Topology Guaranteeing Manifold Reconstruction using Distance Function to Noisy Data

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ABSTRACT

Given a smooth compact codimension one submanifold S of $\mathbb{R}^k$ and a compact approximation K of S, we prove that it is possible to reconstruct S and to approximate the medial axis of S with topological guarantees using unions of balls centered on K. We consider two notions of noisy-approximation that generalize sampling conditions introduced by Amenta & al. and Dey & al. Our results are based upon critical point theory for distance functions. For the two approximation conditions, we prove that the connected components of the boundary of unions of balls centered on K are isotopic to S. Our results allow to consider balls of different radii. For the first approximation condition, we also prove that a subset (known as the λ-medial axis) of the medial axis of $\mathbb{R}^k \setminus K$ is homotopy equivalent to the medial axis of S. We obtain similar results for smooth compact submanifolds S of $\mathbb{R}^k$ of any codimension.

Categories and Subject Descriptors

I.3.5 [**Computer Graphics**]: Computational Geometry and Object Modeling

General Terms

Theory, Algorithms

Keywords

distance function, sampling, surface and manifold reconstruction

1. INTRODUCTION

Motivation and previous works. Algorithms for surface reconstruction from point samples are required in many application areas such as reverse engineering, medical imaging or, more generally, each time a geometric model of an object must be built from (finite) measures. In last years, many such algorithms have been designed that, starting from a set of 3D point samples, build a polyhedral approximation of the sampled object.

In the continuation of a paper of Amenta et al [1], a family of reconstruction algorithms that provide topological guarantees have been designed.The most recent of them allow to deal with noisy samples [10, 18] or [13] extend similar techniques to the reconsruction of manifolds of any codimension embedded in $\mathbb{R}^d$. Here topological guarantee means that, under some assumptions on the sampled surface S and the sampling, the algorithm builds a geometric model that is homeomorphic or even isotopic to S. However, in these works, the proofs of the topological correctness are deeply intricated with the details of the algorithms or with some specificities of 3D Voronoi diagrams. In particular, the *poles* introduced in [1] play a central role: they allow to approximate the medial axis and the normals of the surface S from the Voronoi diagram of the sample. The relations between Voronoi diagrams, poles and medial axes show that capturing the topology of the surface or capturing the topology of its medial axis are strongly related problems. This suggests that these topological correctness proofs could be better understood in a more general mathematical framework. The expected outcomes of this framework are conditions and associated algorithms able to produce a topologically correct approximation of an object given partial and inaccurate geometrical approximations, not necessarily by finite sample points, in any dimension and for non-smooth objects. Based upon the critical point theory for distance functions to compact sets, this point of view has already brought some results on the medial axis topology and approximation [17, 5] and the computation of homotopy and homology groups of compact sets [6]. More recently, this approach has allowed to propose sampling conditions guaranteeing a topologically correct reconstruction of non-smooth objects in any dimension [4]. Beside this result for non-smooths objects, the smooth case desserves a specific study because it allows simpler sampling conditions with better constants.

A recent work of S. Smale et al [19] considers the question for smooth submanifolds of any dimension in Euclidean spaces. They introduce a uniform sampling condition related to the *reach* that is the minimum distance between the manifold and its medial axis. Based on this sampling condition, they show that an offset of the sampling bears the homotopy type of the sampled manifold.

Contribution. This paper presents some results obtained as a continuation of [19] in the context of our mathematical framework. First, under similar uniform noisy sampling conditions, we extend the result of [19] to get isotopic ap-

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proximations of hypersurfaces as well as a reconstruction of the medial axis of the manifold with the right homotopy type. Our uniform sampling condition is merely a ratio between the reach of the sampled manifold S and the Hausdorff distance between the sample and S. This sampling model allows a noise level whose amplitude is of the order of magnitude of the sampling density and does not require any sparsity condition. It is the same as in [19] except that it allows approximation by any compact set rather than finite point samples.

It requires no sparsity condition, and allows approximation by any compact sets, such as finite sets of geometric primitives like triangles (polygon soup).

Secondly, we extend the results to non-uniform sampling conditions. Our non-uniform sampling condition (see section 6) is the same as the notion of noisy r-sample without sparsity of [18] generalized to any compact sampling and not restricted to finite set of points. Theorem 6.2 below can be seen as an extension of the class of algorithms initiated by [1], considering a sampling density related to the local feature size, for any dimension of the manifold and the ambient space. The associated proposed algorithm is extremly simple: it consists of taking a union of balls centered on the sampling set. The radii of the balls may freely vary in prescribed intervals depending upon the local feature size of the manifold. From a more practical point of view it merely means computing an alpha-shape [14].

However, the algorithm suggested by theorem 6.2 requires an oracle: for each sample point we would need a lower bound of the local feature size of the projection of the point on the surface. In fact, usual local feature size based algorithms implicitly assume that one is able to adapt the density and accuracy of the sampling to the local feature size in order to produce a good sampling. So, in practice, ensuring that a sampling is a "good" sampling may require our oracle. Still, if the sampling is assumed good, algorithms described in [2, 12] does not require any oracle. For exact sampling condition, the oracle information is contained in the poles as they are known to approximate the medial axis. In presence of noise, a sparsity condition (see [10]) is required in order to still extract some information about "filtered poles". In order to relax the sparsity condition one needs some kind of oracle to filter the poles. In [18] the classical notion of r-sampling [1] is extended to noisy r-sampling without any sparsity condition. In this context, the proposed filtering of the poles only requires the knowledge of the reach (which plays the role of a global oracle) but constrains the value of r to be bounded by some constant of the order of the ratio between the minimum and maximum of the local feature size function over the surface (similar to the one in theorem 6.1). In this case, the non-uniformity of the sampling can not be fully exploited. In contrast, theorem 6.2 allows a local sampling density independant on this ratio. Of course, there is no hope to get rid of the oracle if we consider a noisy sampling without any sparsity condition. For example, given points sampled on a surface, if one replaces each point by a dense sampling of a tiny sphere, the relevant topology depends on the scale at which one observes the resulting points cloud. Our oracle plays the role of a local scale parameter.

We believe that, for the sake of clarity, the problem of reconstruction under non-uniform, local feature size related sampling conditions, could be split into two simpler independant problems. First, assuming minimal noise and/or sparsity conditions, how can one derive a lower bound on the local feature size exploiting only the point set. Secondly, starting from the sample point and the oracle, how can one produce a topologically correct reconstruction. Theorem 6.2 answers the second problem in a general setting.

The paper is organized as follow. Section 2 gives some definitions and recall results on distance functions and medial axis. In section 3 one defines uniform noisy sampling and studies distance function to such sampling. Section 4 presents topology guaranteeing algorithms for surface reconstruction with uniform sampling conditions. Section 5 gives results about toplogy guarantying algorithms for Medial axis approximation. Section 6 states result for surface recontruction with non-uniform sampling conditions.

Some technical proofs are not detailled here. They can be found in the longer research report version of this paper (see [7]).

2. PRELIMINARIES AND DISTANCE FUNCTIONS

Throughout the paper, we use the following notations. For any set $X \subset \mathbb{R}^k$, $\overline{X}$, X^c and ∂X denote respectively the closure, the complement and the boundary of X. For any $x \in \mathbb{R}^k$ and any $r > 0$, $\mathbb{B}(x, r)$ is the open ball of center x and radius r. Given two spaces X and Y, two maps $f : X \to Y$ and $g : X \to Y$ are said *homotopic* if there is a continuous map H, $H : [0, 1] \times X \to Y$, such that $\forall x \in X, H(0, x) = f(x)$ and $H(1, x) = g(x)$. X and Y are said *homotopy equivalent* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y. Homotopy equivalence between topological sets enforces a one-to-one correspondance between topological features of the two sets (connected components, cycles, holes,...) as well as the way these features are related. More precisely, if X and Y have same homotopy type, then their homotopy and homology groups are isomorphic. When $Y \subset X$, one says that Y is a *deformation retract* of X if one can continuously deform X onto Y i.e. there exists a continuous map $H : [0, 1] \times X \to X$ such that for any $x \in X$, $H(0, x) = x$ and $H(1, x) \in Y$ and for any $y \in Y, t \in [0, 1]$, $H(t, y) = y$. In this case, X and Y are homotopy equivalent. Two subsets X and Y of $\mathbb{R}^k$ are *isotopic* if there is a continuous map $F : X \times [0, 1] \to \mathbb{R}^k$ such that $F(., 0)$ is the identity of X, $F(X, 1) = Y$, and for each $t \in [0, 1]$, $F(., t)$ is a homeomorphism onto its image. Notice that isotopy is a stronger condition than homeomorphy.

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^k$ with compact boundary $K = \overline{\mathcal{O}} \cap \mathcal{O}^c$ and let $\mathcal{R}$ be the function defined on $\mathcal{O}$ by $\mathcal{R}(x) = d(x, K)$ for all $x \in \mathcal{O}$. For any point $x \in \mathcal{O}$, we denotes by $\Gamma(x)$ the set of closest boundary points:

$$\Gamma(x) = \{y \in K : d(x, y) = d(x, K)\}.$$

The *medial axis* $\mathcal{M}$ of $\mathcal{O}$ is the set of points $x \in \mathcal{O}$ that have at least two closest boundary points:

$$\mathcal{M} = \{x \in \mathcal{O} : \mid \Gamma(x) \mid \geq 2\}.$$

For a compact subset K of $\mathbb{R}^k$, the medial axis $\mathcal{M}(K)$ of K is the medial axis of its complement $\mathbb{R}^k \setminus K$. The function $\mathcal{R}$ is differentiable on $\mathcal{O} \setminus \overline{\mathcal{M}}$ (see [15]). Intuitively one can consider that a point x is regular for $\mathcal{R}$ if one can find a direction issued from x such that $\mathcal{R}$ is locally increasing linearly along this direction. Otherwise, x is said to be critical. Such an intuition coincides with the notion of critical point classicaly used in non-smooth analysis ([9]) and in Riemannian geometry ([8]):

Definition 2.1. *A point $x \in \mathcal{O}$ is critical for $\mathcal{R}$ if and only if it is contained in the convex hull of $\Gamma(x)$.*

Some of the properties of the distance function to a compact set are quite similar to the smooth functions ones. In particular, they satisfy an Isotopy Lemma [16] that we reproduce below. For any $\rho \in \mathbb{R}_+$ one denotes by $\mathcal{O}_\rho$ the open offset $\mathcal{O}_\rho = \{x \in \mathcal{O} : \mathcal{R}(x) > \rho\}$.

Proposition 2.2. *If $0 < \rho_1 < \rho_2$ are such that $(\overline{\mathcal{O}}_{\rho_1} \setminus \mathcal{O}_{\rho_2})$ does not contain any critical point of $\mathcal{R}$, then all the levels $\mathcal{R}^{-1}(\rho)$, $\rho \in [\rho_1, \rho_2]$, are homeomorphic topological manifolds and*

$$\overline{\mathcal{O}}_{\rho_1} \setminus \mathcal{O}_{\rho_2} = \{x \in \mathcal{O} : \rho_1 \leq \mathcal{R}(x) \leq \rho_2\}$$

is homeomorphic to $\mathcal{R}^{-1}(\rho_1) \times [\rho_1, \rho_2]$. As a consequence, $\mathcal{O}\rho_1$ and $\mathcal{O}\rho_2$ are homeomorphic.

In the following, we also consider some subset of the medial axis known as λ-medial axis [5]. For any point $x \in \mathcal{O}$ one denotes by $\mathcal{F}(x)$ the radius of the smallest ball containing $\Gamma(x)$. We thus define a function $\mathcal{F} : \mathcal{O} \to \mathbb{R}_+$ which is upper semi-continuous (see [5]) and satisfies $\mathcal{F}(x) \neq 0$ if and only if $x \in \mathcal{M}$. Given a positive real $\lambda > 0$ one defines the λ-medial axis of $\mathcal{O}$ as the closed subset $\mathcal{M}_\lambda$ of $\mathcal{M}$:

$$\mathcal{M}_\lambda = \{x \in \mathcal{O} : \mathcal{F}(x) \geq \lambda\}$$

Topological properties of the medial axis and its subsets have been studied in [17, 5] for bounded open sets. In the following we consider unbounded open sets that are the complements of compact subsets of $\mathbb{R}^k$. To avoid problems with non-bounded open sets, we consider the complement of these compact restricted to a sufficiently big ball.

Definition 2.3. *Let $K \subset \mathbb{R}^k$ be a compact subset of $\mathbb{R}^k$ and let $D > 0$ be the distance between the origin O of $\mathbb{R}^k$ and the farthest point of K from O. The bounded medial axis of K (resp. the bounded λ-medial axis), is the medial axis (resp. λ-medial axis) of the complement of K intersected with the open ball $\mathbb{B}(O, 10D)$ of center O and radius $10D$:*

$$\begin{aligned} \mathcal{BM}(K) &= \mathcal{M}(\mathbb{B}(0, 10D) \setminus K) \\ \mathcal{BM}_\lambda(K) &= \mathcal{M}_\lambda(\mathbb{B}(0, 10D) \setminus K) \end{aligned}$$

Such a definition should be considered as a technical trick to avoid unboundness problems. Remark that $\mathbb{B}(0, 10D) \setminus K$ and $\mathbb{R}^k \setminus K$ are homeomorphic and thus homotopy equivalent. $\mathcal{BM}(K)$ and $\mathbb{B}(0, 10D) \setminus K$ are homotopy equivalent [17] and if $\lambda < reach(K)$ (see next section for a definition of reach), then $\mathcal{BM}_\lambda$ and $\mathbb{B}(0, 10D) \setminus K$ are homotopy equivalent [5].

3. UNIFORM NOISY APPROXIMATION

In this section we study the location of the critical points of the distance function to a compact that approximate uniformly a given smooth manifold. The results restate and extend some results of [19]. Our approach allow to derive in a simple way results on reconstruction and medial axis approximation.

Let $S \subset \mathbb{R}^k$ be a compact smooth manifold of any codimension and let $\mathcal{M}$ be its medial axis. The *local feature size* of S is the function $\mathrm{lfs} : S \to \mathbb{R}_+$ defined by

$$lfs(x) = d(x, \mathcal{M}) = \inf\{d(x, y) : y \in \mathcal{M}\}.$$

Notice that since lfs is a distance function, it is 1-Lipschitz. The infimum of lfs is known as the *reach* of S ([15]) and is denoted *reach*(S). The distance of a point $x \in \mathbb{R}^k$ to S is denoted by $\mathcal{R}(x) = \inf\{d(x, y) : y \in S\}$. For any point $x \in \mathbb{R}^k \setminus \mathcal{M}$, the projection $\Pi(x)$ of x on S is the unique point on S such that $d(x, \Pi(x)) = \mathcal{R}(x)$.

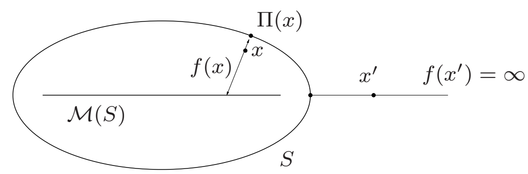

Figure 1: Definition of function f

We also denote by $f : \mathbb{R}^k \setminus (S \cup \mathcal{M}) \to \mathbb{R}_+ \cup \{+\infty\}$ the function defined by $f(x)$ is the distance between $\Pi(x)$ and the first intersection point of the half-line $[\Pi(x), x)$ (which is normal to S) with the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ (see figure 1). Remark that lfs and f are related by

$$reach(S) \leq \mathrm{lfs}(\Pi(x)) \leq f(x) \quad \text{for} \quad x \notin (S \cup \mathcal{M}). \qquad (1)$$

Definition 3.1. *A compact set $\mathcal{K} \subset \mathbb{R}^k$ is a uniform noisy ε-approximation of S if $d_H(\mathcal{K}, S) < \varepsilon reach(S)$.*

Notice that we do not make any assumption neither on the finiteness nor on the geometric structure of $\mathcal{K}$. The case when $\mathcal{K}$ is a finite set of points is of particular interest for applications, but as mentionned in the introduction, considering some other sets may be relevant from a practical point of view.

In the following $\mathcal{K} \subset \mathbb{R}^k$ denotes a uniform noisy ε-approximation of S. Let $\tilde{\mathcal{R}}$ be the distance function to the compact set $\mathcal{K}$. The functions $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are related by the following inequality

$$| \mathcal{R}(x) - \tilde{\mathcal{R}}(x) | < \varepsilon\tau \quad \text{for any} \quad x \in \mathbb{R}^k. \qquad (2)$$

Next lemma is an extension of proposition 7.1 in [19]. A result of the same kind is also obtained in [11] under more restrictive hypothesis (only finite sets of point sampled exactely on a surface $S \subset \mathbb{R}^3$ are considered).

Proposition 3.2. *Let $\varepsilon < 1/6 \simeq 0.1667$, let $\mathcal{K}$ be a uniform noisy ε-approximation of S and let $\tau = reach(S)$. Let $x \in \mathbb{R}^n \setminus S$ satisfaying one of the two following conditions:*

condition 1: *$f(x)$ is finite and*

$$\frac{5}{2}\varepsilon\tau < \mathcal{R}(x) < (1 - \frac{7}{2}\varepsilon) f(x).$$

condition 2: *$f(x) = +\infty$ and $\mathcal{R}(x) > \varepsilon\tau$.*

Then x is not a critical point of $\tilde{\mathcal{R}}$.
Moreover, if one denotes by $N_{\Pi(x)}$ the half-line passing throught x and normal to S at $\Pi(x)$, then, in case 1, the

function $\tilde{\mathcal{R}}$ is strictly increasing along the connected component of $N_{\Pi(x)} \cap \mathcal{R}^{-1}\left([\frac{5}{2}\varepsilon\tau, (1-\frac{7}{2}\varepsilon)f(x)]\right)$ that contains $\Pi(x)$.
In case 2, $\tilde{\mathcal{R}}$ is strictly increasing along the half-line $N_{\Pi(x)} \cap \mathcal{R}^{-1}\left(]\varepsilon\tau, +\infty[\right)$.

Proof. To simplify notations, one introduces $E = \varepsilon\tau$. Let S_E be the offset manifold $S_E = \{x \in \mathcal{R}^k : d(x,S) = E\}$. $\mathcal{K}$ is contained in the tubular neighborhood $Tub_E(S) = \{x \in \mathbb{R}^k : d(x,S) < E\}$.
First, suppose that x satisfies condition 1. Let $c \in \overline{\mathcal{M}}$ be such that $d(c, \Pi(x)) = f(x)$, x is contained in the segment $[c, \Pi(x)]$ and the ball $\mathbb{B}(c, f(x))$ of center c and radius $f(x)$ is tangent to S at $\Pi(x)$ (see figure 2). Notice that, since the open ball $\mathbb{B}(c, f(x))$ is contained in $\mathbb{R}^k \setminus S$, the ball $\mathbb{B}(c, f(x) - E)$ is contained in the complement of $Tub_E(S)$.

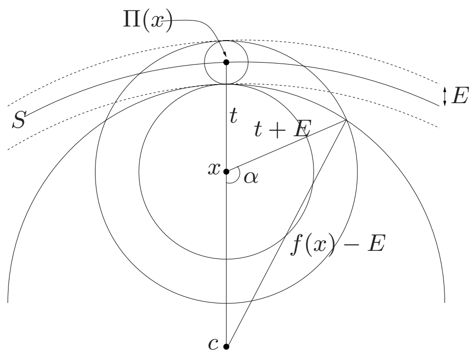

Figure 2:

Denoting by $t = d(x, \Pi(x)) = \mathcal{R}(x)$, it follows from hypothesis that the ball $\mathbb{B}(x, t-E)$ does not contain any point of $\mathcal{K}$. Since $\mathbb{B}(\Pi(x), E)$ intersects $\mathcal{K}$ and is contained in $\mathbb{B}(x, t+E)$, the ball $\mathbb{B}(x, t+E)$ contains at least one point of $\mathcal{K}$. The radius of the maximal ball $\mathbb{B}_{max}(x)$ contained in $\mathbb{R}^k \setminus \mathcal{K}$ with center x is thus contained in $[t-E; t+E]$. Moreover, the points of $\mathcal{K}$ which are on the boundary of $\mathbb{B}_{max}(x)$ are contained in $\mathbb{B}(x, t+E) \setminus \mathbb{B}(c, f(x) - E)$.
It follows from the definition of critical point that whenever the part of the sphere $\mathbb{S}(x, t+E)$ which is not contained in $\mathbb{B}(c, f(x) - E)$ is less than an hemisphere, x is a regular point of $\tilde{\mathcal{R}}$. This condition is equivalent to $\cos\alpha < 0$ where α is the angle between $[xc]$ and any segment joining x to a point of the $(k-2)$-sphere $\mathbb{S}(x, t+E) \cap \mathbb{S}(c, f(x) - E)$ (see figure 2).
Using the relations between the lengths of the edges of a triangle, α satisfies the following relation

$$(f(x)-E)^2 = (f(x)-t)^2 + (t+E)^2 - 2(f(x)-t)(t+E)\cos\alpha.$$

Since $(f(x)-t)(t+E) > 0$, $\cos\alpha < 0$ if and only if

$$(f(x)-t)^2 + (t+E)^2 - (f(x)-E)^2 < 0$$

or:

$$2(t^2 + t(E - f(x)) + 2f(x)E) < 0.$$

The discriminant of this equation is equal to $E^2 - 6f(x)E + f(x)^2 = f(x)^2((\tau\varepsilon/f(x))^2 - 6(\tau\varepsilon/f(x)) + 1)$. It is positive whenever $\varepsilon < 3 - 2\sqrt{2} \simeq 0.1715$ because $\tau \leq f(x)$. In this case, the roots of equation $t^2 + t(E - f(x)) + 2f(x)E = 0$ are given by

$$t_{\pm} = \frac{f(x) - E \pm f(x)\sqrt{1 - 6E/f(x) + E^2/f(x)^2}}{2}$$

Using that $\sqrt{1-u} > 1-u$ for any $u \in]0,1]$ and that $\tau \leq f(x)$, one immediately deduces that

$$\begin{aligned} t_+ &> f(x) - \frac{7}{2}E + \frac{E^2}{2f(x)} \\ &= f(x)(1 - \frac{7}{2}\frac{\tau}{f(x)}\varepsilon + \left(\frac{\tau}{f(x)}\right)^2 \frac{\varepsilon^2}{2}) \\ &> (1 - \frac{7}{2}\varepsilon)f(x) \end{aligned}$$

and

$$\begin{aligned} t_- &< \frac{5}{2}E - \frac{E^2}{2f(x)} \\ &= f(x)(\frac{5}{2}\frac{\tau}{f(x)}\varepsilon - \left(\frac{\tau}{f(x)}\right)^2 \frac{\varepsilon^2}{2}) \\ &< \frac{5}{2}\varepsilon\tau \end{aligned}$$

The first statement of the lemma follows from that if $t = \mathcal{R}(x) \in]t_-, t_+[$, then x is a regular value of $\tilde{\mathcal{R}}$.

Suppose now that x satisfies condition 2. From $f(x) = +\infty$, one deduces that x is not contained in the convex hull of S. Since moreover $\mathcal{R}(x) > \varepsilon\tau$, one has that x is not contained in the convex hull of $\mathcal{K}$. From the remark following the definition of critical point, it follows that x is not a critical point.

To prove the second part of the lemma, we just have to remark that, under both conditions, the angle between the vector colinear to $N_{\Pi(x)}$ and pointing away from S and any segment joining x to a point of $\mathcal{K} \cap \mathbb{S}(x, \tilde{\mathcal{R}}(x))$ is greater than $\pi/2$. $\mathcal{K} \cap \mathbb{S}(x, \tilde{\mathcal{R}}(x))$ being compact, the infimum of these angles is greater than $\pi/2$. It follows from [16], lemma 1.5 that the function $\tilde{\mathcal{R}}$ restricted to $N_{\Pi(x)}$ is strictly increasing around x. □

Proposition 3.2 implies the following result that was first proven in [19].

Corollary 3.3. **([19], prop. 7.1)** *Let $\varepsilon < 1/8 \simeq 0.125$ and let $\mathcal{K}$ be a uniform noisy ε-approximation of S. If $\alpha \in [\frac{7}{2}\varepsilon\tau, (1-\frac{9}{2}\varepsilon)\tau]$, then S is a deformation retract of the union of balls $U_\alpha = \bigcup_{e \in \mathcal{K}} \mathbb{B}(e, \alpha) = \tilde{\mathcal{R}}^{-1}([0, \alpha[)$*

Proof. See [7]. □

4. APPLICATIONS TO HYPERSURFACE RECONSTRUCTION

Proposition 3.2 implies the following result for hypersurfaces embedded in $\mathbb{R}^k$.

Theorem 4.1. *Let S be a smooth compact connected hypersurface embedded in $\mathbb{R}^k$ with positive reach $\tau > 0$. Let $0 < \varepsilon < 1/10$ and $\mathcal{K}$ be a uniform noisy ε-approximation of S. For any value $\alpha \in [\frac{7}{2}\varepsilon\tau, (1-\frac{9}{2}\varepsilon)\tau]$ the boundary of the union U_α of balls of radii α and centers the point of $\mathcal{K}$, $U_\alpha = \bigcup_{e \in \mathcal{K}} \mathbb{B}(e, \alpha)$, contains two connected components, each of one isotopic to S.*

Proof. We use the second part of proposition 3.2 of previous section to prove that the restriction of Π to any connected component $\tilde{S}_\alpha$ of the boundary of U_α is an homeomorphism. The isotopy between S and $\tilde{S}_\alpha$ is then realised by "pushing" $\tilde{S}_\alpha$ onto S along the normals of S. Since S is a connected hypersurface, $\mathbb{R}^k \setminus S$ contains two connected components denoted by $\mathcal{O}_i$ and $\mathcal{O}_e$. Let consider, for example, the component $\tilde{S}_\alpha$ of the boundary of U_α which is contained in $\mathcal{O}_i$. Since α is a regular value of $\tilde{\mathcal{R}}$, $\tilde{S}_\alpha$ is a compact $\mathcal{C}^0$ hypersurface in $\mathcal{O}_i$ (proposition 2.2).

Claim: *For any $p \in S$, the half line N_p issued from p, normal to S and pointing into $\mathcal{O}_i$ meets $\tilde{S}_\alpha$ in exactly one point in $\mathcal{R}^{-1}\left([\frac{5}{2}\varepsilon\tau, (1-\frac{7}{2}\varepsilon)\tau]\right)$.*

First notice that $\tilde{\mathcal{R}}(p) < \varepsilon\tau < \alpha$. Since $\tilde{\mathcal{R}}$ is continuous and unbounded on N_p there exists some point $x \in N_p$ such that $\tilde{\mathcal{R}}(x) = \text{lfs}(p) > \alpha$ and N_p intersects $\tilde{S}_\alpha$. Now, let $y \in N_p$ be such that $\tilde{\mathcal{R}}(y) = \alpha$. Inequality (2) implies $\frac{5}{2}\varepsilon\tau < \mathcal{R}(y) < (1-\frac{7}{2}\varepsilon)\tau$. It follows from proposition 3.2 of previous section that $\tilde{\mathcal{R}}$ is strictly increasing along the segment $N_p \cap \mathcal{R}^{-1}\left([\frac{5}{2}\varepsilon\tau, (1-\frac{7}{2}\varepsilon)\tau]\right)$, so y is the unique point of N_p satisfaying $\tilde{\mathcal{R}}(y) = \alpha$. This proves the claim.

The end of the proof of theorem now follows easily from the claim: the restriction of Π to $\tilde{S}_\alpha$ is thus a continuous bijective map. The hypersurfaces S and $\tilde{S}_\alpha$ being compact, it is thus an homeomorphism. □

Remark that assuming connectedness of S in previous theorem is not necessary. By taking care of the definition of S_α, one can easily give a similar statement when S contains several components. The previous proof can be adapted to manifolds S of any codimension. One thus obtains that $\tilde{S}_\alpha$ is a $\mathcal{C}^0$ hypersurface isotopic to the boundary of the tubular neighborhood of S of sufficiently small radius.

5. APPROXIMATION OF λ-MEDIAL AXIS WITH TOPOLOGICAL GUARANTIES

Let S be a smooth compact submanifold of $\mathbb{R}^k$ of any codimension and let $\mathcal{BM}_\lambda$ be the bounded λ-medial axis of S. Given $\mathcal{K}$ a uniform noisy ε-approximation of S and $\lambda > 0$, one denotes by $\mathcal{BM}_\lambda(\mathcal{K})$ the bounded λ-medial axis of $\mathcal{K}$ and by $U_\lambda = \tilde{\mathcal{R}}^{-1}([0, \lambda[)$ the union of balls of radii λ and centers the points of $\mathcal{K}$.

Lemma 5.1. *Let $\varepsilon < 1/8 \simeq 0.125$ and let $\mathcal{K}$ be a uniform noisy ε-approximation of S. If $\lambda \in [\frac{7}{2}\varepsilon\tau, (1-\frac{9}{2}\varepsilon)\tau]$, then $\mathbb{R}^k \setminus U_\lambda$ is a deformation retract of $\mathbb{R}^k \setminus S$.*

Proof. See [7] □

It follows from previous lemma that $\mathbb{R}^k \setminus S$ and $\mathbb{R}^k \setminus U_\lambda$ are homotopy equivalent. Using results from [6], we can relate the homotopy type of $\mathbb{R}^k \setminus U_\lambda$ to the one of $\mathcal{BM}_\lambda(\mathcal{K})$.

Theorem 5.2. *Let S be a smooth compact submanifold of $\mathbb{R}^k$. Let $\varepsilon < 1/8$ and let $\mathcal{K}$ be a uniform noisy ε-approximation of S. For any value $\lambda \in [\frac{7}{2}\varepsilon\tau, (1-\frac{9}{2}\varepsilon)\tau]$, $\mathcal{BM}_\lambda(\mathcal{K})$ and $\mathbb{R}^k \setminus S$ are homotopy equivalent.*

Proof. It is proven in [5], theorem 2, that if λ is not a critical value of $\tilde{\mathcal{R}}$, then the open set $\mathbb{R}^k \setminus \overline{U_\lambda}$ and the bounded λ-medial axis $\mathcal{BM}_\lambda(\mathcal{K})$ are homotopy equivalent. Since, in our case, $\lambda \in [\frac{7}{2}\varepsilon\tau, (1-\frac{9}{2}\varepsilon)\tau]$, it follows from proposition 3.2 that λ is a regular value of $\tilde{\mathcal{R}}$. The theorem is thus an immediate consequence of lemma 5.1. □

An important particular case is when $\mathcal{K}$ is a finite sample of points and S is the boundary of a bounded open set $\mathcal{O}$. In this case, the bounded λ-medial axis of $\mathcal{K}$ is a subcomplex $Vor_\lambda(\mathcal{K})$ of the Voronoï diagram $Vor(\mathcal{K})$ of $\mathcal{K}$ (see [5]).

Corollary 5.3. *Let S be a smooth compact hypersurface of $\mathbb{R}^k$ that is the boundary of a bounded open set $\mathcal{O}$. Let $\varepsilon < 1/8$ and let $\mathcal{K}$ be a finite set of points which is a uniform noisy ε-approximation of S. For any value $\lambda \in [\frac{7}{2}\varepsilon\tau, (1-\frac{9}{2}\varepsilon)\tau]$, $Vor_\lambda(\mathcal{K})$ and $\mathcal{O}$ are homotopy equivalent.*

In [5] we proved that, under hypothesis of previous lemma, $Vor_\lambda(\mathcal{K})$ is an approximation of $\mathcal{M}_\lambda(\mathcal{O})$ for Hausdorff distance that may be easily computed from the Voronoï diagram of $\mathcal{K}$. So, previous lemma insures that the algorithm given in [5] to approximate the λ-medial axis of $\mathcal{O}$ provides an output which has the homotopy type of $\mathbb{R}^k \setminus S$. The parameter λ being choosen smaller than τ, it follows that $\mathcal{O}$ has the homotopy type of $\mathcal{M}(\mathcal{O})$ that has itself the homotopy type of $\mathcal{M}_\lambda(\mathcal{O})$ ([5], theorem 2).

6. NON-UNIFORM APPROXIMATIONS

In practical applications it may be useful to use non-uniform approximations. For example, one may want to have more precise approximation in the areas where the manifold has a small lfs and less precise approximation in the areas where the manifold has big lfs. We thus now consider the following notion of approximation.

Definition 6.1. *Let $S \subset \mathbb{R}^k$ be a compact manifold with positive reach and let $\varepsilon > 0$. A compact set $\mathcal{K} \subset \mathbb{R}^k$ is a (non-uniform) noisy ε-approximation of S if it satisfies the following conditions:*

- *for any $e \in \mathcal{K}$, $d(e, \Pi(e)) < \varepsilon \text{lfs}(\Pi(e))$,*
- *for any $p \in S$, there exists a point $e \in \mathcal{K}$ such that $d(p, \Pi(e)) < \varepsilon \text{lfs}(p)$.*

Remark that in the case where $\mathcal{K}$ is a finite sample of points, the second condition is equivalent to $\Pi(\mathcal{K})$ is an ε-sample of S as defined in [1] and our sampling condition is almost the same as the one introduced in [10] and [18]. Using that lfs is 1-Lipschitz, the proof of proposition 3.2 adapts quite easily to give a result on the location of the critical points of the distance function to $\mathcal{K}$ (see [7] lemma 6.2). This leads to the following result on hypersurface reconstruction.

Theorem 6.1. *Let S be a smooth compact connected hypersurface embedded in $\mathbb{R}^k$ with positive reach $\tau > 0$ and let $M = \sup_{X \in S} \text{lfs}(X)$. Let $0 < \varepsilon < 1/32$ be such that $(27M + 37\tau)\varepsilon < 2\tau$ and let $\mathcal{K}$ be a noisy ε-approximation of S. For any value $\alpha \in [\frac{27}{2}\varepsilon M, (1-\frac{37}{2}\varepsilon)\tau]$ the boundary of the union of balls $U_\alpha = \bigcup_{e \in \mathcal{K}} \mathbb{B}(e, \alpha)$, contains two connected components, each of one isotopic to S.*

Proof. See [7] □

The main drawback of theorem 6.1 is twofold. First, it imposes to consider balls of constant radius. Second, the condition on ε involves the ratio between the minimum and the maximum of the the lfs function. It thus follows that if ε fullfills condition of previous theorem, then the compact $\mathcal{K}$ is in fact a uniform noisy approximation of S. The following theorem improves theorem 6.1.

Let S be a smooth compact submanifold of $\mathbb{R}^k$. Let $\varepsilon > 0$ and $\mathcal{K}$ be a noisy ε-approximation of S. For any family $r = (r_e)_{e\in\mathcal{K}}$ of positive real numbers such that $r_e = \alpha_e \mathrm{lfs}(\Pi(e))$, $0 < \alpha_e < 1$, one denotes by $\mathcal{K}(r)$ the union of balls

$$\mathcal{K}(r) = \bigcup_{e\in\mathcal{K}} \mathbb{B}(e, r_e).$$

THEOREM 6.2. *Let $\varepsilon < 1/160$, $a = 1/20$, $b = 1/10$ and let $\mathcal{K}$ be a noisy ε-approximation of S. If $r = (r_e)_{e\in\mathcal{K}}$ is such that $a \le \alpha_e \le b$ then*

- *S is a deformation retract of $\mathcal{K}(r)$,*
- *$\mathcal{K}(r)$ is homeomorphic to any tubular neighborhood $\mathcal{R}^{-1}([0, d))$ of S, $d < reach(S)$,*
- *the boundary $\partial\mathcal{K}(r)$ of $\mathcal{K}(r)$ is an hypersurface isotopic to $\mathcal{R}^{-1}(d)$.*

Notice that when S is a codimension one submanifold, $\partial\mathcal{K}(r)$ is isotopic to two copies of S. Values given here for a, b and ε have been choosen arbitrarily. We show in the proof of theorem that conclusion still holds for any triplet (a, b, ε) of values that verify an explicit but technical inequality.
Notice that the theorem is still valid if $\mathrm{lfs}(\pi(x))$ is replaced by a 1-Lipschitz lower bound of it. Note also that given a lower bound of $\mathrm{lfs}(\pi(x))$ for each point, it is not diffcult to propagate a 1-Lipschitz one.

PROOF. The proof follows from a sequence of lemmas.

Let $g : \mathbb{R}^k \setminus \overline{\mathcal{M}(S)} \to \mathbb{R}_+$ defined by $g(x) = \mathcal{R}(x)/\mathrm{lfs}(\Pi(x))$. Since $\mathcal{R}$, lfs and Π are continuous functions, $g = \mathcal{R}/(\mathrm{lfs} \circ \Pi)$ is a continuous fonction.

LEMMA 6.2. *(Isotopy lemma) Let $a, b \in]0, 1[$ be such that $a < b$. The level set $g^{-1}(a)$ is an hypersurface isotopic to $\mathcal{R}^{-1}(d)$ for any $0 < d < reach(S)$ and $g^{-1}([a, b])$ is homeomorphic to $g^{-1}(a) \times [a, b]$. Moreover for any $x \in S$ and any normal half-line N_x issued from x, g is strictly increasing along $g^{-1}([a, b]) \cap N_x$.*

PROOF. See [7]. □

Next lemma shows that the boundary of $\mathcal{K}(r)$ is enclosed between two sublevel sets of g.

LEMMA 6.3. *Let $a' = (a-\varepsilon)(1-\varepsilon)-\varepsilon$ and $b' = \frac{b+\varepsilon}{1-2(b+\varepsilon)}$. One has*

$$g^{-1}([0, a']) \subset \mathcal{K}(r) \subset g^{-1}([0, b'[).$$

PROOF. It is based upon the fact that lfs is a 1-Lipschitz function. See [7]. □

The next lemma (and its corollary) is the key argument for the proof of theorem 6.2. It shows that distance function to $\mathcal{K}(r)$ restricted to the normals of S is strictly increasing between $\mathcal{K}(r)$ and $g^{-1}(b')$.

LEMMA 6.4. *Let $x \in g^{-1}([a', b']) \setminus \mathcal{K}(r)$, let $X = \Pi(x)$ and let $l_X = [X, x)$ be the half-line normal to S and issued from X passing through x. For any $e \in \mathcal{K}$ such that the ball of maximal radius centered on x and contained in $\mathcal{K}(r)^c$ meets $\overline{\mathbb{B}}(e, r_e)$, the distance to e restricted to l_X is strictly increasing in a neighborhood of x.*

PROOF. To prove the lemma, one introduces a few notations. Let $E = \Pi(e)$, $d = d(x, \mathbb{B}(e, r_e)) \ge 0$, $t = d(x, X)$ and let $c \in l_X$ be the center of the ball of radius $\mathrm{lfs}(X)$ tangent to S at X (see figure 3). Notice that $\mathbb{B}(c, \mathrm{lfs}(X)) \cap S = \emptyset$ and that $a'\mathrm{lfs}(X) \le t < b'\mathrm{lfs}(X)$.

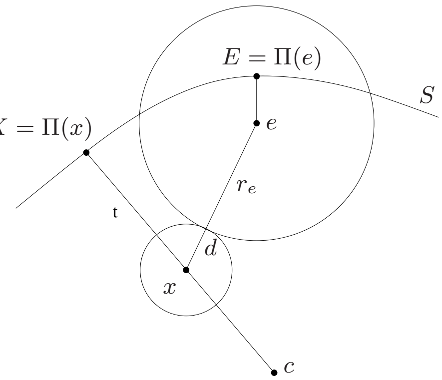

Figure 3:

To prove that the distance to e restricted to l_X is strictly increasing in a neighborhood of x, it sufficies to show that the angle between the vectors $\overrightarrow{xc}$ and $\overrightarrow{xe}$ is greater than $\pi/2$. Such a condition is satisfied as soon as

$$d(x, e)^2 + d(c, x)^2 < d(c, e)^2. \quad (3)$$

Since $\mathbb{B}(c, \mathrm{lfs}(X)) \cap S = \emptyset$, $d(c, E) > \mathrm{lfs}(X)$. Using triangular inequality it follows that $d(c, e) > \mathrm{lfs}(X) - \varepsilon\mathrm{lfs}(E)$. So inequality (3) is satisfied as soon as

$$d(x, e)^2 + d(c, x)^2 < (\mathrm{lfs}(X) - \varepsilon\mathrm{lfs}(E))^2. \quad (4)$$

Now, using that $d(c, x) = \mathrm{lfs}(X) - t < (1 - a')\mathrm{lfs}(X)$, one obtains that inequality (4) is satisfied as soon as

$$d(x, e)^2 + (1 - a')^2\mathrm{lfs}(X)^2 < (\mathrm{lfs}(X) - \varepsilon\mathrm{lfs}(E))^2. \quad (5)$$

It now remains to bound $d(x, e)$ and $\mathrm{lfs}(E)$.
One has $d(x, e) = d + r_e < d + b\mathrm{lfs}(E)$. Using lemma 6.3, one deduces that $d < t - a'\mathrm{lfs}(X) < (b' - a')\mathrm{lfs}(X)$. It follows that

$$d(x, e) < (b' - a')\mathrm{lfs}(X) + b\mathrm{lfs}(E). \quad (6)$$

Recall that lfs is 1-Lipschitz so $\mid \mathrm{lfs}(X) - \mathrm{lfs}(E) \mid \le d(X, E) < t + d + r_e + \varepsilon\mathrm{lfs}(E)$. It follows that

$$\mid \mathrm{lfs}(X) - \mathrm{lfs}(E) \mid < (2b' - a')\mathrm{lfs}(X) + (b + \varepsilon)\mathrm{lfs}(E)$$

which implies

$$\mathrm{lfs}(E) < \frac{1 + 2b' - a'}{1 - (b + \varepsilon)}\mathrm{lfs}(X) \quad (7)$$

Combining this last inequality with (6), one obtains

$$d(x, e) < \left((b' - a') + \frac{b(1 + 2b' - a')}{1 - (b + \varepsilon)}\right)\mathrm{lfs}(X) \quad (8)$$

One also deduces from (7) that

$$\mathrm{lfs}(X) - \varepsilon\mathrm{lfs}(E) > \left(1 - \frac{\varepsilon(1 + 2b' - a')}{1 - (b + \varepsilon)}\right)\mathrm{lfs}(X) \quad (9)$$

From (8) and (9) and dividing by $\mathrm{lfs}(X)^2$ one finally obtains the following inequality.

PROPOSITION 6.5. *Inequality (5) is satisfied as soon as*

$$(1-a')^2 \quad + \quad \left((b'-a') + \frac{b(1+2b'-a')}{1-(b+\varepsilon)}\right)^2 \quad < \quad \left(1 - \frac{\varepsilon(1+2b'-a')}{1-(b+\varepsilon)}\right)^2$$

An easy numerical computation shows that this last condition is satisfied for $\varepsilon = 1/160$, $a = 1/20$ and $b = 1/10$. This concludes the proof of lemma 6.4. □

From lemma 6.4 one deduces the following corollary.

COROLLARY 6.6. *Let x be as in lemma 6.4. Then the distance function to $\mathcal{K}(r)$ restricted to l_X is strictly increasing in a neighborhood of x.*

PROOF. It is based upon a compactness argument and is detailed in [7]. □

LEMMA 6.7. *Restricted to $g^{-1}([0,b'])$, any half-line l_X normal to S and issued from a point $X \in S$ intersects $\partial\mathcal{K}(r)$ in a unique point.*

PROOF. Denote by N_X the connected component of $l_X \cap g^{-1}([0,b'])$ that contains X and denote by $t \to x(t) = X + tb'\mathrm{lfs}(X)\vec{n_X}$ a parametrization of N_X where n_X is the unitary vector normal to S at X which spans l_X. It follows from lemma 6.3 that $x(0) \in \mathcal{K}(r)$ and $x(1) \in \mathcal{K}(r)^c$. So N_X intersects $\partial\mathcal{K}(r)$. It follows from corollary 6.6 that once $x(t) \in \mathcal{K}(r)^c$ distance function to $\mathcal{K}(r)$ restricted to N_X is strictly increasing in a neighborhood of x. So $x(t)$ cannot re-enter into $\mathcal{K}(r)$. This proves the lemma. □

We are now able to prove theorem 6.2. This is done by "pushing" $\partial\mathcal{K}(r)$ onto $g^{-1}(a')$ along the normals to S. For any $x \in g^{-1}(a')$ denote by $X = \Pi(x)$ and by $\varphi(x)$ the first intersection point of the half-line $l_X = [X, x)$ with $\partial\mathcal{K}(r)$. One thus defines a map $\varphi : g^{-1}(a') \to \partial\mathcal{K}(r)$. Using continuity of Π and of the field of half-lines normal to S, one easily check that φ is continuous. Remark that restricted to $g^{-1}([0,b'])$, l_X intersects $g^{-1}(a')$ in a unique point. It follows that φ is a bijection. The subsets $g^{-1}(a')$ and $\partial\mathcal{K}(r)$ being compact, φ is thus an homeomorphism. The map $\Phi : \partial\mathcal{K}(r) \times [0,1] \to \mathbb{R}^k$ defined by $\Phi(x,t) = \varphi(x) - t\overrightarrow{x\varphi(x)}$ is an isotopy between $\partial\mathcal{K}(r)$ and $g^{-1}(a')$. The same map can be easily extended and used to define a deformation retraction of $\mathcal{K}(r)$ onto $g^{-1}([0,a'])$ Proof of theorem 6.2 now follows from lemma 6.2.

It is important to remark that the proof of theorem 6.2 does not restrict to the numerical values of a, b and ε given in its statement. For example, it also works with $a = 0.09$, $b = 0.1$ and $\varepsilon = 1/60 \simeq 0.0167$ or $a = b = 0.11$ and $\varepsilon = 1/50 \simeq 0.02$.

THEOREM 6.8. *For any values ε, a and b that satisfies the inequality in proposition 6.5 with $a' = (a-\varepsilon)(1-\varepsilon) - \varepsilon$ and $b' = \frac{b+\varepsilon}{1-2(b+\varepsilon)}$, conclusion of theorem 6.2 still holds.*

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