Comparison Geometry for the Smooth Metric Measure Spaces

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Abstract

For smooth metric measure spaces the Bakry-Emery Ricci tensor is a natural generalization of the classical Ricci tensor. It occurs naturally in the study of diffusion processes, Ricci flow, the Sobolev inequality, and conformal geometry. Recent developments show that many topological and geometric results for Ricci curvature can be extended to the Bakry-Emery Ricci tensor. In this article we survey some of these results.

1. Introduction

A smooth metric measure space is a Riemannian manifold with a measure conformal to the Riemannian measure. Formally it is a triple $(M^n, g, e^{-f} dvol_g)$, where M is a complete n-dimensional Riemannian manifold with metric g, f is a smooth real valued function on M, and $dvol_g$ is the Riemannian volume density on M. This is also sometimes called a manifold with density.

A basic principle in classical Riemannian geometry is that a lower bound on the Ricci curvature implies that the Riemannian measure is bounded above by the measure in a corresponding model space. There are various ways to expand this principle to the setting of smooth metric measure spaces. In this paper we will consider the corresponding Ricci tensor to be the N-Bakry-Emery Ricci tensor

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{N} df \otimes df \quad \text{for } N > 0. \tag{1.1}$$

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As we will discuss below, N is related to the dimension of the model space. We allow N to be infinite, in this case we denote $\mathrm{Ric}_f = \mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess} f$. Note that when f is a constant function $\mathrm{Ric}_f^N = \mathrm{Ric}$ for all N and we can take $N = 0$ in this case. Moreover, if $M \geq N$ then $\mathrm{Ric}_f^N \geq \mathrm{Ric}_f^M$ so that $\mathrm{Ric}_f^N \geq \lambda g$ implies $\mathrm{Ric}_f \geq \lambda g$.

The Bakry Emery Ricci tensor (for N finite and and infinite) has a natural extension to non-smooth metric measure spaces [19, 30, 31] and diffusion operators [4]. Moreover, the equation $\mathrm{Ric}_f = \lambda g$ for some constant λ is exactly the gradient Ricci soliton equation, which plays an important role in the theory of Ricci flow. See [7] for a modification of the Ricci tensor which is conformal invariant.

We are interested in investigating what geometric and topological results for the Ricci tensor extend to the Bakry-Emery Ricci tensor. This was studied by Lichnerowicz [17, 18] almost forty years ago, though this work does not seem to be widely known. Recently this has been actively investigated and there are many interesting results in this direction which we will discuss below, see for example [15, 27, 20, 5, 26, 6, 14, 24, 25, 10, 33, 32]. In this note we first recall the Bochner formulas for Bakry-Emery Ricci tensors (stated a little differently from how they have appeared in the literature). The derivation of these from the classical Bochner formula is elementary, so we present the proof. Then we quickly derive the first eigenvalue comparison from the Bochner formulas as in the classical case. In the rest of the paper we focus on mean curvature and volume comparison theorems and their applications. When N is finite, this work is mainly from [27, 6], and when N is infinite, it's mainly from our recent work [32].

2. Bochner formulas for the N-Bakry-Emery Ricci tensor

Comparison theorems for lower bound on Ricci curvature can be derived from the Bochner formula for the usual Laplace-Beltrami operator Δ. Recall that for any smooth function u on a complete Riemannian manifold (M^n, g) the Bochner formula is

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u). \qquad (2.1)$$

Applying the Schwarz inequality $|\mathrm{Hess}\, u|^2 \geq \frac{(\Delta u)^2}{n}$ we obtain the following inequality

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u). \qquad (2.2)$$

If $\mathrm{Ric} \geq (n-1)H$, then

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle\nabla u, \nabla(\Delta u)\rangle + (n-1)H|\nabla u|^2. \qquad (2.3)$$

In fact, this inequality characterizes a lower bound on Ricci curvature for Riemannian manifolds. Namely, if (2.3) holds for all functions $u \in C^3(M)$, then $\mathrm{Ric} \geq (n-1)H$. This can be seen as follows. Given any $x_0 \in M$ and $v_0 \in T_{x_0}M$, let u be a C^3 function such that $\nabla u(x_0) = v_0$ and $\mathrm{Hess}\, u(x_0) = \lambda_0 I_n$. Then from (2.1) and (2.3), we have $\mathrm{Ric}(v_0, v_0) \geq (n-1)H|v_0|^2$, so $\mathrm{Ric} \geq (n-1)H$.

With respect to the measure $e^{-f}dvol$ the natural self-adjoint f-Laplacian is $\Delta_f = \Delta - \nabla f \cdot \nabla$. In this case we have

$$\Delta_f|\nabla u|^2 = \Delta|\nabla u|^2 - 2\mathrm{Hess}\, u(\nabla u, \nabla f),$$
$$\langle\nabla u, \nabla(\Delta_f u)\rangle = \langle\nabla u, \nabla(\Delta u)\rangle - \mathrm{Hess}\, u(\nabla u, \nabla f) - \mathrm{Hess} f(\nabla u, \nabla u).$$

Plugging these into (2.1) we immediately get the following Bochner formula for the N-Bakry-Emery Ricci tensor.

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle\nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f^N(\nabla u, \nabla u) + \frac{1}{N}|\langle\nabla f, \nabla u\rangle|^2.$$
$$(2.4)$$

When $N = \infty$, we have

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle\nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \qquad (2.5)$$

This formula is virtually the same as (2.1) except for the important fact that $tr(\mathrm{Hess}\, u) = \Delta u$ not $\Delta_f(u)$. In the case where N is finite, however, we can get around this difficulty by using the inequality

$$\frac{(\Delta u)^2}{n} + \frac{1}{N}|\langle\nabla f, \nabla u\rangle|^2 \geq \frac{(\Delta_f(u))^2}{N+n} \qquad (2.6)$$

which implies

$$\frac{1}{2}\Delta_f|\nabla u|^2 \geq \frac{(\Delta_f(u))^2}{N+n} + \langle\nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f^N(\nabla u, \nabla u). \qquad (2.7)$$

In other words, a Bochner formula holds for Ric_f^N that looks like the Bochner formula for the Ricci tensor of an $n + N$ dimensional manifold. Note that (2.6) is an equality if and only if $\Delta u = \frac{n}{N}\langle\nabla f, \nabla u\rangle$, so equality in (2.7) is seldom achieved when f is nontrivial. When f is constant, we can take $N = 0$ so (2.7) recovers (2.2).

3. Eigenvalue and Mean Curvature Comparison

From the Bochner formulas we can now prove eigenvalue and mean curvature comparisons which generalize the clasical ones. First we consider the eigenvalue comparison.

Let M^n be a complete Riemannian manifold with $\mathrm{Ric}_f^N \geq (n-1)H > 0$. Applying (2.7) to the first eigenfunction u of Δ_f, $\Delta_f u = -\lambda_1 u$, and integrating with respect to the measure $e^{-f} dvol$, we have

$$0 \geq \int_M \left(\frac{(\lambda_1 u)^2}{N+n} - \lambda_1 |\nabla u|^2 + (n-1)H|\nabla u|^2 \right) e^{-f} dvol.$$

Since $\int_M |\nabla u|^2 e^{-f} dvol = \lambda_1 \int_M u^2 e^{-f} dvol$, we deduce the eigenvalue estimate [3]

$$\lambda_1 \geq (n-1)H \left(1 + \frac{1}{N+n-1} \right). \tag{3.1}$$

When f is constant, taking $N = 0$ gives the classical Lichnerowicz's first eigenvalue estimate $\lambda_1 \geq nH$ [16]. When $N = \infty$, we have [4]

$$\lambda_1 \geq (n-1)H. \tag{3.2}$$

This also can be derived from (2.5) directly. One may expect that the estimate (3.2) is weaker than the classical one. In fact (3.2) is optimal as the following example shows.

Example 3.1 *Let $M = \mathbb{R}^1 \times S^2$ with standard product metric g_0, $f(x,y) = \frac{1}{2}x^2$. Then $\mathrm{Hess} f(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 1$ and zero on all other directions. We have $\mathrm{Ric}_f = 1 g_0$. Now for the linear function $u(x,y) = x$, $\Delta_f u = -x$. So $\lambda_1 = 1$.*

On the other hand (3.2) is never optimal for compact manifolds since equality in (3.2) implies $\mathrm{Hess}\, u = 0$. Note that $\mathrm{Ric}_f \geq (n-1)H > 0$ on a compact manifold implies $\mathrm{Ric}_f^N \geq (n-1)H' > 0$ for some N big, hence one can use estimate (3.1).

Now we turn to the mean curvature (or Laplacian) comparison. Recall that the mean curvature measures the relative rate of change of the volume element. Therefore, for the measure $e^{-f} dvol$, the associated mean curvature is $m_f = m - \partial_r f$, where m is the mean curvature of the geodesic sphere with inward pointing normal vector. Also $m_f = \Delta_f(r)$, where r is the distance function.

Let m_H^k be the mean curvature of the geodesic sphere in the model space M_H^k, the complete simply connected k-manifold of constant curvature H. When we drop the superscript k and write m_H we mean

the mean curvature from the model space whose dimension matches the dimension of the manifold. Since $\mathrm{Hess}\, r$ is zero along the radial direction, applying the Bochner formula (2.4) to the distance function r, the Schwarz inequality $|\mathrm{Hess} r|^2 \geq \frac{(\Delta r)^2}{n-1}$ and (2.6) gives

$$m_f' \leq -\frac{(m_f)^2}{n+N-1} - \mathrm{Ric}_f^N(\partial_r, \partial_r). \tag{3.3}$$

Thus, using the standard Sturm-Liouville comparison argument, one has the mean curvature comparison [6].

Theorem 3.2 (Mean curvature comparison for N-Bakry-Emery)
If $\mathrm{Ric}_f^N \geq (n+N-1)H$, *then*

$$m_f(r) \leq m_H^{n+N}(r). \tag{3.4}$$

Namely the mean curvature is less or equal to the one of the model with dimension $n+N$. This does not give any information when N is infinite.

In fact, such a strong, uniform estimate is not possible when N is infinite. To see this note that, when $H > 0$, the model space M_H^{n+N} is a round sphere so that $m_H^{n+N}(r)$ goes to $-\infty$ as r goes to $\frac{\pi}{\sqrt{H}}$. Thus (3.4) implies that if N is finite and $\mathrm{Ric}_f^N \geq \lambda > 0$ then M is compact (See Theorem 4.5 in the next section for the diameter bound). On the other hand, this is not true when $N = \infty$ as the following example shows.

Example 3.3 *Let* $M = \mathbb{R}^n$ *with Euclidean metric* g_0, $f(x) = \frac{\lambda}{2}|x|^2$. *Then* $\mathrm{Hess} f = \lambda g_0$ *and* $\mathrm{Ric}_f = \lambda g_0$.

Thus, when N is infinite, one can not expect such a strong mean curvature comparison to be true. However, we can show a weaker, nonuniform estimate and also give some uniform estimates if we make additional assumptions on f such as f being bounded or $\partial_r f$ bounded from below. In these cases we have the following mean curvature comparisons [32] which generalizes the classical one.

Theorem 3.4 (Mean Curvature Comparison for ∞-Bakry-Emery)
Let $p \in M^n$. *Assume* $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$,
a) given any minimal geodesic segment and $r_0 > 0$,

$$m_f(r) \leq m_f(r_0) - (n-1)H(r-r_0) \quad \text{for } r \geq r_0. \tag{3.5}$$

b) if $\partial_r f \geq -a$ *along a minimal geodesic segment from* p *(when* $H > 0$ *assume* $r \leq \pi/2\sqrt{H}$*) then*

$$m_f(r) - m_H(r) \leq a \tag{3.6}$$

In [20] Lott shows that if M is compact (or just $|\nabla f|$ is bounded) with $\mathrm{Ric}_f^N \geq \lambda$ for some positive integer $2 \leq N < \infty$, then, in fact, there is a family of warped product metrics on $M \times S^N$ with Ricci curvature bounded below by λ, recovering the comparison theorems for Ric_f^N.

When $N = \infty$ we have the following volume comparison results which generalize the classical one. Part a) is originally due to Morgan [23] where it follows from a hypersurface volume estimate(also see [24]). For the proofs of parts b) and c) see [32].

Theorem 4.2 (Volume Comparison for ∞-Bakry-Emery)   *Let* $(M^n, g, e^{-f} dvol_g)$ *be complete smooth metric measure space with* $\mathrm{Ric}_f \geq (n-1)H$. *Fix* $p \in M^n$.
a) If $H > 0$, *then* $\mathrm{Vol}_f(M)$ *is finite.*
b) If $\partial_r f \geq -a$ *along all minimal geodesic segments from* p *then for* $R \geq r > 0$ *(assume* $R \leq \pi/2\sqrt{H}$ *if* $H > 0$*),*

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq e^{aR} \frac{\mathrm{Vol}_H^n(R)}{\mathrm{Vol}_H^n(r)}. \tag{4.4}$$

Moreover, equality holds if and only if the radial sectional curvatures are equal to H *and* $\partial_r f \equiv -a$. *In particular if* $\partial_r f \geq 0$ *and* $\mathrm{Ric}_f \geq 0$ *then* M *has* f-*volume growth of degree at most* n.
c) If $|f(x)| \leq k$ *then for* $R \geq r > 0$ *(assume* $R \leq \pi/4\sqrt{H}$ *if* $H > 0$*),*

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq \frac{\mathrm{Vol}_H^{n+4k}(R)}{\mathrm{Vol}_H^{n+4k}(r)}. \tag{4.5}$$

In particular, if f *is bounded and* $\mathrm{Ric}_f \geq 0$ *then* M *has polynomial* f-*volume growth.*

Part a) should be viewed as a weak Myers' theorem for Ric_f. Namely if $\mathrm{Ric}_f > \lambda > 0$ then the manifold may not be compact but the measure must be finite. In particular the lifted measure on the universal cover is finite. Since this measure is invariant under the deck transformations, this weaker Myers' theorem is enough to recover the main topological corollary of the classical Myers' theorem.

Corollary 4.3 *If* M *is complete and* $Ric_f \geq \lambda > 0$ *then* M *has finite fundamental group.*

Using a different approach the second author has proven that the fundamental group is, in fact, finite for spaces satisfying $\mathrm{Ric} + \mathcal{L}_X g \geq \lambda > 0$ for some vector field X [33]. This had earlier been shown under the additional assumption that the Ricci curvature is bounded by Zhang [35]. See also [25]. When M is compact the finiteness of fundamental

group was first shown by X. Li [15, Corollary 3] using a probabilistic method.

On the other hand, the volume comparison Theorem 4.1 and Theorem 4.2 Part c) also give the following generalization of Calabi-Yau's theorem [34].

Theorem 4.4 *If M is a noncompact, complete manifold with $\mathrm{Ric}_f^N \geq 0$, assume f is bounded when N is infinite, then M has at least linear f-volume growth.*

Theorem 4.2 Part a) and Theorem 4.4 then together show that any manifold with $\mathrm{Ric}_f^N \geq \lambda > 0$ and f bounded if N is infinite must be compact. In fact, from the mean curvature estimates one can prove this directly and obtain an upper bound on the diameter. For finite N this is due to Qian [27], for Part b) see [32].

Theorem 4.5 (Myers' Theorem) *Let M be a complete Riemannian manifold with $\mathrm{Ric}_f^N \geq (n-1)H > 0$,*

a) when N is finite, then M is compact and $\mathrm{diam}_M \leq \sqrt{\frac{n+N-1}{n-1}}\frac{\pi}{\sqrt{H}}$.
b) when N is infinite and $|f| \leq k$ then M is compact and $\mathrm{diam}_M \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}$.

For some other Myers' Theorems for manifolds with measure see [10] and [24]. The relative volume comparison Theorem 4.2 also implies the following extensions of theorems of Gromov [11] and Anderson [2].

Theorem 4.6 *For the class of manifolds M^n with $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{diam}_M \leq D$ and $|f| \leq k$ ($|\nabla f| \leq a$), the first Betti number $b_1 \leq C(n + 4k, HD^2)$ ($C(n, HD^2, aD)$).*

Theorem 4.7 *For the class of manifolds M^n with $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{Vol}_f \geq V$, $\mathrm{diam}_M \leq D$ and $|f| \leq k$ ($|\nabla f| \leq a$) there are only finitely many isomorphism types of $\pi_1(M)$.*

5. Splitting theorem and excess estimate

An important application of the mean curvature comparison is the extension of the Cheeger-Gromoll splitting theorem. When N is finite and $\mathrm{Ric}_f^N \geq 0$, (3.4) gives

$$m_f(r) \leq \frac{n+N-1}{r}.$$

Using this, or (3.9) when N is infinite, and Bochner formulas for the Bakry-Emery Ricci tensor, and the arguments of the original proof of Cheeger-Gromoll's splitting theorem one can show the following splitting theorem.

Theorem 5.1 (Splitting Theorem) *Suppose M contains a line and $\mathrm{Ric}_f^N \geq 0$. When N is infinite, assume also f is bounded from above. Then $M = N^{n-1} \times \mathbb{R}$ and f is linear along the line.*

This version is due to Fang, Li, and Zhang [9]. In the case N is infinite if one assumes a two sided bound on f the splitting theorem was proven earlier by Lichneorwicz [17, 18]. Also see [32].

The following example shows that the upper bound on f is necessary for the theorem.

Example 5.2 *Let $M = \mathbb{H}^n$ be hyperbolic space. Fix any $p \in M$ and let $f(x) = (n-1)r^2 = (n-1)d^2(p,x)$. $\mathrm{Hess}\, r^2 = 2|\nabla r|^2 + 2r\mathrm{Hess}r \geq 2I$, therefore $\mathrm{Ric}_f \geq (n-1)$ and the space has many lines that do not split.*

Using the clever covering arguments in [8], Theorem 5.1 implies the following structure theorem for compact manifolds with $\mathrm{Ric}_f \geq 0$, which is a weaker assumption than $\mathrm{Ric}_f^N \geq 0$ for finite N.

Theorem 5.3 *[17] If M is compact and $\mathrm{Ric}_f \geq 0$ then M is finitely covered by $N \times T^k$ where N is a compact simply connected manifold and f is constant on the flat torus T^k.*

Theorem 5.3 has the following topological consequences.

Corollary 5.4 *[17] Let M be compact with $\mathrm{Ric}_f \geq 0$ then*

  1. $b_1(M) \leq n$.
  2. $\pi_1(M)$ has a free abelian subgroup of finite index of rank $\leq n$.
  3. $b_1(M)$ or $\pi_1(M)$ has a free abelian subgroup of rank n if and only if M is a flat torus and f is a constant function.
  4. $\pi_1(M)$ is finite if $\mathrm{Ric}_f > 0$ at one point.

For noncompact manifolds with positive Ricci curvature the splitting theorem has been used by Cheeger and Gromoll [8] and Sormani [29] to give some other topological obstructions. These results also can be extended.

Theorem 5.5 *Suppose M is a complete manifold with $\mathrm{Ric}_f^N > 0$, when N is infinite, assume f is bounded from above, then*

  1. M has only one end and
  2. M has the loops to infinity property.

In particular, if M is simply connected at infinity then M is simply connected.

The mean curvature comparisons can also be used to prove excess estimates. Recall that for $p, q \in M$ the excess function is $e_{p,q}(x) =$

$d(p, x) + d(q, x) - d(p, q)$. Let $h(x) = d(x, \gamma)$ where γ is a fixed minimal geodesic from p to q, then (3.8) along with the arguments in [1, Proposition 2.3] imply the following version of the Abresch-Gromoll excess estimate.

Theorem 5.6 (Excess Estimate) *Let* $\mathrm{Ric}_f \geq 0$, $|f| \leq k$ *and* $h(x) < \min\{d(p, x), d(q, x)\}$ *then*

$$e_{p,q}(x) \leq 2 \left(\frac{n + 4k - 1}{n + 4k - 2} \right) \left(\frac{1}{2} Ch^{n+4k} \right)^{\frac{1}{n+4k-1}}$$

where

$$C = 2 \left(\frac{n + 4k - 1}{n + 4k} \right) \left(\frac{1}{d(p, x) - h(x)} + \frac{1}{d(q, x) - h(x)} \right)$$

The version for $\mathrm{Ric}_f^N \geq 0$ and N finite is exactly the same, if we replace $4k$ by N. Example 5.2 shows that the assumption of bounded f is necessary.

Theorem 5.6 gives extensions of theorems of Abresch-Gromoll [1] and Sormani [28] to Ric_f^N.

Theorem 5.7 *Let be* M *a complete noncompact manifold with* $\mathrm{Ric}_f^N \geq 0$ *and assume* f *is bounded when* N *is infinite.*

1. *If* M *has bounded diameter growth and sectional curvature bounded below then* M *is homeomorphic to the interior of a compact manifold with boundary.*
2. *If* M *has sublinear diameter growth then* M *has finitely generated fundamental group.*

6. Scalar Curvature and Comments

For a compact manifold Corollaries 4.3 and 5.3 show that the topological obstructions on the fundamental group to having a metric with positive or nonnegative Ric_f are the same as for positive or nonnegative Ric. This raises the following question.

Question 6.1 *If* M^n *is a compact Riemannian manifold with a measure such that* $\mathrm{Ric}_f \geq (>)0$, *does* M^n *have a metric on it with* $\mathrm{Ric} \geq (>)0$?

There are compact shrinking soliton metrics which are not Einstein, but these examples have positive Ricci curvature. One could try to see if the $K3$ surface has a metric with $\mathrm{Ric}_f > 0$. If this were true it would give a negative answer to Question 6.1 because a $K3$ surface can not have a

metric on it with positive scalar curvature. Thus, it is also natural to consider the scalar curvature with measure.

As pointed out by Perelman in [26, 1.3], in order for the Lichnerovicz formula to hold, the corresponding scalar curvature equation is $S_f = 2\Delta f - |\nabla f|^2 + S$. Note that this is different than taking the trace of Ric_f which is $\Delta f + S$. Even though the Lichnerovicz theorem naturally extends to S_f, $\text{Ric}_f \geq 0$ doesn't immediately imply $S_f \geq 0$ so we have the following question.

Question 6.2 *If M^n is a compact spin manifold with $\text{Ric}_f > 0$, is the $\hat{A}$-genus zero?*

An affirmative answer to this question would show that a $K3$ surface does not have a metric on it with $\text{Ric}_f > 0$. We note that compact shrinking solitons have positive scalar curvature, so the answer to Question 6.2 is clearly yes for solitons.

In [21] Lott studied the N dimensional scalar curvature

$$S_f^N = S_f - \frac{1}{N}|\nabla f|^2.$$

He showed that analog of O'Neill's theorem holds for these modified scalar curvatures.

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