) be defined by

$$\mathrm{Ric}_N = \begin{cases} \mathrm{Ric} + \mathrm{Hess}(\Psi) & \text{if } N = \infty, \\ \mathrm{Ric} + \mathrm{Hess}(\Psi) - \frac{1}{N-n} d\Psi \otimes d\Psi & \text{if } n < N < \infty, \\ \mathrm{Ric} + \mathrm{Hess}(\Psi) - \infty(d\Psi \otimes d\Psi) & \text{if } N = n, \\ -\infty & \text{if } N < n, \end{cases} \tag{4.21}$$

where by convention $\infty \cdot 0 = 0$.

Theorem 4.22. *For $N \in [1, \infty]$, the measured length space (M, g, ν) has N-Ricci curvature bounded below by K if and only if* $\mathrm{Ric}_N \geqslant Kg$.

Proof. The proof is similar to that of [10, Theorems 7.3 and 7.42] and [14, Theorem 1.9]. We only sketch a few points of the proof, in order to clarify the role played by the function U.

Suppose that $N \in (1, \infty)$ and $\mathrm{Ric}_N \geqslant Kg$. We want to show that the condition in Definition 4.7 holds. As in [10, Theorem 7.3], we can reduce to the case when μ_0 and μ_1 are absolutely continuous with respect to ν. The unique Wasserstein geodesic between them is of the form $\mu_t = (F_t)_* \mu_0$ for certain maps $F_t : M \to M$. Put

$$C(y, t) = e^{-\frac{\Psi(F_t(y))}{N}} \det{}^{\frac{1}{N}}(dF_t)(y) \tag{4.23}$$

and

$$\eta_0 = \frac{d\mu_0}{\operatorname{dvol}_M}. \tag{4.24}$$

Then in terms of the function ψ of (4.1) there is an equation [10, (7.19)]

$$U_\nu(\mu_t) = \int_M \psi\big(C(y, t)\eta_0^{-\frac{1}{N}}(y)\big)\, d\mu_0(y). \tag{4.25}$$

With the notation (4.5) in use, define

$$\tau_{K,N}^{(t)}\big(d(x_0, x_1)\big) = \begin{cases} t^{\frac{1}{N}} (\frac{\sin(t\alpha)}{\sin\alpha})^{1-\frac{1}{N}} & \text{if } K > 0, \\ t & \text{if } K = 0, \\ t^{\frac{1}{N}} (\frac{\sinh(t\alpha)}{\sinh\alpha})^{1-\frac{1}{N}} & \text{if } K < 0, \end{cases} \tag{4.26}$$

one can show by combining [10, Section 7] and [14, Section 5] that

$$C(y, t) \geqslant \tau_{K,N}^{(1-t)}\big(d\big(y, F_1(y)\big)\big) C(y, 0) + \tau_{K,N}^{(t)}\big(d\big(y, F_1(y)\big)\big) C(y, 1). \tag{4.27}$$

As ψ is nonincreasing, we obtain

$$U_\nu(\mu_t) \leqslant \int_M \psi\left(\frac{\tau_{K,N}^{(1-t)}(d(y, F_1(y)))C(y,0) + \tau_{K,N}^{(t)}(d(y, F_1(y)))C(y,1)}{\eta_0^{1/N}(y)}\right) d\mu_0(y). \tag{4.28}$$

Since ψ is convex by assumption, we deduce

$$\begin{aligned} U_\nu(\mu_t) \leqslant (1-t) \int_M \psi\left(\frac{\tau_{K,N}^{(1-t)}(d(y, F_1(y)))}{1-t} \frac{C(y,0)}{\eta_0^{1/N}(y)}\right) d\mu_0(y) \\ + t \int_M \psi\left(\frac{\tau_{K,N}^{(t)}(d(y, F_1(y)))}{t} \frac{C(y,1)}{\eta_0^{1/N}(y)}\right) d\mu_0(y). \end{aligned} \tag{4.29}$$

After using the definition of ψ again, along with (4.25) in the cases $t = 0$ and $t = 1$, one arrives at (4.10). □

The next result is an analog of [10, Theorem 5.52].

Theorem 4.30. *If (X, d, ν) has N-Ricci curvature bounded below by K then for any $\mu_0, \mu_1 \in P(X)$ that are absolutely continuous with respect to ν, the Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ of Definition* 4.7 *has the property that μ_t is absolutely continuous with respect to ν, for all $t \in [0, 1]$.*

Proof. The proof is along the lines of [10, Theorem 5.52]. □

Theorem 4.30 will be needed in Eq. (5.7) below. This is one reason why we require (4.8) to hold for all $U \in \mathcal{DC}_N$, as opposed to just U_N. (We note that the distinction between these two definitions disappears in nonbranching spaces, as will be shown elsewhere.)

5. Sobolev inequality and global Poincaré inequality

Definition 5.1. Given $f \in \mathrm{Lip}(X)$, put

$$\left|\nabla^- f\right|(x) = \limsup_{y\to x} \frac{[f(y) - f(x)]_-}{d(x,y)} = \limsup_{y\to x} \frac{[f(x) - f(y)]_+}{d(x,y)}. \tag{5.2}$$

Here $a_+ = \max(a, 0)$ and $a_- = \max(-a, 0)$. Note that $|\nabla^- f|(x) \leqslant |\nabla f|(x)$, the latter being defined as in (2.2).

Theorem 5.3. *Given $N \in (1, \infty)$ and $K > 0$, suppose that (X, d, ν) has N-Ricci curvature bounded below by K. Then for any positive Lipschitz function $\rho_0 \in \mathrm{Lip}(X)$ with $\int_X \rho_0\, d\nu = 1$, one has*

$$N - N \int_X \rho_0^{1-\frac{1}{N}}\, d\nu \leqslant \int_X \theta^{(N,K)}\left(\rho_0, \left|\nabla^- \rho_0\right|\right) d\nu, \tag{5.4}$$

where

$$\theta^{(N,K)}(r,g)=r\sup_{\alpha\in[0,\pi]}\biggl[\frac{N-1}{N}\frac{g}{r^{1+\frac{1}{N}}}\sqrt{\frac{N-1}{K}}\alpha+N\biggl(1-\biggl(\frac{\alpha}{\sin(\alpha)}\biggr)^{1-\frac{1}{N}}\biggr)$$
$$+(N-1)\biggl(\frac{\alpha}{\tan(\alpha)}-1\biggr)r^{-\frac{1}{N}}\biggr]. \tag{5.5}$$

Proof. We recall the definitions of U_N and $H_{N,\nu}$ from (4.12) and (4.13). Applying Definition 4.7 with $U=U_N$, any two probability measures $\mu_0=\rho_0\nu$ and $\mu_1=\rho_1\nu$ can be joined by a Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$, arising from an optimal dynamical transference plan, along which the following inequality holds:

$$H_{N,\nu}(\mu_t)\leqslant N-N\int_{X\times X}\bigl[\tau_{K,N}^{(1-t)}\bigl(d(x_0,x_1)\bigr)\cdot\rho_0^{-\frac{1}{N}}(x_0)$$
$$+\tau_{K,N}^{(t)}\bigl(d(x_0,x_1)\bigr)\cdot\rho_1^{-\frac{1}{N}}(x_1)\bigr]\,d\pi(x_0,x_1). \tag{5.6}$$

By Theorem 4.30, μ_t is absolutely continuous with respect to ν.

Given a positive function $\rho_0\in\operatorname{Lip}(X)$, put $\mu_0=\rho_0\nu$ and $\mu_1=\nu$. Put $\phi(t)=H_{N,\nu}(\mu_t)$. In the proof of [10, Proposition 3.36] it was shown that

$$-\limsup_{t\to0}\frac{\phi(t)-\phi(0)}{t}\leqslant\int_{\Gamma}U_N''\bigl(\rho\bigl(\gamma(0)\bigr)\bigr)\bigl|\nabla^-\rho_0\bigr|\bigl(\gamma(0)\bigr)\,d\bigl(\gamma(0),\gamma(1)\bigr)\,d\Pi(\gamma)$$
$$=\frac{N-1}{N}\int_X\frac{|\nabla^-\rho_0|(x_0)}{\rho_0(x_0)^{1+\frac{1}{N}}}\,d(x_0,x_1)\,d\pi(x_0,x_1). \tag{5.7}$$

On the other hand, from (5.6),

$$\phi(t)\leqslant N-N\int_{X\times X}\bigl[\tau_{K,N}^{(1-t)}\bigl(d(x_0,x_1)\bigr)\cdot\rho_0^{-\frac{1}{N}}(x_0)+\tau_{K,N}^{(t)}\bigl(d(x_0,x_1)\bigr)\bigr]\,d\pi(x_0,x_1) \tag{5.8}$$

and so

$$\frac{\phi(t)-\phi(0)}{t}\leqslant-N\int_{X\times X}\biggl[\frac{\tau_{K,N}^{(1-t)}(d(x_0,x_1))-1}{t}\cdot\rho_0^{-\frac{1}{N}}(x_0)+\frac{\tau_{K,N}^{(t)}(d(x_0,x_1))}{t}\biggr]\,d\pi(x_0,x_1). \tag{5.9}$$

Then

$$\limsup_{t\to0}\frac{\phi(t)-\phi(0)}{t}\leqslant-N\int_{X\times X}\Biggl(\frac{\sqrt{\frac{K}{N-1}}\,d(x_0,x_1)}{\sin(\sqrt{\frac{K}{N-1}}\,d(x_0,x_1))}\Biggr)^{1-\frac{1}{N}}\,d\pi(x_0,x_1)$$

$$+\int_{X\times X}\left[1+(N-1)\sqrt{\frac{K}{N-1}}\,d(x_0,x_1)\cot\left(\sqrt{\frac{K}{N-1}}\,d(x_0,x_1)\right)\right]\rho_0^{-\frac{1}{N}}(x_0)\,d\pi(x_0,x_1). \tag{5.10}$$

Combining (5.7) and (5.10), and slightly rewriting the result, gives

$$\begin{aligned}
&-\frac{N-1}{N}\int_X \frac{|\nabla^-\rho_0|(x_0)}{\rho_0(x_0)^{1+\frac{1}{N}}}\,d(x_0,x_1)\,d\pi(x_0,x_1)\\
&\leqslant N\int_{X\times X}\left[1-\left(\frac{\sqrt{\frac{K}{N-1}}\,d(x_0,x_1)}{\sin(\sqrt{\frac{K}{N-1}}\,d(x_0,x_1))}\right)^{1-\frac{1}{N}}\right]d\pi(x_0,x_1)\\
&\quad+(N-1)\int_{X\times X}\left[\sqrt{\frac{K}{N-1}}d(x_0,x_1)\cot\left(\sqrt{\frac{K}{N-1}}d(x_0,x_1)\right)-1\right]\rho_0^{-\frac{1}{N}}(x_0)\,d\pi(x_0,x_1)\\
&\quad-H_{N,\nu}(\mu),
\end{aligned} \tag{5.11}$$

or

$$\begin{aligned}
&H_{N,\nu}(\mu)\\
&\quad\leqslant \frac{N-1}{N}\int_X \frac{|\nabla^-\rho_0|(x_0)}{\rho_0(x_0)^{1+\frac{1}{N}}}d(x_0,x_1)\,d\pi(x_0,x_1)\\
&\quad+N\int_{X\times X}\left[1-\left(\frac{\sqrt{\frac{K}{N-1}}\,d(x_0,x_1)}{\sin(\sqrt{\frac{K}{N-1}}\,d(x_0,x_1))}\right)^{1-\frac{1}{N}}\right]d\pi(x_0,x_1)\\
&\quad+(N-1)\int_{X\times X}\left[\sqrt{\frac{K}{N-1}}d(x_0,x_1)\cot\left(\sqrt{\frac{K}{N-1}}d(x_0,x_1)\right)-1\right]\rho_0^{-\frac{1}{N}}(x_0)\,d\pi(x_0,x_1).
\end{aligned} \tag{5.12}$$

Replacing $\sqrt{\frac{K}{N-1}}d(x_0,x_1)$ by α, we get only a weaker inequality by taking the sup over $\alpha\in[0,\pi]$. The theorem follows. □

In order to clarify the nature of the inequality of Theorem 5.3, we derive a slightly weaker inequality. First, we prove an elementary estimate.

Lemma 5.13. *For $x\in[0,\pi]$, one has*

$$\frac{x}{\tan(x)}\leqslant 1-\frac{x^2}{3} \tag{5.14}$$

and

$$1-\left(\frac{x}{\sin(x)}\right)^{1-\frac{1}{N}} \leqslant -\left(1-\frac{1}{N}\right)\frac{x^2}{6}. \tag{5.15}$$

Proof. Put

$$F(x)=x-\frac{\sin(x)\cos(x)}{1-\frac{2}{3}\sin^2(x)}. \tag{5.16}$$

Then

$$F'(x)=\frac{4}{9}\frac{\sin^4(x)}{(1-\frac{2}{3}\sin^2(x))^2} \geqslant 0. \tag{5.17}$$

As $F(0)=0$, it follows that $F(x)\geqslant 0$ for $x\in[0,\pi]$, so

$$x\left(1-\frac{2}{3}\sin^2(x)\right) \geqslant \sin(x)\cos(x). \tag{5.18}$$

Putting

$$G(x)=\frac{x}{\tan(x)}+\frac{1}{3}x^2 \tag{5.19}$$

and using (5.18), one obtains

$$G'(x)=-\frac{x}{\sin^2(x)}+\frac{1}{\tan(x)}+\frac{2}{3}x \leqslant 0. \tag{5.20}$$

As $G(0)=1$, we have

$$\frac{x}{\tan(x)}+\frac{1}{3}x^2 \leqslant 1, \tag{5.21}$$

which proves (5.14).

Next, from (5.14) we have

$$\frac{1}{x}-\frac{1}{\tan(x)} \geqslant \frac{x}{3}. \tag{5.22}$$

Integrating gives

$$\ln\left(\frac{x}{\sin(x)}\right) \geqslant \frac{x^2}{6}, \tag{5.23}$$

so

$$\frac{x}{\sin(x)} \geqslant e^{\frac{x^2}{6}} \geqslant \left(1+\left(1-\frac{1}{N}\right)\frac{x^2}{6}\right)^{\frac{1}{1-\frac{1}{N}}}. \tag{5.24}$$

Thus

$$\left(\frac{x}{\sin(x)}\right)^{1-\frac{1}{N}} \geqslant 1+\left(1-\frac{1}{N}\right)\frac{x^2}{6}. \tag{5.25}$$

This proves (5.15). □

We now prove a Sobolev-type inequality.

Theorem 5.26. *Given $N \in (1,\infty)$ and $K > 0$, suppose that (X,d,ν) has N-Ricci curvature bounded below by K. Then for any nonnegative Lipschitz function $\rho_0 \in \mathrm{Lip}(X)$ with $\int_X \rho_0\, d\nu = 1$, one has*

$$N - N\int_X \rho_0^{1-\frac{1}{N}}\, d\nu \leqslant \frac{1}{2K}\left(\frac{N-1}{N}\right)^2 \int_X \frac{\rho_0^{-1-\frac{2}{N}}}{\frac{1}{3}+\frac{2}{3}\rho_0^{-1/N}}\left|\nabla^-\rho_0\right|^2 d\nu. \tag{5.27}$$

Proof. If ρ_0 is positive then using Lemma 5.13, we can estimate the function $\theta^{(N,K)}(r,g)$ of (5.5) by

$$\begin{aligned}\theta^{(N,K)}(r,g) &\leqslant r\sup_{\alpha\in[0,\pi]}\left[\frac{N-1}{N}\frac{g}{r^{1+1/N}}\sqrt{\frac{N-1}{K}}\alpha - \frac{N-1}{6}\alpha^2\left(1+2r^{-\frac{1}{N}}\right)\right]\\ &\leqslant \frac{1}{2K}\left(\frac{N-1}{N}\right)^2\frac{r^{-1-\frac{2}{N}}}{\frac{1}{3}+\frac{2}{3}r^{-1/N}}g^2.\end{aligned} \tag{5.28}$$

The theorem in this case follows from Theorem 5.3. The case when ρ_0 is nonnegative can be handled by approximation with positive functions. □

To put Theorem 5.26 into a more conventional form, we prove a slightly weaker inequality.

Theorem 5.29. *Given $N \in (2,\infty)$ and $K > 0$, suppose that (X,d,ν) has N-Ricci curvature bounded below by K. Then for any nonnegative Lipschitz function $f \in \mathrm{Lip}(X)$ with $\int_X f^{\frac{2N}{N-2}} d\nu = 1$, one has*

$$1-\left(\int_X f\, d\nu\right)^{\frac{2}{N+2}} \leqslant \frac{6}{KN}\left(\frac{N}{N-2}\right)^2\int_X \left|\nabla^- f\right|^2 d\nu. \tag{5.30}$$

Proof. Put $\rho_0 = f^{\frac{2N}{N-2}}$. From (5.27) we have

$$N - N\int_X \rho_0^{1-\frac{1}{N}}\, d\nu \leqslant \frac{3}{2K}\left(\frac{N-1}{N}\right)^2\int_X \rho_0^{-1-\frac{2}{N}}\left|\nabla^-\rho_0\right|^2 d\nu, \tag{5.31}$$

which gives

$$1 - \int\limits_X f^{2(\frac{N-1}{N-2})}\,d\nu \leqslant \frac{6}{KN}\left(\frac{N-1}{N-2}\right)^2 \int\limits_X |\nabla^- f|^2\,d\nu. \tag{5.32}$$

By Hölder's inequality,

$$\int\limits_X f^{2(\frac{N-1}{N-2})}\,d\nu \leqslant \left(\int\limits_X f^{\frac{2N}{N-2}}\,d\nu\right)^{\frac{N}{N+2}} \left(\int\limits_X f\,d\nu\right)^{\frac{2}{N+2}}. \tag{5.33}$$

The theorem follows. □

Putting (5.30) into a homogeneous form reveals the content of Theorem 5.29: there is a bound of the form $\|f\|_{\frac{2N}{N-2}} \leqslant F(\|f\|_1, \|\nabla^- f\|_2)$ for some appropriate function F. This is an example of Sobolev embedding. Of course, Eq. (5.30) is not sharp, due to the many approximations made.

Finally, we prove a sharp global Poincaré inequality.

Theorem 5.34. *Given $N \in (1,\infty)$ and $K > 0$, suppose that (X,d,ν) has N-Ricci curvature bounded below by K. Suppose that $f \in \mathrm{Lip}(X)$ has $\int_X f\,d\nu = 0$. Then*

$$\int\limits_X f^2\,d\nu \leqslant \frac{N-1}{KN} \int\limits_X |\nabla^- f|^2\,d\nu. \tag{5.35}$$

Proof. Without loss of generality we may assume that $\max|f| \leqslant 1$. Given $\epsilon \in (-1,1)$, put $\rho_0 = 1 + \epsilon f$. Then $\rho_0 > 0$ and $\int_X \rho_0\,d\nu = 1$. For small ϵ,

$$N - N\int\limits_X \rho_0^{1-\frac{1}{N}}\,d\nu = \epsilon^2 \frac{N-1}{2N} \int\limits_X f^2\,d\nu + O(\epsilon^3) \tag{5.36}$$

and

$$\frac{1}{2K}\left(\frac{N-1}{N}\right)^2 \int\limits_X \frac{\rho_0^{-1-\frac{2}{N}}}{\frac{1}{3} + \frac{2}{3}\rho_0^{-1/N}} |\nabla^- \rho_0|^2\,d\nu = \frac{\epsilon^2}{2K}\left(\frac{N-1}{N}\right)^2 \int\limits_X |\nabla^- f|^2\,d\nu + O(\epsilon^3). \tag{5.37}$$

Then the result follows from Theorem 5.26. □

Remark 5.38.

1. In the case of an N-dimensional Riemannian manifold with $\mathrm{Ric} \geqslant Kg$, one recovers the Lichnerowicz inequality for the lowest positive eigenvalue of the Laplacian [9]. It is sharp on round spheres.
2. The case $N = \infty$ was treated by similar means in [10, Theorem 6.18].

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Appendix A. Stability of dynamical transference plans

In this appendix we prove a general compactness theorem for probability measures on geodesic paths. This theorem is used to show that the condition $\mathrm{DM}(C)$ is preserved under measured Gromov–Hausdorff limits.

Lemma A.39. *Let X be a compact length space. Given $\epsilon > 0$, there is a $\delta > 0$ with the following property. Suppose that Y is a compact length space and $f : Y \to X$ is a δ-approximation. Let $\gamma : [0,1] \to Y$ be a geodesic. Then there is a geodesic $T(\gamma) : [0,1] \to X$ so that for all $t \in [0,1]$, $d_X(T(\gamma)(t), f(\gamma(t))) \leqslant \epsilon$.*

Proof. Suppose that the lemma is not true. Then there is some $\epsilon > 0$ along with:

1. A sequence of compact metric spaces $\{Y_i\}_{i=1}^\infty$,
2. $\frac{1}{i}$-approximations $f_i : Y_i \to X$, and
3. Geodesics $\gamma_i : [0,1] \to Y_i$

so that for each geodesic $\gamma' : [0,1] \to X$, there is some $t_{i,\gamma'} \in [0,1]$ with

$$d_X\big(\gamma'(t_{i,\gamma'}), f_i\big(\gamma_i(t_{i,\gamma'})\big)\big) > \epsilon. \tag{A.40}$$

After passing to a subsequence, we may assume that $\{f_i \circ \gamma_i\}_{i=1}^\infty$ converges uniformly to a geodesic $\gamma_\infty : [0,1] \to X$. After passing to a further subsequence, we may assume that $\lim_{i\to\infty} t_{i,\gamma_\infty} = t_\infty$ for some $t_\infty \in [0,1]$. Then

$$\begin{aligned} d_X\big(\gamma_\infty(t_{i,\gamma_\infty}), f_i\big(\gamma_i(t_{i,\gamma_\infty})\big)\big) &\leqslant d_X\big(\gamma_\infty(t_{i,\gamma_\infty}), \gamma_\infty(t_\infty)\big) + d_X\big(\gamma_\infty(t_\infty), f_i\big(\gamma_i(t_\infty)\big)\big) \\ &\quad + d_X\big(f_i\big(\gamma_i(t_\infty)\big), f_i\big(\gamma_i(t_{i,\gamma_\infty})\big)\big) \\ &\leqslant \mathrm{diam}(X)|t_{i,\gamma_\infty} - t_\infty| + d_X\big(\gamma_\infty(t_\infty), f_i\big(\gamma_i(t_\infty)\big)\big) \\ &\quad + \frac{1}{i} + \mathrm{diam}(Y_i)|t_{i,\gamma_\infty} - t_\infty|. \end{aligned} \tag{A.41}$$

Then the right-hand side converges to 0 as $i \to \infty$, which contradicts (A.40) with $\gamma' = \gamma_\infty$. □

Lemma A.42. *One can choose the map T in Lemma* A.39 *to be a measurable map from $\Gamma(Y)$ to $\Gamma(X)$.*

Proof. This follows from [16, Theorem A.5], as

$$\big\{(\gamma_1,\gamma_2)\in \Gamma(X)\times\Gamma(Y)\colon \text{ for all } t\in[0,1], d_X\big(\gamma_1(t), f\big(\gamma_2(t)\big)\big)\leqslant \epsilon\big\} \tag{A.43}$$

is a Borel subset of $\Gamma(X)\times\Gamma(Y)$ and for each $\gamma_2\in\Gamma(Y)$,

$$\big\{\gamma_1\in \Gamma(X)\colon \text{ for all } t\in[0,1], d_X\big(\gamma_1(t), f\big(\gamma_2(t)\big)\big)\leqslant \epsilon\big\} \tag{A.44}$$

is compact. □

Theorem A.45. *Let $\{(X_i,d_i)\}_{i=1}^{\infty}$ be a sequence of compact length spaces that converges in the Gromov–Hausdorff topology to a compact length space (X,d). Let $f_i: X_i\to X$ be ε_i-approximations, with $\varepsilon_i\to 0$, that realize the Gromov–Hausdorff convergence. For each i, let Π_i be a Borel probability measure on $\Gamma(X_i)$; and let π_i and $\{\mu_{i,t}\}_{t\in[0,1]}$ be the associated transference plan and measure-valued path. Then after passing to a subsequence, there is a dynamical transference plan Π on X, with associated transference plan π, and measure-valued path $\{\mu_t\}_{t\in[0.1]}$, such that*:

(i) $\lim_{i\to\infty}(f_i,f_i)_*\pi_i=\pi$ *in the weak-$*$ topology on $P(X\times X)$;*
(ii) $\lim_{i\to\infty}(f_i)_*\mu_{i,t}=\mu_t$ *for all $t\in[0,1]$.*

Proof. Let $T_i:\Gamma(X_i)\to\Gamma(X)$ be the map constructed by means of Lemma A.39 with $\varepsilon=\varepsilon_i$, $Y=X_i$, $f=f_i$. After passing to a convergent subsequence, we can assume that $\lim_{i\to\infty}(T_i)_*\Pi_i=\Pi$ in the weak-$*$ topology, for some $\Pi\in P(\Gamma(X))$. Given $F\in C(X\times X)$, we have

$$\int_{X\times X} F\,d\pi=\int_{\Gamma(X)} F\big(\gamma(0),\gamma(1)\big)\,d\Pi(\gamma)=\lim_{i\to\infty}\int_{\Gamma(X_i)} F\big(T_i(\gamma_i)(0),T_i(\gamma_i)(1)\big)\,d\Pi_i(\gamma_i). \tag{A.46}$$

By Lemma A.39 and the uniform continuity of F,

$$\begin{aligned}\lim_{i\to\infty}\int_{\Gamma(X_i)} F\big(T_i(\gamma_i)(0),T_i(\gamma_i)(1)\big)\,d\Pi_i(\gamma_i)&=\lim_{i\to\infty}\int_{\Gamma(X_i)} F\big(f_i\big(\gamma_i(0)\big),f_i\big(\gamma_i(1)\big)\big)\,d\Pi_i(\gamma_i)\\&=\lim_{i\to\infty}\int_{X_i\times X_i} F\big(f_i(x_i),f_i\big(x_i'\big)\big)\,d\pi_i\big(x_i,x_i'\big)\\&=\lim_{i\to\infty}\int_{X\times X} F\,d(f_i,f_i)_*\pi_i.\end{aligned} \tag{A.47}$$

This proves (i). Similarly, for $t\in[0,1]$ and $F\in C(X)$,

$$\int_X F\,d\mu_t=\int_{\Gamma(X)} F\big(\gamma(t)\big)\,d\Pi(\gamma)=\lim_{i\to\infty}\int_{\Gamma(X_i)} F\big(T_i(\gamma_i)(t)\big)\,d\Pi_i(\gamma_i). \tag{A.48}$$

By Lemma A.39 and the uniform continuity of F,

$$\begin{aligned}\lim_{i\to\infty}\int\limits_{\Gamma(X_i)} F\big(T_i(\gamma_i)(t)\big)\,d\Pi_i(\gamma_i) &= \lim_{i\to\infty}\int\limits_{\Gamma(X_i)} F\big(f_i\big(\gamma_i(t)\big)\big)\,d\Pi_i(\gamma_i)\\ &= \lim_{i\to\infty}\int\limits_{X_i} F\big(f_i(x_i)\big)\,d\mu_{i,t}(x_i)\\ &= \lim_{i\to\infty}\int\limits_{X} F\,d(f_i)_*\mu_{i,t}. \end{aligned} \tag{A.49}$$

This proves (ii). □

For reference we give a slight variation of Lemma A.39, although it is not needed in the body of the paper.

Lemma A.50. *Let X be a compact length space. Choose points $x, x' \in X$. Given $\epsilon > 0$, there is a $\delta = \delta(x, x') > 0$ with the following property. Suppose that Y is a compact length space and $f : Y \to X$ is a δ-approximation. Given $y \in f^{-1}(x)$ and $y' \in f^{-1}(x')$, let $\gamma : [0, 1] \to Y$ be a geodesic joining them. Then there is a geodesic $T(\gamma) : [0, 1] \to X$ from x to x' so that for all $t \in [0, 1]$, $d_X(T(\gamma)(t), f(\gamma(t))) \leqslant \epsilon$.*

Proof. The proof is along the same lines as that of Lemma A.39. □

Remark A.51. In general, one cannot take δ to be independent of x and x'.

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