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GLOBAL BOUNDS FOR THE BETTI NUMBERS OF REGULAR FIBERS OF DIFFERENTIABLE MAPPINGS

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INTRODUCTION

IT IS well known that the Betti numbers of any fiber $p^{-1}(\xi)$ of a polynomial mapping p: $R^n \to R^m$ are bounded by some constants, depending only on n, m and the degree of p (see e.g. [7]).

Now let f be a k times differentiable mapping of a bounded domain, with all the derivatives of order k bounded by a constant M_k. We can think of M_k as a measure of the deviation of f from a polynomial mapping of degree $k-1$; as far as the deviation in a C^j-norm is concerned, $j \leq k-1$, the Taylor formula gives the precise expression for it.

The important general phenomenon is that also in much more delicate questions, concerning the topology and the geometry of the mapping f, its "deviation" from the "polynomial behavior" can be bounded in terms of M_k.

In [11] this fact was established for the structure of critical points and values of f, and in [12] for some geometric properties of its fibers.

The aim of the present paper is to extend in the same spirit to k-smooth mappings the property of polynomial ones, given above: the boundness of the Betti numbers of the fibers.

Clearly it is impossible to bound the Betti numbers of each fiber: any closed set can be the set of zeroes of a C^∞-smooth function. So the proper way to generalize the above property of polynomials is the following:

First, we prove for any f the existence of fibers with the Betti numbers bounded by constants, depending only on M_k (and, of course, on k and on the dimensions and the size of the domain and image of f).

Secondly, we estimate, in the same terms, the integrals over the image of the Betti numbers of the fibers of f. In particular, we answer a question concerning the conditions of integrability of the Banach indicatrix of a differentiable mapping, which was open for a long time (see [1], [2], [9]).

All the inequalities below have the following form: they consist of a term, corresponding to the case of polynomials, and of a "correction term", containing the factor M_k. Thus, for $M_k = 0$, i.e. for f a polynomial of degree $k-1$, we obtain, up to constants, the usual bounds.

The results below, as well as the results of [11] and [12] can be considered as the description of "the worst" possible behavior of k-smooth mappings. However, mainly they intend to answer another question: what can be said about the topology of a smooth or polynomial (of high degree) mapping, if the only information on its derivatives of order $\geq k$ (where k is fixed and "small") we want to use, concerns their uniform bounds.

Thus, we can reformulate most of results below (and of [11], [12]) for polynomials only, without mentioning differentiable functions at all. In this setting they show how to work with polynomials of high degree, as if they were polynomials of low degree.

Another important remark concerns the existence results below: in many cases we prove the existence of at least one value ξ in the image of f, for which the Betti numbers of the fiber $f^{-1}(\xi)$ are bounded by suitable constants. Although we do not touch in this paper the question of explicitly finding such values, we should mention that the corresponding results can be brought to a rather effective form: for instance, we can prove that in any regular net with a sufficiently small (explicitly given) step, there are points ξ with the required properties.

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§1. CONNECTION BETWEEN TOPOLOGY OF FIBERS AND GEOMETRY OF CRITICAL VALUES

Although all the results below remain valid, with minor modifications, for any compact manifold, we shall consider only mappings defined on a closed ball B_r^n of radius r in R^n. In this case all the constants involved can be given explicitly.

We say that the mapping $f: B_r^n \to R^m$ is q-smooth, where $q = p+\alpha$, $p \geqq 1$ an integer, $0 < \alpha \leqq 1$, if f is p times continuously differentiable on B_r^n, and the pth derivative $d^p f$ satisfies on B_r^n the Hölder condition:

$$\|d^p f(x) - d^p f(y)\| \leqq L\|x-y\|^\alpha, \tag{1}$$

with some constant L.

Let
$$M_i(f) = \max_{y\in B_r^n} \|d^i f(y)\|, \quad i = 0, 1, \ldots, p,$$

$M_q(f) =$ infinum of L in (1), and let $R_j(f) = M_j(f)r^j$, $j = 0, 1, \ldots, p, q$. (All the Euclidean spaces R^s and the spaces of their linear and multilinear mappings are considered with the usual Euclidean norms).

We always assume below that $n \geqq m$. Let $\Sigma(f)$ be the set of critical points of f, i.e. of points $x \in B_r^n$, where rank $df(x) < m$, or, if x belongs to the boundary S_r^{n-1} of B_r^n, rank $d(f/S_r^{n-1}) < m$. Let $\Delta(f) = F(\Sigma(f)) \subset R^m$ be the set of critical values of f.

For $\xi \in R^m$ we denote by Y_ξ the fiber $f^{-1}(\xi)$ of f over ξ. If ξ is a regular value of f, i.e. $\xi \notin \Delta(f)$, Y_ξ is a compact $n-m$-dimensional manifold. We denote by $b_i(Y_\xi)$, $i = 0, \ldots, n-m$, the ith Betti number of Y_ξ.

Let $\rho(\xi) = d(\xi, \Delta(f))$ be the distance from ξ to $\Delta(f)$.

THEOREM 1.1. *Let $f: B_r^n \to R^m$ be a q-smooth mapping, $q = p+\alpha$. Then for any regular value $\xi \in R^m$ of f, and $i = 0, \ldots, n-m$,*

$$b_i(Y_\xi) \leqq \begin{cases} B_i, & \rho(\xi) \geqq R_q(f) \\ B_i(R_q(f)/\rho(\xi))^{n/q}, & \rho(\xi) \leqq R_q(f), \end{cases}$$

where the constants B_i, $i = 0, \ldots, n-m$, depend only on n, m and p.

Proof. Below K_j denote constants depending only on n, m, p. We also omit sometimes the index f in the notations of $\Delta(f)$, $M_i(f)$ and $R_i(f)$.

Denote by B an open ball of radius $\rho(\xi)$, centered at the given regular value $\xi \in R^m - \Delta(f)$. All the points $\xi' \in B$ are regular values both of f and of the restriction f/S_r^{n-1}. Hence $f: N \to B$, where $N = f^{-1}(B)$, is a trivial fibration, and, in particular, we can find a retraction $\pi: N \to Y_\xi$, $\pi/Y = Id$.

We shall construct a semialgebraic set $S \subset N$, containing Y_ξ, such that the Betti numbers of S satisfy inequalities of theorem 1.1. The existence of a retraction $\pi: S \to Y_\xi$ then shows that the Betti numbers of Y_ξ do not exceed those of S.

For a given $\delta > 0$ let $I^\delta_{k_1\ldots k_n}$ be the cube $\{x = (x_1, \ldots, x_n) \in R^n / k_j\delta \leqq x_j \leqq (k_j+1)\delta, j = 1, \ldots, n\}$, $k_j \in Z$. Let I^δ_β, $\beta = 1, \ldots, K(\delta)$, be those of the cubes $I^\delta_{k_1\ldots k_n}$, which intersect B_r^n. Clearly, for $\delta \leqq r$, $K(\delta) \leqq K_1(r/\delta)^n$.

For each $\beta = 1, \ldots, K(\delta)$, take some point $x_\beta \in I^\delta_\beta \cap B_r^n$, and let P^δ_β be the Taylor polynomial of degree p of f at x_β. By Taylor formula we have for each $x \in I^\delta_\beta$: $\|f(x) - P^\delta_\beta(x)\| \leqq K_2 M_q \delta^q$.

Now take $\delta = \min(r, (\rho(\xi)/4K_2 M_q)^{1/q})$ and let

$$S_\beta = \{x \in I^\delta_\beta \cap B_r^n / \|P^\delta_\beta(x) - \xi\| \leqq \tfrac{1}{2}\rho(\xi)\}, \quad S = \bigcup_{1 \leqq \beta \leqq K(\delta)} S_\beta.$$

S is a semialgebraic set and we have $Y_\xi \subset S \subset N$. Indeed, by the choice of δ, $\|f(x) - P^\delta_\beta(x)\| \leqq \frac{1}{4}\rho(\xi)$ for $x \in I^\delta_\beta \cap B_r^n$. Hence, if $x \in Y_\xi \cap I^\delta_\beta$, $\|P^\delta_\beta(x) - \xi\| = \|P^\delta_\beta(x) - f(x)\| \leqq \frac{1}{4}\rho(\xi)$, i.e. $x \in S_\beta \subset S$.

Conversely, for $x \in S_\beta$, $\|f(x) - \xi\| \leqq \|f(x) - P^\delta_\beta(x)\| + \|P^\delta_\beta(x) - \xi\| \leqq \frac{1}{4}\rho(\xi) + \frac{1}{2}\rho(\xi) < \rho(\xi)$. i.e. $x \in N$.

It remains to estimate the Betti numbers of S. Each S_β is defined by polynomial inequalities, whose number depends only on n, and whose degrees do not exceed $2p$.

The same is true for any nonempty intersection of S_β (which occurs only if the corresponding cubes I_β^δ intersect). Hence for each $i = 0, \ldots, n-m$, $b_i(S_{\beta_1} \cap \ldots \cap S_{\beta_j}) \leqq B_i'$, where the constants B_i' depend only on n, m and p (Some explicit estimates of B_i' can be found by the methods of [6], [7], [8] or [10]).

Using the Mayer–Vietoris sequence, we obtain immediately that $b_i(S) \leqq B_i' K_3 K(\delta) \leqq B_i' K_3 K_1 (r/\delta)^n = B_i' K_3 K_1 (4K_2)^{n/q} (M_q r^q/\rho(\xi))^{n/q} = B_i (R_q/\rho(\xi))^{n/q}$.

These computations are valid for $\rho(\xi) \leqq R_q \leqq 4K_2 R_q$, since in this case $r \geqq (\rho(\xi)/4K_2 M_q)^{1/q}$, and we take δ equal to the last number. But for $\rho(\xi) > R_q$ we can restrict our consideration to the ball of radius R_q at ξ. Theorem 1.1 is proved.

Easy examples show that the bound of theorem 1.1 is sharp, up to constants.

§2. EXISTENCE OF FIBERS WITH SIMPLE TOPOLOGY

In this section we combine the result of theorem 1.1 with the information on the geometry of critical values of f, obtained in [11].

For a q-smooth f: $B_r^n \to R^m$ define $R_{1q}(f)$ as follows: $R_{1q}(f) = 2(V_m A(R_1(f) + R_q(f))^{m-1} R_q(f))^{1/m}$, where V_m is the volume of the unit ball in R^m, and $A = A(n, m, p)$, depending only on n, m, p, is twice the maximum of the constants $\overline{A}_i(n, m, p)$, $i = 0, \ldots, m$, defined in theorem 1.1, [11].

Denote $n - m + 1$ by s. Below we assume that the smoothness q of f is greater than s, and hence, by the Sard theorem, almost all values of f are regular.

THEOREM 2.1. *Let f: $B_r^n \to R^m$ be a q-smooth mapping, $q > s$. Then in any set $G \subset R^m$ with $m(G) > 0$ there is a regular value ξ of f, such that for $i = 0, \ldots, n-m$,*

$$b_i(Y_\xi) \leqq \begin{cases} B_i, & m(G) \geqq R_{1q}^m(f) \\ B_i (R_{1q}^m(f)/m(G))^{n/(q-s)}, & m(G) \leqq R_{1q}^m(f), \end{cases}$$

where $m(G)$ denotes the Lebesgue measure of G.

Proof. Let $G \subset R^m$ with $m(G) = \eta > 0$ be given. According to theorem 1.1, it is sufficient to find a point $\xi \in G$, which is "far away" from $\Delta(f)$.

We shall use theorem 1.1, [11], which gives an upper bound for the ε entropy of $\Delta(f)$ (see [2]), or, which is the same, for the minimal number $M(\varepsilon, \Delta(f))$ of balls of a given radius $\varepsilon > 0$, covering $\Delta(f)$. The following form of this bound, which can be deduced easily from the original general one, is appropriate for our case: For any $\varepsilon \leqq R_q(f)$,

$$M(\varepsilon, \Delta(f)) \leqq A(1/\varepsilon)^{m-1} (R_q(f)/\varepsilon)^{s/q} (R_1(f) + R_q(f))^{m-1}. \quad (2)$$

Now let $\varepsilon > 0$, $\varepsilon \leqq R_q$, be fixed. Cover Δ by $M(\varepsilon, \Delta)$ balls of radius ε, and let Ω_ε be the union of open balls of radius 2ε, centered at the same points. Ω_ε contains an ε neighborhood of Δ, and hence for any $\xi \in R^m \setminus \Omega_\varepsilon$, $d(\xi, \Delta) \geqq \varepsilon$ and by theorem 1.1, $b_i(Y_\xi) \leqq B_i(R_q/\varepsilon)^{n/q}$.

Denote by $C_i(t)$ the set of points $\xi \in R^m$, for which $b_i(Y_\xi) > t$. We obtain $C_i(B_i(R_q/\varepsilon)^{n/q}) \subset \Omega_\varepsilon$, for $\varepsilon \leqq R_q$, or $C_i(t) \subset \Omega_{\varepsilon(t)}$, where $\varepsilon(t) = R_q(B_i/t)^{q/n}$, $t \geqq B_i$.

By (2) for the measure of Ω_ε we have: $m(\Omega_\varepsilon) \leqq V_m 2^m \varepsilon^m M(\varepsilon, \Delta) \leqq V_m 2^m A(\varepsilon/R_q)^{1-s/q} (R_1 + R_q)^{m-1} R_q$, or

$$m(\Omega_\varepsilon) \leqq R_{1q}^m (\varepsilon/R_q)^{1-s/q}. \quad (3)$$

Substituting here the value of $\varepsilon(t)$ as above, we obtain the following:

PROPOSITION 2.2.

$$m(C_i(t)) \leqq \begin{cases} V_m R_1^m, & 0 \leqq t < B_i, \\ R_{1q}^m (B_i/t)^{(q-s)/n}, & t \geqq B_i \end{cases}.$$

The first inequality here means simply, that $b_i(Y) > 0$ only for $\xi \in f(B_r^n)$, and $f(B_r^n)$ clearly is contained in a ball of radius R_1.

Now if $m(C_i(t)) < \eta = m(G)$, then G contains some points $\xi \notin C_i(t)$, i.e. with $b_i(Y_\xi) \leq t$. It remains to note that by propoposition 2.2, $m(C_i(t)) < \eta$ is satisfied for $t = B_i$, if $\eta > R_{1q}^m$, and for any $t > B_i(R_{1q}^m/\eta)^{n/q-s}$, for $\eta \leq R_{1q}^m$. Theorem 2.1 is proved.

Notice that the use of the ε-entropy of critical values instead of the Lebesgue or the Hausdorff measure, and, respectively, the use of the stronger theorem 1.1 [11] instead of the Sard theorem, is the crucial point here: no bounds on the measure of $\Delta(f)$ allow to find points "far away" from this set.

The fiber Y_ξ in theorem 2.1 can be empty, for instance, if all the points of B_r^n are critical for f. Now we consider situations where nonempty fibers with simple topological structure can be found.

COROLLARY 2.3. *Let $f: B_r^n \to R^m$ be q-smooth, $q > s$, and let $m(f(B_r^n)) = \eta > 0$. Then there exists a nonempty fiber Y_ξ of f with*

$$b_i(Y_\xi) \leq \begin{cases} B_i, & \eta \geq R_{1q}^m(f) \\ B_i(R_{1q}^m(f)/\eta)^{n/(q-s)}, & \eta \leq R_{1q}^m(f). \end{cases}$$

These inequalities have specially simple form in the case $m = 1$:

COROLLARY 2.4. *Let $f: B_r^n \to R$ be a q-smooth function, $q > n$. Then in any set $G \subset R$ with $m(G) > 0$, there is a point c with*

$$b_i(Y_c) \leq \begin{cases} B_i, & m(G) \geq 4AR_q(f) \\ B_i(4AR_q(f)/m(G))^{n/q-n}, \cdot & m(G) \leq 4AR_q(f) \end{cases}$$

In particular, for $a = \min f$, $b = \max f$, there is c, $a < c < b$, such that

$$b_i(Y_c) \leq \begin{cases} B_i, & b-a \geq 4AR_q(f) \\ B_i(4AR_q(f)/(b-a))^{n/q-n}, & b-a \leq 4AR_q(f). \end{cases}$$

Let us formulate separately one important special case:

COROLLARY 2.5. *Let $f: R_r^n \to R$ be a q-smooth function, $q > n$, and let $\max f - \min f \geq 4AR_q(f)$. Then there exists c, $\min f < c < \max f$, such that $b_i(Y_c) \leq B_i$, where the constants B_i depend only on n and p.*

This corollary can be interpreted as the appearance of a "near-polynomiality" effect: if f is sufficiently close to a polynomial, in the sense that $R_q(f)$ is sufficiently small with respect to $\max f - \min f$, then the Betti numbers of at least one nonempty fiber of f satisfy exactly the same kind of inequalities as the Betti numbers of the polynomial fibers.

It is interesting to compare this fact with the result of [12], which indicates another appearance of the same effect: if for a q-smooth f, $\max f - \min f \geq 2^{q+1} R_q(f)$, then *any* fiber Y_c of f is similar to the fibers of a polynomial of degree p in the following sense: Y_c is contained in a countable union of compact smooth hypersurfaces in R^n, "many" straight lines cross Y_c in at most p points, and the $n-1$ volume $v(Y_c)$ is bounded by Kr^{n-1}, where K depends only on n and p. However, easy examples show that the Betti numbers of some fibers of f can be infinite.

The inequality of theorem 2.1 is rather precise. In example 1, §6, VI, [2], for any n and $q > n$ the function $f: B_1^n \to R$ is built with the following properties:

(i) f is q-smooth.

(ii) For any $\eta > 0$ there is an interval

$I_\eta \subset R$ of length η, such that for any $c \in I_\eta$, $b_i(Y_c) \geq K(1/\eta)^{n/q}$, $i = 0, \ldots, n-1$.

Hence the degree of $1/m(G)$ in the bounds for b_i cannot be smaller than n/q. Our value $n/q - s$ is "asymptotically" sharp, for $q \to \infty$.

Theorem 2.1 implies also the following fact: if there is at least one point $x \in B_r^n$, where the rank of $df(x)$ is maximal (equal to m), then the Betti numbers of some nonempty fibers of f

can be effectively bounded. As usual in our "quantitative" approach, we must not simply assume the nondegeneracy of the differential of f, but measure the degree of this nondegeneracy.

For a linear mapping $L: R^n \to R^m$, let $\omega(L)$ be the minimal semiaxis of the ellipsoid $L(B_1^n) \subset R^m$. For a smooth $f: B_r^n \to R^m$ define $\gamma(f)$ as $r \max_{x \in B_r^n} \omega(\mathrm{d}f(x))$. We also denote by $R_{12_q}(f)$ the constant $\sqrt{(20)}(1/V_m)^{1/2m}(R_{1_q}(f)R_2(f))^{1/2}$.

To simplify the expressions below, we assume, that $\gamma(f) \leqq R_2(f)$.

THEOREM 2.6. *Let $f: B_r^n \to R^m$ be a q-smooth mapping, $q > s$, with $0 < \gamma(f) \leqq R_2(f)$. Then there exists a nonempty fiber Y_ξ of f with*

$$b_i(Y_\xi) \leqq \begin{cases} B_i, & \gamma(f) \geqq R_{12_q}(f) \\ B_i(R_{12_q}(f)/\gamma(f))^{2mn/q-s}, & \gamma(f) \leqq R_{12_q}(f) \end{cases}$$

Proof. Fix some $x \in B_r^n$ with $\omega = \omega(\mathrm{d}f(x))$ maximal. Now let P be some m-dimensional plane through x, for which $\omega(\mathrm{d}f(x)/P) = \omega$.

Easy estimates, repeating the proof of the inverse function theorem, show that the ball of radius $\omega/3M_2$ in P (or the part of this ball, containing in B_r^n) is mapped by f diffeomorphically, and its image contains the ball of radius $\omega^2/20M_2 = \gamma(f)^2/20R_2(f)$. Hence $m(f(B_r^n)) \geqq V_m[\gamma(f)^2/20R_2(f)]^m$
. Substituting this value in the inequality of corollary 2.3, we obtain the required result.

Studying in more detail the structure of f in the case when rank $\mathrm{d}f < m$ everywhere, one can prove the existence of a nonempty fiber of f with Betti numbers bounded by constants depending only on $R_q(f)$ and the geometry of the image $f(B_r^n)$, for any sufficiently smooth mapping $f: B_r^n \to R^m$, with no assumptions of nondegeneracy. This proof requires considerations somewhat different from the ones used in this paper, and it will appear separately.

§3. AVERAGE COMPLEXITY OF THE FIBERS

In this section we give the bounds for the integrals of $b_i(Y_\xi)$, when ξ runs over R^m.

THEOREM 3.1. *Let $f: B_r^n \to R^m$ be a q-smooth mapping, $q > s$, and let $\upsilon > 0$ be given. Then for $i = 0, \ldots, n-m$,*

$$\int_{R^m} b_i^\upsilon(Y_\xi)\,\mathrm{d}\xi \leqq B_i^\upsilon[V_m R_1^m(f) + R_{1q}^m(f)\int_1^\infty (1/t)^{q-s/n\upsilon}\,\mathrm{d}t].$$

Proof. By the Fubini theorem, $\int_{R^m} b_i^\upsilon(Y_\xi)\,\mathrm{d}\xi = \int_0^\infty m(C_i(t^{1/\upsilon}))\,\mathrm{d}t$, and by proposition 2.2, the last integral is bounded by

$$\int_0^{B_i^\upsilon} V_m R_1^m\,\mathrm{d}t + \int_{B_i^\upsilon}^\infty R_{1q}^m(B_i/t^{1/\upsilon})^{q-s/n}\,\mathrm{d}t = B_i^\upsilon V_m R_1^m + B_i^\upsilon R_{1q}^m \int_1^\infty (1/t')^{q-s/n\upsilon}\,\mathrm{d}t'.$$

Theorem 3.1 is proved.

THEOREM 3.1 in particular answers the following question, which sometimes is called the question of integrability of the Banach indicatrix: for given $n \geqq m$ and $\upsilon > 0$ to find $q(n,m,\upsilon)$ such that for any q-smooth mapping $f: R^n \to R^m$ with compact support, $q > q(n,m,\upsilon)$, $\int_{R^m} b_0^\upsilon(Y_\xi)\,\mathrm{d}\xi < \infty$ (and, in particular, to prove the existence of such a $q(n,m,\upsilon)$).

Some special cases have been settled: the case $m = \upsilon = 1$, n arbitrary—in [9], the case $m = 1$, n and υ arbitrary—in [2], the cases $\upsilon = 1$, $n \geqq m$ arbitrary and $n = m$, υ arbitrary—in [1].

Theorem 3.1 implies immediately the following:

COROLLARY 3.2. *For $f\colon B_r^n \to R^m$ — a q-smooth mapping, $q > s$, and for a given υ, $0 \leqq \upsilon < q - s/n$,*

$$\int_{R^m} b_i^\upsilon(Y_\xi)\,d\xi \leqq B_i^\upsilon\left[V_m R_1^m(f) + R_{1q}^m(f)\frac{n\upsilon}{q-n\upsilon-s}\right] < \infty.$$

In particular, $q(n,m,\upsilon) \leqq \upsilon n + s = (\upsilon+1)n - m + 1$.

Examples of [2] show, that $q(n,m,\upsilon) \geqq \upsilon n$, so our bound for $q(n,m,\upsilon)$ is sharp asymptotically, for $\upsilon \to \infty$.

§4. VOLUME OF THE FIBERS

In this section, using the results of §3, we study the distribution of the volume of regular fibers of f. Here it is convenient first to obtain average bounds, and then to deduce the existence of fibers with "small" volume.

Let for ξ a regular value of $f\colon B_r^n \to R^m$, $v(Y_\xi)$ denote the $n-m$-dimensional volume of the compact $n-m$-dimensional submanifold Y_ξ in R^n.

THEOREM 4.1. *Let $f\colon B_r^n \to R^m$ be a q-smooth mapping, $\upsilon \geqq 1$. Assume that $q > m\upsilon + 1$. Then*

$$\int_{R^m} [v(Y_\xi)]^\upsilon\,d\xi \leqq B_0^\upsilon C^\upsilon\, r^{(n-m)\upsilon}\left[V_m R_1^m(f) + R_{1q}^m(f)\frac{m\upsilon}{q-m\upsilon-1}\right] < \infty,$$

where the constant C depends only on n and m.

Proof. By the standard integral-geometric formula, $v(Y_\xi) = \int_{G_n^m} b_0(Y_\xi \cap L)\,dL$, where G_n^m is the space of all the m-dimensional planes in R^n with the standard measure dL. $b_0(Y_\xi \cap L)$ here for almost all L is simply the number of points in $Y_\xi \cap L$.

The integration above runs, in fact, only over the set $H \subset G_n^m$ of planes L intersecting the ball B_r^n, and the measure of H in G_n^m is equal to Cr^{n-m}, where C depends only on n and m.

Hence $$\int_{R^m} [v(Y_\xi)]^\upsilon\,d\xi = \int_{R^m} d\xi\left[\int_H b_0(Y_\xi \cap L)\,dL\right]^\upsilon.$$

By the Hölder inequality,

$$\int_H b_0(Y_\xi \cap L)\,dL \leqq \left(\int_H [b_0(Y_\xi \cap L)]^\upsilon dL\right)^{1/\upsilon}\left(\int_H 1\,dL\right)^{1/\upsilon'},$$

where $\upsilon' = \dfrac{v}{v-1}$. Hence

$$\left[\int_H b_0(Y_\xi \cap L)\,dL\right]^\upsilon \leqq C^{\upsilon-1} r^{(n-m)(\upsilon-1)}\int_H [b_0(Y_\xi \cap L)]^\upsilon\,dL,$$

and by the Fubini theorem

$$\int_{R^m} [v(Y_\xi)]^\upsilon d\xi \leqq C^{\upsilon-1} r^{(n-m)(\upsilon-1)}\int_H dL \int_{R^m} [b_0(Y_\xi \cap L)]^\upsilon\,d\xi.$$

Now since $L \cap B_r^n$ is the ball of radius $\leqq r$ in $L \cong R^m$ and since all the derivatives of the restriction f/L do not exceed those of f, we have by corollary 3.2:

$$\int_{R^m} [b_0(Y_\xi \cap L)]^\upsilon d\xi \leqq B^0\left[V_m R_1^m + R_{1q}^m\frac{m\upsilon}{q-m\upsilon-1}\right],$$

and

$$\int_{R^m} [v(Y_\xi)]^\upsilon d\xi \leqq C^\upsilon r^{(n-m)\upsilon} B^0\left[V_m R_1^m + R_{1q}^m\frac{m\upsilon}{q-m\upsilon-1}\right].$$

Theorem 4.1 is proved.

The question of integrability of $v(Y_\xi)^\upsilon$ was also studied for a long time: for $n = 2, m = 1$ it was settled in [4], and in a general case in [5]. However, our estimate of maximal υ, for which the integral $\int_{R^m}[v(Y_\xi)]^\upsilon d\xi$ converges, namely, $\upsilon = q - 1/m$, is very close to the best possible, $\upsilon \leqq q/m$, and is approximately twice better than the Merkov's one [5]: $\upsilon < q/2m + 1$.

Using the inequality of theorem 4.1, we can obtain the existence of regular fibers with the "small" volume:

THEOREM 4.2. *Let $f: B_r^n \to R^m$ be a q-smooth mapping, $q > m+1$. Then for any $\beta < q - 1/m$, there is a constant K, depending only on $R_1(f)$, $R_q(f)$, β, n, m and p, such that in any $G \subset R^m$ there is ξ with*

$$v(Y_\xi) \leqq K(1/m(G))^{1/\beta}.$$

Proof. It follows immediately from the inequality of theorem 4.1, if we put

$$K = Cr^{n-m}B_0\left[V_m R_1^m + R_{1q}^m \frac{m\beta}{q - m\beta - 1}\right]^{1/\beta}.$$

The results of this section include the situations where the smoothness q of the mapping f is smaller than $s = n - m + 1$. In these cases all the values of f may be critical and, respectively, all the fibers Y_ξ of f may not be the regular $n-m$-dimensional manifolds. Here we understand $v(Y_\xi)$ as the $n-m$-dimensional Hausdorff measure.

§5. SOME INEQUALITIES BETWEEN THE DERIVATIVES OF f

In this section we show that all the constants in the inequalities above can be expressed in terms of the only two parameters of the mapping $f: B_r^n \to R^m$: the remainder term $R_q(f)$ and the diameter $R_0(f)$ of the image $f(B_r^n) \subset R^m$.

PROPOSITION 5.1. *There are constants $N_j, j = 1, \ldots, p$, depending only on n, m and p, such that for any $q = p + \alpha$-smooth mapping $f: B_r^n \to R^m$,*

$$R_j(f) \leqq N_j(R_0(f) + R_q(f)), \quad j = 1, \ldots, p.$$

Proof. For any polynomial mapping $h: B_r^n \to R^m$ of degree p the following Markov inequality is satisfied (see e.g. [3]):

$$R_j(h) \leqq N'_j R_0(h), \quad j = 1, \ldots, p. \qquad (*)$$

Now let h be the Taylor polynomial of f at the center of B_r^n. The Taylor formula shows that

$$R_j(h) - N''_j R_q(f) \leqq R_j(f) \leqq R_j(h) + N''_j R_q(f), j = 1, \ldots, p. \qquad (**)$$

Combining (*) and (**), we obtain the required inequalities.

COROLLARY 5.2. *There is a constant D, depending only on n, m, p, such that for any $q = p + \alpha$-smooth mapping $f: B_r^n \to R^m$*

(a) If $R_0(f) \geqq R_q(f)$, then

$$R_{1q}(f) \leqq D[R_0^{m-1}(f)R_q(f)]^{1/m},$$
$$R_{12q}(f) \leqq D[R_0^{2m-1}(f)R_q(f)]^{1/2m}.$$

(b) If $R_0(f) \leqq R_q(f)$, then

$$R_{1q}(f) \leqq DR_q(f),$$
$$R_{12q}(f) \leqq DR_q(f).$$

REFERENCES

1. S. A. GILEVIC: On the necessary condition of integrability of a Banach indicatrix of a differentiable mapping, *Isvest. AN SSSR, Ser. Mat.* **47**, 5 (1983), 1114–1134.
2. L. D. IVANOV: *Variazii mnozestv i funkzii*, Nauka, 1975.
3. O. D. KELLOGG: On bounded polynomials in several variables, *Math. Zschrift*, **27** (1928), 55–64.
4. E. M. LANDIS: On the length of level curves of a function of two variables, *Dokl. AN SSSR*, **79**, 3 (1951), 393–395.
5. A. B. MERKOV: On some properties of smooth mappings, *Vestnik mosk. Univ. Ser. Mat.* **2** (1979), 63–69.
6. J. MILNOR: On the Betti numbers of real varieties, *Proc. Am. Math. Soc.* **15** (1964), 275–280.
7. I. G. PETROVSKII: O. A. OLEINIK: On the topology of real algebraic surfaces, *Ivest. AN SSSR, Ser. Mat.* **13** (1949), 389–402.
8. R. THOM: Sur l'homologie des varietes algebriques reelles, In: *Differential and Combinatorial Topology. A Symposium in honour of Marston Morse*, Princeton Univ. Press, 1965, 255–265.
9. A. G. VITUSHKIN: *O mnogomernyh variaziah*, Gostehisdat, 1955.
10. H. WHITNEY: Elementary structure of real algebraic varieties, *Ann. Math.* **66** (1957), 545–556.
11. Y. YOMDIN: The geometry of critical and near-critical values of differentiable mappings, *Math. Ann.* **264** (1983), 495–515.
12. Y. YOMDIN: The set of zeros of an "almost polynomial" function, *Proc. Am. Math. Soc.*, **90**, (N4), (1984), 538–542.

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