# SCALABLE SPACES

ALEKSANDR BERDNIKOV AND FEDOR MANIN

Abstract. *Scalable spaces* are simply connected compact manifolds or finite complexes whose real cohomology algebra embeds in their algebra of (flat) differential forms. This is a rational homotopy invariant property and all scalable spaces are formal; indeed, scalability can be thought of as a metric version of formality. They are also characterized by particularly nice behavior from the point of view of quantitative homotopy theory. Among other results, we show that spaces which are formal but not scalable provide counterexamples to Gromov's long-standing conjecture on distortion in higher homotopy groups.

## 1. Introduction

This paper resolves two distinct problems in quantitative homotopy theory using a common analytic method. Quantitative homotopy theory is a program which seeks to understand the geometry of the maps and homotopies whose existence is mandated by the powerful, but often indirect methods of algebraic topology. Specifically, we seek to understand the Lipschitz constant of such maps: the Lipschitz constant tells us the scale at which the map becomes homotopically trivial, and therefore is a good measure of homotopical information. Besides the inherent appeal of this program, it is important for achieving an understanding of broader questions in quantitative geometric topology, for example the questions regarding cobordism theory studied in [FW13] and [CDMW18]. This program has its roots in Gromov's 1978 paper [Gro78] and was outlined in detail in his 1990s works [Gro98, Ch. 7] and [Gro99]; in the past several years, much progress has been made by various people including the current authors; see [FW13] [CDMW18] [CMW18] [Gut17] [Manarb] [MW18] [Manara] [Ber18].

A different, more structure-oriented lens through which to see the paper is that of formal spaces, a notion introduced by Sullivan in [Sul77]. A formal space is one whose rational homotopy type is a "formal consequence" of its rational cohomology ring; that is, there are no higher-order relations between the cohomology classes. However, Sullivan gives two other characterizations: one in terms of quasi-isomorphisms (maps preserving cohomology) and another in terms of rational self-maps. Scalability, the main notion introduced here, satisfies two similar equivalent conditions, but with a metric flavor. This, it turns out, has important consequences for other quantitative properties.

1.1. **Growth, distortion, Lipschitz homotopy.** Let $X$ and $Y$ be sufficiently nice compact metric spaces, for example Riemannian manifolds or simplicial complexes. In [Gro99], Gromov outlines a number of homotopical invariants concerning the asymptotic behavior of the Lipschitz functional on the mapping space $\operatorname{Map}(X, Y)$.

The most basic such question pertains to *growth*: how many elements of the set of homotopy classes $[X, Y]$ have representatives with Lipschitz constant $\leq L$? This line of inquiry goes back to [Gro78], in which Gromov proved the following:

**Theorem.** *For a simply connected compact Riemannian manifold $Y$, the growth of $\pi_n(Y)$ is at most polynomial in $L$.*

The proof derives from rational homotopy theory. Sullivan, following K.-T. Chen, had showed that all real-valued invariants $\pi_n(Y) \to \mathbb{R}$ could be computed by pulling back differential forms along a map $f : S^n \to Y$, taking wedges and antidifferentials, and finally integrating a resulting $n$-form over the sphere. Gromov remarked that all steps of this procedure could be bounded polynomially in terms of the Lipschitz constant of the original map.

In [Gro99], Gromov conjectured that the upper bounds on the homotopical complexity of $L$-Lipschitz maps obtained in this way are sharp. To make this precise, it is natural to define the *distortion* of an element $\alpha \in \pi_n(Y)$ to be

$$\delta_\alpha(L) = \max\{k : k\alpha \text{ has an } L\text{-Lipschitz representative}\}^1.$$

Then Gromov's conjecture would imply that the distortion of any element is $\Theta(L^r)$ where $r$ is an integer. Moreover, an easily stated consequence is:

**Conjecture** (Gromov)**.** *The distortion of an element $\alpha \in \pi_n(Y)$ is $\Theta(L^n)$ if and only if $\alpha$ has nontrivial image under the rational Hurewicz homomorphism. Moreover, otherwise its distortion is $\Omega(L^{n+1})$.*

The "if" here is easy to see using a degree argument; the "only if" has been open until now, and Gromov remarked that even a proof that the distortion is $\omega(L^n)$ would be remarkable.

Finally, Gromov also defined a related relative invariant: given two homotopic $L$-Lipschitz maps, we can ask for bounds on the Lipschitz constants of the intermediate maps of a homotopy. For example, given nice compact spaces $X$ and $Y$, when can we expect two homotopic $L$-Lipschitz maps $X \to Y$ to be homotopic through $KL$-Lipschitz maps, for some constant $K = K(X, Y)$? Ferry and Weinberger noted that for the applications they were considering, it was more useful to also bound the Lipschitz constant in the time direction. Hence:

**Question.** *For what spaces $Y$ is there always a constant $K = K(X, Y)$, for any compact metric simplicial complex $X$, such that any two homotopic $L$-Lipschitz maps $X \to Y$ have a $K(L + 1)$-Lipschitz homotopy?*

Ferry and Weinberger characterized spaces satisfying a more restrictive condition, where the constant only depends on the dimension $d$ of $X$. In that case, all homotopy groups of $Y$ must be finite. On the other hand, it was shown in [CDMW18] that spaces satisfying the above condition include those that are rationally products of Eilenberg–MacLane spaces, including for example odd-dimensional spheres. This paper also includes the first example of a target space $Y$ which does not have this property. Moreover, in [CMW18] it was shown that even even-dimensional spheres do not have the property as stated; to include them in our class, we must consider only nullhomotopic maps.

A number of weaker, polynomial bounds on sizes of homotopies and nullhomotopies appear in [CMW18] and [Manarb], but before this paper, linearity had only been additionally proven in the case of maps $S^3 \to S^2$, by the first author [Ber18].

The various quantities described here are intimately connected. For example, in [CDMW18], it is shown that if one attaches a cell along an element of $\pi_n(Y)$ with sufficiently large distortion, then the resulting complex is forced to have nonlinear nullhomotopies. Conversely, the argument of [Manara] describing the growth of $[X, Y]$ for certain $X$ and $Y$ relies on estimates on the sizes of Lipschitz homotopies.

[1] This is essentially the inverse function of the notion used in [Gro99], but accords with the notion of distortion used in geometric group theory.

| **Symmetric spaces** | **Scalable spaces** | **Formal spaces** |
|---|---|---|
| $S^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ | $\mathbb{C}P^2 \# \mathbb{C}P^2$ | $(\mathbb{C}P^2)^{\#r}$, $r \geq 4$ |
| Grassmannians | $(\mathbb{C}P^2)^{\#3} \# (\overline{\mathbb{C}P^2})^{\#3}$ | $\mathbb{C}P^n \# \mathbb{C}P^n$, $n \geq 3$ |
| | $(S^n \times S^n)^{\#r}$, $r \leq \binom{2n}{n}/2$ | $(S^n \times S^n)^{\#r}$, $r > \binom{2n}{n}/2$ |

Table 1. A Venn diagram of simply connected manifolds.

1.2. **Main results.** The main result of this paper defines a new class of spaces in which the answers to these questions are particularly nice.

**Theorem A.** *The following are equivalent for a* formal *simply connected finite complex $Y$:*

*(i) There is a homomorphism $H^*(Y) \to \Omega^*_\flat Y$ of differential graded algebras which sends each cohomology class to a representative of that class. Here $\Omega^*_\flat Y$ denotes the flat forms, an algebra of not-necessarily-smooth differential forms studied by Whitney.*

*(ii) There is a constant $C(Y)$ and infinitely many (indeed, a logarithmically dense set of) $p \in \mathbb{N}$ such that there is a $C(Y)(p+1)$-Lipschitz self-map which induces multiplication by $p^n$ on $H^n(Y;\mathbb{R})$.*

*(iii) For all finite simplicial complexes $X$, nullhomotopic $L$-Lipschitz maps $X \to Y$ have $C(X,Y)(L+1)$-Lipschitz nullhomotopies.*

*(iv) For all $n < \dim Y$, homotopic $L$-Lipschitz maps $S^n \to Y$ have $C(X,Y)(L+1)$-Lipschitz homotopies.*

**Remark 1.1.** The conditions (i) and (ii) imply formality of $Y$ almost immediately and in fact can be seen as geometric strengthenings of two equivalent characterizations of formality given by Sullivan. In §6, we give an example of a non-formal space which satisfies (iv) but not (iii). It is not clear whether (iii) implies formality.

On the other hand, condition (i) is strictly weaker than the notion of "geometric formality" introduced by Kotschick [Kot01] based on Sullivan's observation that it is satisfied by symmetric spaces, and studied by several others. For example, all simply connected geometrically formal 4-manifolds are rationally equivalent to $S^4$, $\mathbb{C}P^2$, or $S^2 \times S^2$.

We call spaces satisfying (i)–(iv) *scalable* based on the scaling maps of (ii). Examples of scalable spaces include spheres, projective spaces, and other symmetric spaces of compact type. More examples of spaces known to be scalable and those known not to be scalable are given in Table 1.

We summarize some properties of scalable spaces below.

**Theorem B** (Properties of scalable spaces)**.**

*(a) Scalability is invariant under rational homotopy equivalence.*

*(b) The class of scalable spaces is closed under products and wedge products.*

*(c) All skeleta of scalable complexes are scalable.*

*(d) Scalable spaces satisfy Gromov's distortion conjecture; in fact, the distortion function of an element of $\pi_n(Y)$ is easily deduced from the Sullivan minimal model of $Y$ if $Y$ is scalable.*

On the other hand, we show that the distortion conjecture does not always hold for non-scalable spaces, even those that are formal:

**Theorem C.** *The class of the puncture in $\pi_5([(\mathbb{C}P^2)^{\#4} \times S^2]^\circ)$ has distortion $o(L^6)$.*

We do not, however, know any matching lower bounds on distortion besides the trivial $L^5$, nor do we have upper bounds stronger than the already known $L^6$. We merely show that the known upper bound cannot be sharp. This is similar to the situation for Lipschitz homotopies of non-scalable formal spaces: we show that they cannot have linear Lipschitz constant, but we do not give any other lower bound for the sizes of homotopies. This contrasts with the examples given in [CDMW18] and [CMW18], which include an explicit lower bound.

Finally, applying Theorem A to maps between wedges of spheres yields the following:

**Corollary 1.2.** *For every rational number $r \geq 4$, there are spaces $X_r$ and $Y_r$ such that the growth of $[X_r, Y_r]$ is $\Theta(L^r)$.*

The argument is found in [Manara], where linearity of homotopies in this case was conjectured.

1.3. **Methods.** Here we discuss the techniques used in the proof of Theorem A, as well as in deciding whether a space is scalable in some of the more delicate cases.

To decide scalability, we use condition (i) of Theorem A. To prove that a closed, formal $n$-manifold $Y$ is not scalable, we show a local obstruction. Indeed, by Poincaré duality, for some point $p \in Y$, a map as in (i) restricts to a graded algebra embedding of $H^*(Y;\mathbb{R})$ in $\bigwedge^* T_pY$. We discuss several families of manifolds for which this is impossible. Conversely, in some cases we are able to extend a local embedding of $H^*(Y;\mathbb{R})$ in a single tangent space to an embedding into $\Omega^*(Y)$.

It is tempting to conjecture that this can always be done; that is, that one can always extend an embedding of $H^*(Y;\mathbb{R})$ at one point (when $Y$ is a closed manifold) or several points (otherwise) to an embedding into $\Omega^*(Y)$. This would imply the following additional criterion for scalability:

**Optimistic conjecture.** *A space is scalable if and only if it is formal and $H^*(Y;\mathbb{R})$ embeds in $\bigwedge^* \mathbb{R}^N$ for some finite $N$.*

In particular, scalability would then depend only on real homotopy type—itself an open problem:

**Question.** *Is scalability an $\mathbb{R}$- as well as a $\mathbb{Q}$-homotopy invariant?*

We now discuss techniques used in the proof of Theorem A. The most novel of these is used in showing (ii) $\Rightarrow$ (i). Given a sequence of self-maps, we move to the sequence of induced maps $\mathcal{M}_Y^* \to \Omega^* X$ in the world of rational homotopy. These can be formally scaled so that the corresponding geometric bounds are uniform; then by a compactness theorem, we can find an accumulation point, although this requires us to expand the space of forms to one which is complete. This accumulation point is the map of (i).

The same technique, in combination with the algebraic impossibility discussed above, is used to prove Theorem C. This proof is reminiscent of the work of Wenger [Wen11] showing that there are nilpotent groups whose Dehn function is not exactly polynomial. There the role of the limiting object obtained after scaling is played by the asymptotic cone, and one can use the algebraic structure of the nilpotent group to prove the nonexistence of a filling with certain bounds. Since nilpotent groups can also be studied using rational homotopy theory, it would be interesting to get a stronger handle on the formal similarities between these arguments.

The converse (i) $\Rightarrow$ (ii) is an easy consequence of the shadowing principle of [Manarb]. This allows formal, rational homotopy-theoretic maps and homotopies to be upgraded to actual maps between spaces with only a linear deterioration in geometric bounds.

A more involved application of the shadowing principle is the direction (ii) $\Rightarrow$ (iii). This is a generalization of the first author's argument [Ber18] proving that maps $S^3 \to S^2$ have linear nullhomotopies, which we summarize as follows. A map $f : S^3 \to S^2$, after some local regularization, looks like a bowl of spaghetti: the cross-sections of each spaghetti strand map homeomorphically to $S^2 \setminus$ (south pole) and the air in the bowl maps to the south pole. If the map is nullhomotopic, one can nullhomotope it in linear space and time by gradually combing the spaghetti on larger and larger scales: scale 2, scale 4, and so on up to $2^{\log(\mathrm{Lip}\, f)}$. Each step takes twice as long as the previous one, but there are logarithmically many steps total, making for a linear bound. Finally the last map is well-organized enough to be nullhomotoped by hand.

Alternatively, one can look at these intermediate maps like this: they locally look like they factor through a larger- and larger-degree self-map of $S^2$. To implement a similar process in greater generality, we use the shadowing principle. Given a map $f : X \to Y$, where $Y$ is a scalable space, we create "combed" versions of $f$ by formally scaling it down, finding a nearby genuine map, and then using a scaling self-map of $Y$ to scale it back up. Adjacent such maps are then homotopic via reasonably short formal homotopies, which can again be upgraded to genuine homotopies.

Finally, the direction (iii) $\Rightarrow$ (iv) is obvious and the direction (iv) $\Rightarrow$ (ii) can be done by constructing self-maps skeleton by skeleton.

1.4. **How to read this paper.** Section 2 proves some simple facts about linear algebra which allow us to show that certain spaces are not scalable. Section 3 gives examples of some of the phenomena which occur in non-scalable spaces, one of which is the proof of Theorem C. In section 4, we show that certain high-dimensional manifolds are scalable, beyond the obvious examples of symmetric spaces and their wedges and products. These sections do not require any knowledge of rational homotopy theory and provide examples of most of the phenomena discussed in the paper.

In section 5, we discuss rational homotopy theory and its relationship to quantitative results, introducing necessary facts from [Manarb] and the necessary results on flat differential forms. The remaining sections all use this material in an essential way. The reader who is interested in a slower-paced introduction to the subject is invited to consult [Manarb] for a treatment focusing on quantitative results or a textbook on the subject such as [GM81]. Section 6 discusses an example which demonstrates that our methods don't extend straightforwardly to non-formal spaces. Finally, section 7 gives the proof of Theorem A; one particularly technical point is banished to an additional final section.

1.5. **Acknowledgements.** We would like to thank Robert Young, who pointed out the reference [Wen11], as well as Robin Elliott and Shmuel Weinberger for other useful comments. The second author was supported by the NSF via the individual grant DMS-1906516.

## 2. Obstructions to scalability

Suppose $X^n$ is a scalable closed manifold, equipped with an embedding $i : H^*(X;\mathbb{R}) \hookrightarrow \Omega^* X$. In particular the fundamental class is sent to a form which is nonzero at some point $x \in X$. But then Poincaré duality implies that the restriction $i|_{T_x X}$ is already an embedding of $H^*(X;\mathbb{R})$ into $\bigwedge^* \mathbb{R}^n$. This proves:

**Proposition 2.1.** *If $X$ is a scalable closed $n$-manifold, then the rank of $H^k(X;\mathbb{R})$ is at most $\binom{n}{k}$. In particular, $(S^n \times S^m)^{\# r}$ is not scalable for $r > \binom{m+n}{n}/2$.*

This is clearly only an issue for closed manifolds; for example, arbitrary wedges of spheres and manifold thickenings thereof are scalable.

Here we point out some slightly more subtle reasons that certain cohomology algebras cannot be embedded in the alternating algebra $\bigwedge^* V$ for any finite-dimensional $\mathbb{R}$-vector space $V$.

**Theorem 2.2.** *The following graded algebras cannot be embedded in $\bigwedge^* V$ for any $V = \mathbb{R}^N$:*

*(i) For all $n \geq 1$, the algebra*

$$\Omega_{n,r} = \left\langle a_i^{(n)}, b_i^{(n)}\ (1 \leq i \leq r) \mid a_i b_i = a_j b_j, a_i a_j = b_i b_j = 0\ \forall i, j, a_i b_j = 0\ \forall i \neq j \right\rangle$$

*for $r > \frac{1}{2}\binom{2n}{n}$. (On the other hand, $\Omega_{n,\frac{1}{2}\binom{2n}{n}}$ embeds in $\bigwedge^* \mathbb{R}^{2n}$.)*

*(ii) For all even $n \geq 2$, the algebra*

$$\Sigma_{n,r} = \left\langle a_i^{(n)}\ (1 \leq i \leq r) \mid a_i^2 = a_j^2, a_i a_j = 0\ \forall i \neq j \right\rangle$$

*for all $r > \frac{1}{2}\binom{2n}{n}$. (On the other hand, $\Sigma_{n,\frac{1}{2}\binom{2n}{n}}$ embeds in $\bigwedge^* \mathbb{R}^{2n}$.)*

*(iii) For all $n \geq 3$, the algebra*

$$\Pi_{n,r} = \left\langle a_i^{(2)}\ (1 \leq i \leq r) \mid a_i^n = a_j^n, a_i a_j = 0\ \forall i \neq j \right\rangle$$

*for all $r > 1$.*

**Corollary 2.3.** *The following spaces are not scalable:*

*(i) $(\mathbb{C}P^2)^{\#p} \# (\overline{\mathbb{C}P^2})^{\#q}$ when either $p > 3$ or $q > 3$.*
*(ii) $(\mathbb{H}P^2)^{\#p} \# (\overline{\mathbb{H}P^2})^{\#q}$ when either $p > 35$ or $q > 35$.*
*(iii) $(\mathbb{O}P^2)^{\#p} \# (\overline{\mathbb{O}P^2})^{\#q}$ when either $p > 6435$ or $q > 6435$.*
*(iv) $(\mathbb{C}P^n)^{\#r}$ for $n \geq 3$ and $r > 1$.*

*Proof.* In all the cases, as above, we can restrict an embedding in $\bigwedge^* \mathbb{R}^N$ to a subspace $\mathbb{R}^{2n} \subset \mathbb{R}^N$ on which the top class is nontrivial. Moreover, this restriction is still an embedding since each of the algebras satisfies Poincaré duality, in the sense that its multiplication defines a bilinear pairing between elements of degree $k$ and degree $2n - k$.

*Case (i):* As mentioned above, if $r > \binom{2n}{n}/2$, the number of $n$-dimensional generators is greater than the dimension of $\bigwedge^n \mathbb{R}^{2n}$, and therefore an embedding cannot exist.

Conversely, suppose that $r = \binom{2n}{n}/2$ and let $\mathbb{R}^{2n}$ be generated by $x_1, \ldots, x_{2n}$. Then we can assign the generators to the $\binom{2n}{n}$ degree $n$ monomials generated by $dx_1, \ldots, dx_{2n}$, with $a_i$ and $b_i$ assigned to complementary choices.

*Case (ii):* Again suppose $\mathbb{R}^{2n}$ is generated by $x_1, \ldots, x_{2n}$, and fix a volume form $dx_1 \wedge \cdots \wedge dx_{2n}$. Then $\wedge$ induces a symmetric bilinear form on $\bigwedge^n \mathbb{R}^{2n}$ of signature $\left(\binom{2n}{n}/2, \binom{2n}{n}/2\right)$, with basis vectors

$$dx_I + dx_{I^c} \quad \textit{and} \quad dx_I - dx_{I^c}$$

squaring to 1 and $-1$ respectively. Here $I$ is a choice of $n$ indices between 1 and $2n$ and $I^c$ is its positively oriented complement. Then we can assign $\binom{2n}{n}/2$ generators to forms of the form $dx_I + dx_{I^c}$. On the other hand, if $r > \binom{2n}{n}/2$, then an assignment of these generators would imply the existence of a basis in which the bilinear form has $I_r$ as a minor, which cannot happen.

*Case (iii):* Assume that $n \geq 3$. We would like to show that there cannot be two symplectic forms on $\mathbb{R}^{2n}$ whose wedge product is zero. For some basis $x_1, \ldots, x_{2n}$, one of these is

$$dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$$

and the other one is $\sum_{i<j} a_{ij} dx_i \wedge dx_j$ for some coefficients $a_{ij}$. Making their wedge product zero gives a system of linear equations of the form

$$\begin{aligned} a_{(2i-1)(2i)} + a_{(2j-1)(2j)} &= 0, & &\text{for } i \neq j, \\ a_{k\ell} &= 0, & &\text{when } k,\, \ell \text{ are not } 2i-1,\, 2i \text{ for any } i, \end{aligned}$$

which clearly has no nonzero solution. ☐

## 3. Phenomena in non-scalable spaces

In this section we give two examples in non-scalable spaces in which a rescaling and convergence argument gives new asymptotic lower bounds on the Lipschitz constant of maps. The first is a special case of Theorem A but is proven using a more direct method in order to demonstrate the technique. The second is a counterexample to Gromov's distortion conjecture.

While these examples can be understood perhaps more elegantly via maps from minimal models, we have chosen to make them accessible without any knowledge of rational homotopy theory.

3.1. **Flat differential forms.** Let $\Omega^*_\flat(X)$ denote the flat differential forms on $X$. These can be defined in several ways:

- As the dual normed space to the space of flat chains on $X$ [Whi57, §IX.7].
- As the set of $L^\infty$ forms with $L^\infty$ differential, cf. [GKS82, Thm. 1.5]. Here the differential of a non-smooth form is defined using Stokes' theorem applied to its action on currents.
- As the set of (non-smooth) differential forms satisfying certain complicated "niceness" conditions, see [Whi57, §IX.6].

We also write $\Omega^*_\flat(X, A)$ to denote those flat forms which are identically zero on a subcomplex $A$.

Flat forms have a number of attractive properties:

**Lemma 3.1** (see [GKS82, §3])**.** *The inclusion $\Omega^*(X) \to \Omega^*_\flat(X)$ induces an isomorphism on cohomology.*

**Lemma 3.2** (see [GKS82, Theorem 3.6])**.** *Flat forms pull back to flat forms along Lipschitz maps.*

A sequence of flat forms is said to *weak$^\flat$ converge* if its values on every flat chain converge (this is an instance of weak$^*$ convergence.)

**Lemma 3.3.** *Weak$^\flat$ limits commute with $d$ and $\wedge$.*

*Proof.* The former is true by definition and the latter is shown in [Whi57, §IX.17]. ☐

Finally, we need a version for flat forms of a result originally stated by Gromov and proved among other places as Lemma 2.2 in [Manarb]:

**Lemma 3.4** (Coisoperimetric inequality)**.** *Let $A \subset X$ be a simplicial pair with a linear metric. For every $k$ there is a constant $C(k, X, A)$ such that for every exact form $\omega \in \Omega^{k-1}_\flat(X, A)$, there is an $\alpha \in \Omega^k_\flat(X, A)$ satisfying $d\alpha = \omega$ and $\|\alpha\|_\infty \leq C(k, X, A)\|\omega\|_\infty$.*

The proof given in [Manarb] holds verbatim for flat forms once one defines fiberwise integration for these. This can be done either directly using the $L^\infty$ definition, or by defining a dual notion of shadows of flat chains.

### 3.2. **Nonlinear homotopies.**

**Theorem 3.5.** *Nullhomotopic maps $S^3 \to (\mathbb{C}P^2)^{\#4}$ do not have linear-size nullhomotopies.*

We first note that $(\mathbb{C}P^2)^{\#4}$ can be given a CW structure with four 2-cells corresponding to the copies of $\mathbb{C}P^1$ inside each $\mathbb{C}P^2$, together with one top cell whose attaching map in $\pi_3(\bigvee_4 S^2)$ is the sum of the elements corresponding to the Hopf fibration over each of the spheres.

*Proof.* We start with a specific family of maps $f_N : S^3 \to (\mathbb{C}P^2)^{\#4}$ which are $C_0N$-Lipschitz; we will show by way of contradiction that there is no $C_1$ such that each $f_N$ extends to a $C_1N$-Lipschitz map $D^4 \to (\mathbb{C}P^2)^{\#4}$.

Let $S_i$ be a copy of $\mathbb{C}P^1$ inside the $i$th copy of $\mathbb{C}P^2$. We define the $f_N$ to factor through a graph outside of four fixed balls $B_1, \ldots, B_4$ and maps $\partial B_i$ to the basepoint $*_i$ of $S_i$. On each $B_i$, $f_N$ maps to $S_i$ with Hopf invariant $N^4$; specifically, as a composition

$$B^3 \xrightarrow{\text{Hopf map}} S_i \xrightarrow{\text{degree } N^2} S_i,$$

where the degree $N^2$ map has homeomorphic preimages of $S_i \setminus *_i$ lined up in a square grid within a fixed square inside $S^2$. The maps $f_N$ are nullhomotopic since they are homotopic in $\bigvee_4 S^2$ to $N^4$ times the attaching map of the 4-cell.

Suppose now that, for some $C_1$, every $f_N$ extends to a $C_1N$-Lipschitz map $h_N : D^4 \to (\mathbb{C}P^2)^{\#4}$. Let $\alpha_i$ be forms with disjoint support Poincaré dual to the $S_i$; then for each $i$, $\alpha_i^2$ is a representative of the fundamental class of $(\mathbb{C}P^2)^{\#4}$, so let $\gamma_1, \gamma_2, \gamma_3 \in \Omega^3(\mathbb{C}P^2)^{\#4}$ satisfy $d\gamma_i = \alpha_i^2 - \alpha_4^2$. Since $\|h_N^*\alpha_i\|_\infty = O(N^2)$ and $\|h_N^*\gamma_i\|_\infty = O(N^3)$ we can choose a sequence of $N$ such that

$$\frac{1}{N^2} h_N^* \alpha_i, \quad \frac{1}{N^4} h_N^* \gamma_i$$

simultaneously weak$^\flat$-converge to some form $\alpha_i^\infty$ and 0, respectively. Moreover, since weak$^\flat$ limits commute with $\wedge$ and $d$, this means that $(\alpha_i^\infty)^2 = (\alpha_j^\infty)^2$ for each $i$ and $j$.

On the other hand, by Stokes' theorem

$$\int_{D^4} (h_N^* \alpha_i)^2 = \int_{S^3} f_N^* \alpha_i \wedge \eta$$

where $\eta$ is a form satisfying $d\eta = f_N^*\alpha_1|_{S^3}$; that is, this integral is (Whitehead's definition of) the Hopf invariant of the projection of $f_N$ to $S_1$. Therefore, $\int_{D^4} (\alpha_1^\infty)^2 = 1$; in particular $(\alpha_1^\infty)^2$ is nonzero at some point.

This means that we have constructed an embedding $H^*((\mathbb{C}P^2)^{\#4}; \mathbb{R}) \to \bigwedge^* \mathbb{R}^4$; but by Corollary 2.3, this cannot exist. $\square$

### 3.3. **Proof of Theorem C.**

**Theorem 3.6.** *The distortion of the generator $\alpha \in \pi_5([(\mathbb{C}P^2)^{\#4} \times S^2]^\circ)$ is $o(L^6)$.*

This disproves the strong form of Gromov's conjecture (that any element with trivial Hurewicz image in $\pi_n(Y)$ has distortion $\Omega(L^{n+1})$) and in particular shows that not all formal spaces satisfy the conjecture.

*Proof.* Write $Y = [(\mathbb{C}P^2)^{\#4} \times S^2]^\circ$.We use an argument very similar to the previous one. Take a purported sequence of $CN$-Lipschitz maps $f_N : S^5 \to Y$ representing $N^6\alpha$.

Let $\alpha_1, \ldots, \alpha_4$ and $\beta$ be forms dual to the copies of $\mathbb{C}P^1$ inside the four $\mathbb{C}P^2$'s and to the $S^2$ factor, respectively. We may assume that the $\alpha_i$ have disjoint support and that $\alpha_i^2 \wedge \beta = 0$ (for example by pulling back our original choice along the deformation retraction of $Y$ to its 4-skeleton.) Finally, as before, we define $\gamma_1$, $\gamma_2$, $\gamma_3$ such that $d\gamma_i = \alpha_i^2 - \alpha_4^2$; and choose a sequence of $N$ such that

$$\frac{1}{N^2} f_N^* \alpha_i, \quad \frac{1}{N^2} f_N^* \beta_i, \quad \frac{1}{N^4} f_N^* \gamma_i$$

converge to $\alpha_i^\infty$, $\beta^\infty$, and 0, respectively.

Moreover, the $\alpha_i^\infty$ are nonzero, by the following reasoning. Let $\eta_N \in \Omega^1(S^5)$ be such that $d\eta_N = f_N^*\beta$. Then

$$\int_{S^5} f_N^* \alpha_i^2 \wedge \eta_N = N^6. \tag{3.7}$$

This can be seen for example by Stokes' theorem.[2] The $\alpha_i$ and $\beta$ can be extended to forms $\tilde{\alpha}_i$ and $\tilde{\beta}$ over the unpunctured $\tilde{Y} = (\mathbb{C}P^2)^{\#4} \times S^2$, retaining all their properties except the vanishing of $\alpha_i^2 \wedge \beta$; instead, $\tilde{\alpha}_i^2 \wedge \tilde{\beta}$ represents the fundamental class of $\tilde{Y}$. Let $h_N$ be a nullhomotopy of $f_N$ in $\tilde{Y}$; we know that this nullhomotopy must have degree $N^6$ over the puncture, and therefore $\int_{D^6} h_N^* \tilde{\alpha}_i^2 \wedge \tilde{\beta} = N^6$. By Stokes' theorem, (3.7) holds.

On the other hand, by the coisoperimetric inequality Lemma 3.4, we can take $\eta_N$ so that $\|\eta_N\|_\infty \lesssim N^2$; this allows us to choose a further subsequence in which the $N^{-3}\eta_N$ converge weakly to some $\eta^\infty$ with $d\eta^\infty = \beta^\infty$. Moreover, since $\wedge$ commutes with weak limits, $\int_{S^5} (\alpha_i^\infty)^2 \wedge \eta^\infty = 1$. Therefore, $\int_{S^5} |(\alpha_i^\infty)^2|_\infty d\,\mathrm{vol} \gtrsim 1$, and in particular $(\alpha_i^\infty)^2$ is nonzero.

In other words, $(\alpha_i^\infty)^2 = (\alpha_i^\infty)^2 \neq 0$ for every $i$ and $j$, but $\alpha_i^\infty \wedge \alpha_j^\infty = 0$ for every $i \neq j$. By Theorem 2.2, this cannot happen locally at any point. □

## 4. Examples of scalable spaces

In this section we prove that certain connected sums are in fact scalable by showing that they have the property (i). The basic idea is to use Poincaré duality, building forms supported on normal bundles of certain submanifolds.

**Theorem 4.1.** *For any $n \leq m$ and $r \leq \binom{n+m-1}{n-1}$ the space $(S^n \times S^m)^{\#r}$ is scalable.*

In particular, once we combine this result with Prop. 2.1, we know the exact cutoff for scalability for spaces of the form $(S^n \times S^n)^{\#r}$; for $m \neq n$ there remains a gap. One corollary is as follows:

**Corollary 4.2.** *The following spaces are scalable:*

- $(\mathbb{C}P^2)^{\#p} \# (\overline{\mathbb{C}P^2})^{\#q}$, $0 \leq p, q \leq 3$.

[2] An alternate proof can be given using the homotopy periods of Chen–Sullivan, as described in [Manarb, §3.3].

- $(\mathbb{H}P^2)^{\# p} \# (\overline{\mathbb{H}P^2})^{\# q}$, $0 \leq p, q \leq 35$.
- $(\mathbb{O}P^2)^{\# p} \# (\overline{\mathbb{O}P^2})^{\# q}$, $0 \leq p, q \leq 6435$.

*Proof.* First, when we set $m = n$ in the above theorem and then connect each pair of factors with a cylinder, the resulting space $\Sigma_{n,r}$ is still scalable. (We can think of these as "connected sums of symmetric products".) This is because the inclusion map induces an injection on cohomology, and the corresponding forms are easy to extend over the cylinders.

On the other hand, $(\mathbb{C}P^2)^{\# r}$ is rationally equivalent to $\Sigma_{2,r}$ since both are formal and have the same rational cohomology algebra. Thus by Theorem B(a), $(\mathbb{C}P^2)^{\# 2}$ and $(\mathbb{C}P^2)^{\# 3}$ are scalable.

Similarly, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is formal and has the same rational cohomology algebra as $S^2 \times S^2$. More generally, $(\mathbb{C}P^2)^{\# p} \# (\overline{\mathbb{C}P^2})^{\# q}$ is rationally equivalent to $(S^2 \times S^2)^{\# \max(p,q)}$ with $|q-p|$ of the connected summands "symmetrized".

The quaternionic and octonionic cases are identical. $\square$

On the other hand, our results say nothing about "mixed" connected sums such as $(S^2 \times S^2) \# \mathbb{C}P^2$, since while their real cohomology algebras are isomorphic to ones we understand, their rational cohomology algebras are not. If we knew that scalability is a real homotopy invariant, we would understand it for all simply connected 4-manifolds.

One can make a more general statement than Theorem 4.1 to treat the case of different summands, although the condition on $r$ becomes a bit convoluted and clumsy: we consider families $\mathcal{I}$ of subsets $I \subset [0, 1, \ldots, k]$, that are *intersectful*, meaning that for any $I, J \in \mathcal{I}$ all four intersections of $I$ or $I^c$ with $J$ or $J^c$ are non-empty.

**Theorem 4.3.** *For any intersectful family $\mathcal{I}$ of subsets $I \subset [0, 1, \ldots, k]$ the following space is scalable:*
$$X_{\mathcal{I}} := \#_{I \in \mathcal{I}} \big( S^{|I|} \times S^{k+1-|I|} \big).$$

We think of each pair of spheres as associated to $I$ and $I^c$, respectively, and refer to them as $S_I$ and $S_{I^c}$.

Theorem 4.1 is recovered from this statement by choosing as $\mathcal{I}$ any subcollection of the $\binom{n+m-1}{n-1}$ subsets of $[0, \ldots, n+m-1]$ of cardinality $n$ that contain 0. This, together with the inequality $n \leq m$, ensures that the family is intersectful.

*Proof.* We start by explaining why the combinatorial formulation makes sense. To show that (i) holds we need to present the cohomology ring $H^*(X_{\mathcal{I}})$ by forms $\omega \in \Omega^*_\flat(X_{\mathcal{I}})$. The space $X_{\mathcal{I}}$ has a simple cell decomposition: it is a disk $D^{k+1}$ attached to a wedge of spheres by the sum of Whitehead products $[\mathrm{id}_{S_I}, \mathrm{id}_{S_{I^c}}]$.

So we start building these forms near the center of the disk $D^{k+1} \subset \mathbb{R}^{k+1}$ by sending the generator of $H^{|I|}(S_I)$ to a form $\omega_I := \bigwedge_{i \in I} dx_i$, for each $I \in \mathcal{I} \cup \mathcal{I}^c$ (where $\mathcal{I}^c$ represents the set of all complements of elements of $\mathcal{I}$, not the complement of $\mathcal{I}$). The intersectfulness of the family then implies that any two such forms have a common $dx_i$ and hence multiply to 0, unless they are $\omega_I \wedge \omega_{I^c} = \bigwedge_i dx_i$. This way we get a multiplicative structure isomorphic to that of $H^*(X_{\mathcal{I}})$.

Now it remains to extend the forms $\omega_I$ from a region $[-1, 1]^{k+1} \subsetneq D^{k+1}$ to the rest of the disk so that on the boundary $\partial D^{k+1}$ they turn out to be pullbacks along the attaching map of the volume forms on the $S_I$'s. We summarize this in the following lemma, which we take the rest of the section to prove.

**Lemma 4.4.** *For any intersectful family $\mathcal{I} \subset \{0,1\}^{k+1}$, the forms $\omega_I$, $I \in \mathcal{I} \cup \mathcal{I}^c$, can be extended to closed forms on $D^{k+1}$ so that*

$$\omega_I|_{\partial D^{k+1}} = f^* \alpha_I,$$

*where the forms $\alpha_I$ are the volume forms of*

$$S_I \subset \bigvee_{I \in \mathcal{I}} (S_I \vee S_{I^c}) \subset X_{\mathcal{I}},$$

*and $f$ is the previously mentioned attaching map, and such that the product of the forms is zero outside $[-1,1]^{k+1}$.* □

4.1. **Proof of lemma 4.4.** Let's overview the rough idea of the construction. First we extend the forms $\omega_I$ from the cube $[-1,1]^{k+1}$ to a much larger cube via the same formula

$$\omega_I = \bigwedge_{i \in I} \chi_{x_i \in [-1,1]} dx_i.$$

In other words, on any large sphere around the origin they are concentrated near, and Poincaré dual to, the coordinate spheres

$$S(I^c) := \{\mathbf{x} \in S^k \mid x_i = 0, i \in I\}.$$

Taking $\overline{\mathcal{I}}$ to be the closure of $\mathcal{I} \cup \mathcal{I}^c$ under intersections, the coordinate spheres $S(J)$ form a natural stratification of $S^k$ with strata indexed by $\overline{\mathcal{I}} \setminus \{\emptyset\}$.

Outside this cube, our forms can be thought of as similarly dual to a stratification of $S^k \times [0,T]$ which restricts to the stratification by coordinate spheres on $S^k \times \{0\}$ and such that the strata indexed by all $J \notin \mathcal{I} \cup \mathcal{I}^c$ have trivial intersection with $S^k \times \{T\}$.

We describe this as a kind of stratified framed bordism, that is we examine the intersections of the strata with concentric spheres centered at the origin and describe their evolution as "time", i.e. radius, increases. Over time, the strata are "peeled off" one by one, starting with the maximal ones. These maximal strata are stored aside after being detached, while all subsequent lower ones are peeled off and then collapsed.

Each time a stratum departs, however, it leaves behind a small part of itself, concentated near and held in place by lower strata. We reinterpret the leftover pieces as data associated to fibers over the lower strata: here we actually keep track of the forms rather than the strata themselves. Luckily, the exact shapes that are added this way don't matter, as all the lower-dimensional strata eventually collapse. But we do use the fact that they are globally almost products, in a sense which we now describe.

**Definition.** A *thickening* of the stratification by coordinate spheres described above is determined by a choice of numbers $1 < \varepsilon_I \ll \mathrm{Rad}(S^k)$ for every $I \in \overline{\mathcal{I}}$ which satisfy $\varepsilon_J \gg \varepsilon_I$ whenever $J \subset I$. Then the *(closed) membrane* $\mathbf{S}_I$ is defined to be the $\varepsilon_I$-neighborhood of the coordinate sphere $S(I)$. The *open membrane* $\mathbf{S}_I^\circ$ is $\mathbf{S}_I \setminus \bigcup_{J \subsetneq I} \mathbf{S}_J$.

To start, we must pick the initial $\varepsilon_I$'s small enough that we can pass to significantly thicker membranes a number of times over the course of the argument.

The membrane $\mathbf{S}_I$ is canonically diffeomorphic to $S(I) \times D^{k+1-|I|}$, with coordinates $(x, r, \theta)$ representing the point at distance $r$ along the geodesic from $x \in S(I)$ to $\theta \in S(I^c)$. We say that a form $\omega$ *agrees with* our thickening if on any open membrane

$$\mathbf{S}_I^\circ \cong (S(I) \setminus \bigcup \cdots) \times D(\varepsilon_I)$$

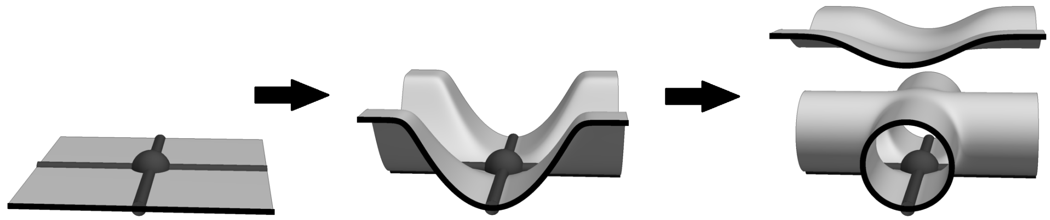

FIGURE 1. An example of pinching off a 2-membrane (gray) with 1- and 0-membranes (black) standing still. The 2-membrane leaves behind some tubes that are inherited by the lower membranes and are incorporated into them.

it depends only on the $D(\varepsilon_I)$ coordinates, i.e., $\omega$ is the pullback of some $\omega_D \in \Omega^*(D(\varepsilon_I))$ under the projection to the second factor.

This notion of agreement is crucial for the description of the construction, so it will be maintained all the way throughout.

As the procedure consists of peeling off the membranes, we need to specify a way to detach them.

**Definition.** Given a thickening and closed forms $\omega_{J,0}$ which agree with it, a *pinching off* of a membrane $\mathbf{S}_I$ in the direction of $p_I \in S(I^c)$ is a new thickening (with new forms $\omega_{J,1}$) such that:

(1) All the change in the forms is supported in a small neighborhood of $\mathbf{S}_I$, and outside all $s_J$ for $J \neq I$.
(2) The membrane $\mathbf{S}_I$ is replaced by a parallel thickened sphere $\mathbf{S}'_I$ which is shifted slightly in the direction of $p_I$ and doesn't cross any other membranes. This carries forms that agree with its product structure. The new thickening does not have a membrane corresponding to $I$.
(3) For $J \subset I$, the $\mathbf{S}_J$ are thickened in a consistent way, and forms changed in such a way that they agree with the new thickening; for other $J$, the thickening does not change.
(4) The forms $\omega_{J,0}$ and $\omega_{J,1}$ extend to closed forms $\omega_{J,t} \in \Omega^*(S^k \times [0,1])$ whose pairwise products are still zero.

**Lemma 4.5.** *Let $\mathcal{J}$ be a set of subsets of $[0,\dots,k]$ which is closed under intersection. For a sufficiently thin thickening of the stratification of $S^k$ by coordinate spheres $S(J)$, $J \in \mathcal{J}$, any maximal membrane $\mathbf{S}_I$ can be pinched off in any direction $p_I$ such that the shift of $\mathbf{S}_I$ in that direction wouldn't intersect other membranes.*

This is the main technical lemma, but its proof is just a wordy description of the picture above. So we put it in a separate section 4.2. With this tool at hand, we are ready to prove the lemma 4.4.

The forms on the cylinder $S^k \times [0,T]$ are constructed on cylinders $S^k \times [t,t+1]$, one after the other, half of which are pinch-off cylinders.

First we pinch off the maximal membranes. Since $\mathcal{I}$ is intersectful, no two sets of indices are contained in one another. Therefore the supports of the forms $\omega_I$ are precisely the maximal membranes. Given a maximal membrane $\mathbf{S}_I$, pick a point $p_I \in S(I^c)$ that is far away from any lower membranes. That ensures that the geodesic disk $D$ with center $p_I$ and boundary

$S(I)$ only intersects the open membranes $\mathbf{S}^\circ_{J\subseteq I}$ and $\mathbf{S}^\circ_{I^c}$. So we pinch off $\mathbf{S}_I$ in the direction of $p_I$ and contract it along $D$ to be a tight loop around $p_I$. Then we pinch off $\mathbf{S}_{I^c}$ in a similar way and contract it along an analogous disk to be tightly linked with the new $\mathbf{S}_I$.

When every maximal membrane is dealt with in this way, we have a set of Whitehead-linked spheres, as required by the conclusion of the lemma. We move them all to a small ball so they can be ignored for the rest of the construction; it remains to kill the remaining membranes.

This is done inductively, from the top down. Any now-maximal membrane $\mathbf{S}_I$ can be pinched off in the direction of the $p_J$ that was picked earlier for one of the original maximal membranes $\mathbf{S}_J$, $J \supset I$. The geodesic disk $D$ with center $p_J$ and boundary $S(I)$ only intersects membranes near its boundary, since we have already gotten rid of $S_{J^\circ}$. Thus after pinching off, we can extend the resulting sphere to a disk in $S^k \times [t, t+1]$. The normal bundle extends to the trivial bundle on this disk, so we can extend the forms to ones on $S^k \times [t, t+1]$ which agree with this bundle structure. On the remaining thickened stratification, the forms do not change on this interval.

After collapsing all the lower-order membranes, all that remains is the linked spheres, as required by the statement of the lemma.

4.2. **Pinch off lemma.** It remains to prove Lemma 4.5.

*Proof.* Recall that the aim is to pinch off a membrane $\mathbf{S}_I$ so that it splits into a parallel disjoint sphere $\mathbf{S}'_I$, plus some leftovers that are brushed under the $\mathbf{S}_{J\subsetneq I}$'s. First, observe that on a neighborhood $K$ of $\mathbf{S}_I$ (which includes $\mathbf{S}'_I$) we can choose coordinates

$$S(I) \times D^{k-|I|} \times [-\varepsilon, \delta + \varepsilon],$$

which preserve trivializations, such that $\mathbf{S}_I \subset S(I) \times D^{k-|I|} \times [-\varepsilon, \varepsilon]$ and $\mathbf{S}'_I$ is just $\mathbf{S}_I$ shifted by $\delta$ in the direction of the last coordinate. Moreover, we can assume that $S(I) \times D^{k-|I|} \times [\varepsilon, \delta + \varepsilon]$ does not intersect any membranes.

We now define forms on $K \times [0, 1]$ which extend the $\omega_J$'s on $K \times \{0\}$ and are time-invariant on the $\mathbf{S}_J$, $J \subsetneq I$. Recall that on $\mathbf{S}^\circ_I$, the $\omega_J$'s are independent of the sphere coordinate, that is they take the form of a pullback of a compactly supported form $\alpha_J \in \Omega^* D^{k+1-|I|}$ along the projection to the disk coordinate. Let $K^\circ = K \setminus \bigcup_{J\subsetneq I} \mathbf{S}_J$. We define $\omega_J$ on $K \times [0, 1]$ via $\omega_J = (\pi \times \tau)^* \alpha_J$, where $\pi(x, y, r, t) = y$ is the projection to $D^{k-|I|}$ and

$$\tau : S(I) \times D^{k-|I|} \times [-\varepsilon, \delta + \varepsilon] \times [0, 1] \to [-\varepsilon, \delta + \varepsilon]$$

is a Lipschitz function satisfying:

(a) $\tau(x, y, r, 0) = r$ for all $x$.
(b) $\tau(x, y, r, 1)$ depends only on $r$ and the distance from $x$ to the nearest point of any $S(J)$, $J \subsetneq I$.
(c) $\tau(x, y, r, 1) = -\varepsilon$ for $r \le 3\varepsilon$ and $x$ more than $\varepsilon$ away from all such $S(J)$.
(d) $\tau(x, y, r, 1) = \begin{cases} r, & -\varepsilon \le r \le \varepsilon \\ 2\varepsilon - r, & \varepsilon \le r \le 3\varepsilon \end{cases}$ if $(x, 0, 0) \notin \mathbf{S}^\circ_I$ (that is, if the fiber of $x$ intersects with a lower membrane).
(e) For $r \ge 3\varepsilon$, $\tau(x, y, r, 1) = \begin{cases} -\varepsilon, & 3\varepsilon \le r \le \delta - \varepsilon \\ r - \delta, & \delta - \varepsilon \le r \le \delta + \varepsilon, \end{cases}$ independent of $x$.

Informally speaking, when $x$ is close to a lower stratum (case (d)), the fiber above $x$ includes three copies of the original fiber, one of them upside-down; that is, it crosses through one of the tubes in Figure 4.1. When $x$ is far from all lower strata (case (c)), the fiber includes only one shifted copy; that is, it misses the tubes. The interpolation between these two creates the walls of the tubes, but it doesn't matter exactly how it's done.

Then we expand all $\mathbf{S}_J$, $J \subsetneq I$, so that they have radius between $3\varepsilon$ and $\delta - \varepsilon$. This can be done as long as $\delta$ is sufficiently larger than $\varepsilon$.

We argue that the new forms at time 1 agree with this new thickening. Indeed, if $(x, y, r)$ is in the newly thickened $\mathbf{S}_J^\circ$, then the closest point to $x$ in any of the lower coordinate spheres is in $S(J)$. Then the values of the forms at $(x, y, r, 1)$ only depend on the distance from $x$ to this point and on $y$ and $r$. All of these only depend only on the fiber coordinate in $\mathbf{S}_J$. On the other hand, outside all of the $\mathbf{S}_J$, the forms are zero except on $\mathbf{S}'_I$.

This completes the proof. □

## 5. Rational homotopy theory

In this section we introduce Sullivan's formulation of rational homotopy theory using differential forms, emphasizing the quantitative aspects outlined in [Manarb]. We also explain why these results apply to flat as well as smooth forms.

The basic category of Sullivan's theory is that of differential graded algebras (DGAs). A DGA is a chain complex over a field (in our case, always $\mathbb{R}$) equipped with a graded-commutative multiplication satisfying the (graded) Leibniz rule. The prototypical examples are:

- The smooth forms $\Omega^*(X)$ on a smooth manifold $X$, or the simplexwise smooth forms on a simplicial complex.
- Sullivan's *minimal DGA* $\mathcal{M}_Y^*$ for a simply connected space $Y$, which is a free algebra generated in degree $n$ by the *indecomposable* elements $\operatorname{Hom}(\pi_n(Y); \mathbb{R})$ and with a differential determined by the $k$-invariants in the Postnikov tower of $Y$.

The cohomology of a DGA is the cohomology of the underlying chain complex. The correct notion of an equivalence between DGAs is a *quasi-isomorphism*, a map which induces an isomorphism on cohomology. In particular, for every simply connected manifold or simplicial complex $Y$ there is a quasi-isomorphism, which we call the *minimal model*,

$$m_Y : \mathcal{M}_Y^* \to \Omega^* Y,$$

constructed by induction on the indecomposable elements of $\mathcal{M}_Y^*$.

When $Y$ is compact, $\Omega^* Y$ is finite-dimensional and $\mathcal{M}_Y^*$ is finitely generated in every degree; so a reductionist perspective is that $m_Y$ is simply a choice of a finite number of forms on $Y$ satisfying certain relations. Nevertheless, the perspective of shifting between maps $f : X \to Y$ and homomorphisms $\varphi : \mathcal{M}_Y^* \to \Omega^* X$ via the correspondence

$$f \mapsto f^* m_Y$$

turns out to be quite powerful.

5.1. **Flat forms and minimal models.** Here we demonstrate the advantages of using flat rather than smooth forms to define our minimal models. First, Lemmas 3.1 and 3.2 imply the following:

- Any minimal model for $\Omega^*(X)$ is also a minimal model for $\Omega_\flat^*(X)$.
- Any minimal model $m_Y : \mathcal{M}_Y^* \to \Omega_\flat^*(Y)$ induces an algebraicization map $f \mapsto f^* m_Y$ sending
$$\operatorname{Map}_{\mathrm{Lip}}(X, Y) \to \operatorname{Hom}(\mathcal{M}_Y^*, \Omega_\flat^*(X)).$$

Given finite complexes $X$ and $Y$, we define a weak$^\flat$ topology on $\mathrm{Hom}(\mathcal{M}_Y^*, \Omega_\flat^*(X))$ generated by the topologies on the restrictions to each indecomposable. In other words, a sequence of maps converges if and only if it converges on every indecomposable.

**Lemma 5.1.** *A sequence of maps in* $\mathrm{Hom}(\mathcal{M}_Y^*, \Omega_\flat^*(X))$ *whose* $L^\infty$ *norm on each indecomposable is bounded has a weak*$^\flat$*-convergent subsequence.*

*Proof.* We note that this also bounds the flat norm on each indecomposable, since the differential is generated by indecomposables in lower degrees. By the Banach–Alaoglu theorem, the restriction of the sequence to every indecomposable has a weak$^\flat$-convergent subsequence. Since we can choose a finite basis of indecomposables of degree $\leq \dim X$, this gives us a subsequence which weak$^\flat$-converges on all indecomposables. By Lemma 3.3, this subsequence in fact converges to a DGA homomorphism. □

Together with Lemma 3.4, these observations are enough to show that the machinery of [Manarb] still works when we substitute flat forms for smooth ones.

5.2. **The shadowing principle.** The quantitative obstruction theory in [Manarb] is built upon a combination of the coisoperimetric lemma 3.4 and algebraic properties of DGAs. Thus all of the results there are true, mutatis mutandis, when one expands the universe to flat forms. In particular, given a geometrically bounded homomorphism $\mathcal{M}_Y^* \to \Omega_\flat^*(X)$, one can produce a nearby genuine map $X \to Y$ with bounded Lipschitz constant.

To state this precisely, we first introduce a few definitions. Let $X$ and $Y$ be finite simplicial complexes or compact Riemannian manifolds such that $Y$ is simply connected and has a minimal model $m_Y : \mathcal{M}_Y^* \to \Omega_\flat^* Y$. Fix norms on the finite-dimensional vector spaces $V_k$ of degree $k$ indecomposables of $\mathcal{M}_Y^*$; then for homomorphisms $\varphi : \mathcal{M}_Y^* \to \Omega_\flat^*(X)$ we define the formal dilatation
$$\mathrm{Dil}(\varphi) = \max_{2 \leq k \leq \dim X} \|\varphi|_{V_k}\|_{\mathrm{op}}^{1/k},$$
where we use the $L^\infty$ norm on $\Omega_\flat^*(X)$. Notice that if $f : X \to Y$ is an $L$-Lipschitz map, then $\mathrm{Dil}(f^* m_Y) \leq C(\dim X, Y)L$ where the exact constant depends on the minimal model and the norms. Thus the dilatation is an algebraic analogue of the Lipschitz constant.

Given a formal homotopy
$$\Phi : \mathcal{M}_Y^* \to \Omega_\flat^*(X \times [0,T]),$$
we can define the dilatation $\mathrm{Dil}_T(\Phi)$ in a similar way. The subscript indicates that we can always rescale $\Phi$ to spread over a smaller or larger interval, changing the dilatation; this is a formal analogue of defining separate Lipschitz constants in the time and space direction, although in the formal world they are not so easily separable.

We note here that in rational homotopy theory, homotopies usually take the form
$$\mathcal{M}_Y^* \to \Omega_\flat^* X \otimes \mathbb{R}\langle t, dt \rangle,$$
that is, they must be polynomial in $t$ and $dt$. This does not make a difference either algebraically or quantitatively, for one because any function of $t$ can be approximated by polynomials. Accordingly, we use the two types of homotopy interchangeably in this paper.

Now we can state some results from [Manarb].

**Theorem 5.2** (A special case of the shadowing principle, [Manarb, Thm. 4.1])**.** *Let* $\varphi : \mathcal{M}_Y^* \to \Omega_\flat^*(X)$ *be a homomorphism with* $\mathrm{Dil}(\varphi) \leq L$ *which is formally homotopic to* $f^* m_Y$ *for some* $f : X \to Y$*. Then there is a* $g : X \to Y$ *which is* $C(X,Y)(L+1)$*-Lipschitz and such that* $g^* m_Y$ *is homotopic to* $\varphi$ *via a homotopy* $\Phi$ *with* $\mathrm{Dil}_{1/L}(\Phi) \leq C(X,Y)(L+1)$*.*

In other words, one can produce a genuine map by a small formal deformation of $\varphi$. We also present one relative version of this result:

**Theorem 5.3** (Cf. [Manarb, Thm. 5.7]). *Let $f, g : X \to Y$ be two nullhomotopic $L$-Lipschitz maps and suppose that $f^* m_Y$ and $g^* m_Y$ are formally homotopic via a homotopy $\Phi : \mathcal{M}_Y^* \to \Omega_\flat^*(X \times [0,T])$ with $\mathrm{Dil}_T(\Phi) \leq L$. Then there is a $C(X,Y)(L+1)$-Lipschitz homotopy $F : X \times [0,T] \to Y$ between $f$ and $g$.*

It is important for this result that the maps be nullhomotopic, rather than just in the same homotopy class. This is because we did not require our formal homotopy to be in the relative homotopy class of a genuine homotopy. In the zero homotopy class, one can always remedy this by a small modification, but in general the size of the modification needed may depend in an opaque way on the homotopy class.

5.3. **Formal spaces.** Many of the spaces we will be discussing in this paper are *formal* in the sense of Sullivan. A space $Y$ is formal if $\Omega^* Y$ is quasi-isomorphic to the cohomology ring $H^*(Y;\mathbb{R})$, viewed as a DGA with zero differential. In other words, there is a map $\mathcal{M}_Y^* \to H^*(Y;\mathbb{R})$ which is a quasi-isomorphism.

Another way of saying this is that for formal spaces, the minimal DGA can be constructed "formally" from the cohomology ring: at stage $k$, one adds generators that kill the relative $(k+1)$st cohomology of the map $\mathcal{M}_Y^*(k-1) \to H^*(Y;\mathbb{R})$.

Spaces known to be formal are the simply connected symmetric spaces [Sul77] and Kähler manifolds [DGMS75], but there are many other examples, some of which are given in Table 1.

There are several ways to characterize formality using different models of rational homotopy theory. For example, from the point of view of DGAs, formal spaces are those whose cohomology is a quotient of $\bigwedge U_0$, where $U_0$ is the subspace of indecomposables in the minimal model which have zero differential.

Any minimal DGA has a canonical filtration

$$0 \subseteq U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots$$

on the minimal model of a formal space $Y$, defined inductively as follows:

- $U_0$ is generated by all indecomposables with zero differential.
- The product respects the filtration: if $u_1 \in U_i$ and $u_2 \in U_j$, then $u_1 u_2 \in U_{i+j}$.
- $U_i$ contains all indecomposables whose differentials are in $U_{i-1}$.

The $U_i$ also induce a dual filtration

$$\pi_*(Y) \otimes \mathbb{Q} = \Lambda_0 \supseteq \Lambda_1 \supseteq \Lambda_2 \subseteq \cdots$$

via the pairing between indecomposables in degree $n$ and $\pi_n(Y)$, and in particular, $\Lambda_1$ is the kernel of the rational Hurewicz map. Finally, Halperin and Stasheff showed [HS79, §3] that for a formal space, one can choose the vector space of indecomposable generators[3] so that the filtration $\{U_i\}$ can be refined, non-canonically, to a bigrading $\mathcal{M}_Y^* = \bigwedge_i W_i$, where

$$(U_i \cap \text{indecomposables}) = W_i \oplus (U_i \cap \text{indecomposables}).$$

An important alternate characterization of formal spaces is that they are those $Y$ for which the *grading automorphisms* $\rho_t : H^*(Y;\mathbb{R}) \to H^*(Y;\mathbb{R})$ taking $w \mapsto t^{\deg w} w$ lift to automorphisms of the minimal model [Sul77, Thm. 12.7]. This lift is homotopically nonunique (for example, maps $S^2 \vee S^3 \to S^2 \to S^3$ are characterized not only by the degrees on $S^2$ and

[3]While the minimal model is unique up to isomorphism, such an isomorphism need not preserve this.

$S^3$ but also by the Hopf invariant of the restriction-projection $S^3 \to S^2$) but all such lifts share certain properties. In particular, all of them send $U_i$ to itself; moreover, given $w \in W_i \cap \mathcal{M}_Y^j$, they send $w \mapsto t^{i+j}w + w'$ where $w' \in U_{i-1}$.

Given a choice of $W_i$, one choice of lift sends every $w \in W_i \cap \mathcal{M}_Y^j$ to $t^{i+j}w$. We refer to this as the automorphism associated to the bigrading $\{W_i\}$.

Similarly, after fixing a quasi-isomorphism $h_Y : \mathcal{M}_Y^* \to H^*(Y)$, the composition $\rho_t h_Y$ lifts to a canonical choice of automorphism of the minimal model. It turns out that we can always find enough genuine maps $Y \to Y$ implementing this choice:

**Lemma 5.4.** *For every $q \in \mathbb{Q}$, there is some $p \in \mathbb{Z}$ such that $\rho_{pq}h_Y$ is realized by a genuine map $Y \to Y$.*

*Proof.* This is obtained using the argument of [Shi79], incidentally giving a correct proof of Shiga's main theorem in that paper. (In the paper, Shiga relies on the incorrect claim that the choice of automorphism of the minimal model is canonical. To fix the argument, it is enough to make a choice of quasi-isomorphism $h_Y : \mathcal{M}_Y^* \to H^*(Y)$ as well as associated choices of quasi-isomorphisms for skeleta of $Y$ which make the diagram in [Shi79, Lemma 3.4] commute.) □

5.4. **Quantitative consequences of formality.** The canonical filtration above allows us to define a notion of "size" for homomorphisms $\varphi : \mathcal{M}_Y^* \to \Omega^* X$, where $Y$ is compact and simply connected and $X$ is any metric complex. This notion depends on the choice of norms on the $V_k$, but this affects it only up to a constant depending on the dimension of $X$, since each $V_k$ is finite-dimensional. Specifically, we define the *$U$-dilatation* of $\varphi$ to be

$$\operatorname{Dil}^U(\varphi) := \max_{\substack{2 \le k \le \dim X \\ 0 \le i < k}} \|\varphi|_{V_k \cap U_i}\|_{\mathrm{op}}^{\frac{1}{k+i}};$$

this is bounded above by the notion of dilatation introduced in [Manarb] but has significant advantages, particularly for formal spaces. In particular, the methods of [Manarb, Prop. 3-9] easily prove the following:

**Proposition 5.5.** *Suppose that $\Phi_k : \mathcal{M}_Y^*(k) \to \Omega^* X \otimes \mathbb{R}\langle t, dt\rangle$ is a partially defined homotopy between $\varphi, \psi : \mathcal{M}_Y^* \to \Omega^* X$, and suppose that $\operatorname{Dil}^U(\varphi)$, $\operatorname{Dil}^U(\psi)$, and $\operatorname{Dil}^U_{1/L}(\Phi_k)$ are all bounded by $L > 0$.*

*(i) The obstruction to extending $\Phi_k$ to a homotopy*

$$\Phi_{k+1} : \mathcal{M}_Y^*(k+1) \to \Omega^* X \otimes \mathbb{R}\langle t, dt\rangle$$

*is a class in $H^k(X; V_{k+1})$ represented by a cochain whose restriction to $V_{k+1} \cap U_i$ has operator norm bounded by $C(k+1, Y)L^{k+1+i}$.*

*(ii) If this obstruction class vanishes, then we can choose $\Phi_{k+1}$ so that $\operatorname{Dil}^U_{1/L}(\Phi_{k+1}) \le C(k, Y)L$.*

In particular, this leads to a neat formulation of Gromov's distortion conjecture as discussed in the introduction:

**Definition.** We say that $\pi_n(Y)$ *satisfies Gromov's distortion conjecture* if any element of $\pi_n(Y) \cap (\Lambda_i \setminus \Lambda_{i+1})$ has distortion $\Theta(L^{n+i})$.

The upper bound on distortion holds in all cases, and is obtained using the above proposition. Suppose $f : S^n \to Y$ is an $L$-Lipschitz map; then we can bound its homotopy class by inductively applying Prop. 5.5(ii) to a putative homotopy from $f^*m_Y$ to the zero map, until we finally hit an obstruction at stage $n$ whose size is bounded by Prop. 5.5(i); see also [Manarb, §3.3]. The existence of a corresponding lower bound is one of the properties distinguishing scalable spaces.

## 6. A non-formal example

In this section we discuss an example space $Y$ which is not formal, but satisfies condition (iv) of Theorem A: for $n < \dim Y$, nullhomotopic $L$-Lipschitz maps $S^n \to Y$ have $O(L)$-Lipschitz nullhomotopies. On the other hand, nullhomotopies of maps from higher-dimensional spheres cannot be made linear. This demonstrates that the method of proof of Theorem A, which relies on induction by skeleta, cannot be straightforwardly extended to show that non-formal spaces never admit linear nullhomotopies. On the other hand, we also do not have a candidate non-formal space which could admit linear nullhomotopies from all domains. Thus the following question remains open:

**Question.** *Do non-formal simply connected targets ever admit linear nullhomotopies of maps from all compact domains? For that matter, from all spheres?*

Our space is 8-dimensional, although a 6-dimensional example can also be constructed. Namely, we take the CW complex

$$Y = (S^3_a \vee S^3_b \vee S^5) \cup_f e^8,$$

where $f : S^7 \to S^3 \vee S^3 \vee S^5$ is given by the iterated Whitehead product

$$\big[\mathrm{id}_a, \mathrm{id}_{S^5} + [\mathrm{id}_a, \mathrm{id}_b]\big],$$

with $\mathrm{id}_a$ and $\mathrm{id}_b$ representing the identity maps on the two copies of $S^3$.

Above and below we use the following conventions to define representatives of homotopy classes with good Lipschitz constants. Let $\varphi : S^k \to Y$ and $\psi : S^\ell \to Y$ be maps with Lipschitz constant $\leq L$. The notation $[\varphi, \psi]$ represents the standard Whitehead product of $\varphi$ and $\psi$, that is the $C(k,\ell)L$-Lipschitz map $S^{k+\ell-1} \to Y$ given by composing $\varphi \vee \psi$ with the attaching map of the $(k+\ell)$-cell of $S^k \times S^\ell$. The notation $N\varphi$ represents the composition of $\varphi$ with a degree $N$, $O(N^{1/k})$-Lipschitz map $S^k \to S^k$. Finally, if $k = \ell$, then $\varphi + \psi$ represents the $C(k)L$-Lipschitz map given by composing $\varphi \vee \psi$ with a map sending the northern and southern hemisphere to different copies of the sphere.

**Proposition 6.1.** *For $n \leq 7$, nullhomotopic maps $S^n \to Y$ have linear nullhomotopies.*

**Proposition 6.2.** *There is a sequence of nullhomotopic maps $g_N : S^{13} \to Y$ with Lipschitz constant $O(N)$ but such that every nullhomotopy of $g_N$ has Lipschitz constant $\Omega(N^{17/16})$.*

*Proof of Prop. 6.1.* For $n \leq 7$, any $L$-Lipschitz map $S^n \to Y$ has an $O(L)$-Lipschitz homotopy to one whose image lies in the 7-skeleton of $Y$, $W = S^3 \vee S^3 \vee S^5$. Moreover, if $n < 7$, such a map is nullhomotopic in $Y$ if and only if it is nullhomotopic in $W$. Since this is a scalable space, any such nullhomotopy can be made $O(L)$-Lipschitz by Theorem A.

There remains the case $n = 7$. Clearly a map $g : S^7 \to W$ is nullhomotopic in $Y$ if and only if it is in the homotopy class $N[f] \in \pi_7(W)$ for some $N$. By Theorem 8.1, the distortion of $[f]$ in $W$ is $\sim L^8$, meaning that if $g$ is $L$-Lipschitz, it is homotopic in $W$ to the $O(L)$-Lipschitz map

$$g' = \big[A\,\mathrm{id}_{S^3_a}, B\big(\mathrm{id}_{S^5} + [\mathrm{id}_a, \mathrm{id}_b]\big)\big] + Cf$$

for $A \lesssim L^3$, $B \lesssim L^5$, and $C \lesssim L^7$, and again by Theorem A, this homotopy can be made $O(L)$-Lipschitz.

Finally, we need to show that $g'$ has an $O(L)$-Lipschitz nullhomotopy in $Y$. So consider a map $p : S^3 \times S^5 \to Y$ sending the $S^3$ factor to $S^3_a$ and the $S^5$ factor to $Y$ via $\mathrm{id}_{S^5} + [\mathrm{id}_a, \mathrm{id}_b]$. Since $S^3 \times S^5$ is scalable, the map

$$[A\,\mathrm{id}_{S^3}, B\,\mathrm{id}_{S^5}] + C[\mathrm{id}_{S^3}, \mathrm{id}_{S^5}]$$

for $A \lesssim L^3$, $B \lesssim L^5$, and $C \lesssim L^7$ has an $O(L)$-Lipschitz nullhomotopy there. Pushing this nullhomotopy to $Y$ via $p$ gives an $O(L)$-Lipschitz nullhomotopy of $g'$. □

*Proof of Prop. 6.2.* The map

$$g_N = [[N^3\,\mathrm{id}_a, N^5\,\mathrm{id}_{S^5}], [N^3\,\mathrm{id}_a, [N^3\,\mathrm{id}_a, N^3\,\mathrm{id}_b]]]$$

is $O(N)$-Lipschitz, and it is homotopic in $S^3_a \vee S^3_b \vee S^5$ to $[[N^3\,\mathrm{id}_a, N^5\,\mathrm{id}_{S^5}], N^9 f]$ and therefore nullhomotopic. We will now show that any nullhomotopy has Lipschitz constant $\Omega(N^{17/16})$.

We will need to understand some of the rational homotopy theory of the subspace

$$W = S^3_a \vee S^3_b \vee S^5 \subset Y.$$

We note that $W$ is formal and therefore its minimal DGA can be computed formally; some of the generators in low dimensions are

$$\mathcal{M}^*_W \supset \left\langle \begin{array}{r|l} a^{(3)}, b^{(3)}, c^{(5)}, u_b^{(5)} & da = db = dc = 0, du_b = ab \\ u_c^{(7)}, v_b^{(7)}, w_b^{(9)}, v_c^{(9)} & du_c = ac, dv_b = au_b, dw_b = av_b, dv_c = au_c \\ w_c^{(11)}, z^{(13)} & dw_c = av_c, dz = u_c v_b + v_c u_b + c w_b + b w_c \end{array} \right\rangle$$

and moreover the pairing between generators of the minimal model and elements of $\pi_n(W)$ gives $\langle z, [g_N] \rangle \sim N^{17}$ due to the duality between Whitehead products and the differential discussed in [FHT12, §13(e)].

By [FHT12, §13(d)], a minimal model $m_W : (\mathcal{M}^*_W, d) \to \Omega^* W$ can be we extended to a (non-minimal) quasi-isomorphic model $m_Y : (\mathcal{M}^*_W \oplus \mathbb{R}y, d') \to \Omega^* Y$ for $Y$. Here $y$ satisfies $y^2 = 0 = dy$ and $xy = 0$ for every $x \in \mathcal{M}^*_W$, and $d' = d$ except for 7-dimensional indecomposables $x$ in $\mathcal{M}^*_W$, for which

$$d'x = dx + \langle x, [f] \rangle y.$$

In particular, $m_Y y$ is a closed form concentrated in the interior of the 8-cell, representing the fundamental class of $H^8(Y, W; \mathbb{R})$.

Now suppose that $F : (D^{14}, \partial D^{14}) \to (Y, W)$ is an $L$-Lipschitz map. We would like to argue that $\langle z, [F|_\partial] \rangle = O(L^{16})$. This is enough to prove the proposition.

One way of computing this pairing is as follows; cf. [Sul77, §11] and [Manarb, §3.3]. We attempt to extend $(F|_\partial)^* m_W$ to a map $\varepsilon : \mathcal{M}^*_W \to \Omega^* D^{14}$. Since the relative cohomology is zero through dimension 13, we do not encounter an obstruction until we try to extend to 13-dimensional indecomposables. At that point, regardless of previous choices, the obstruction to extending to $z$ is given by the pairing, that is,

$$\int_{D^{14}} \varepsilon(dz) = \langle z, [F|_\partial] \rangle.$$

One way of doing the extension is by first sending

$$a \mapsto F^* m_Y a, \quad b \mapsto F^* m_Y b, \quad c \mapsto f^* m_Y c, \quad u_b \mapsto F^* m_Y u_b;$$

then choosing a 7-form $\omega \in \Omega^*(D^{14}, \partial D^{14})$ satisfying $d\omega = F^* m_Y y$ and $\|\omega\|_\infty = O(L^8)$ and sending

$$u_c \mapsto F^* m_Y u_c - \langle u_c, [f] \rangle \omega, \quad v_b \mapsto F^* m_Y v_b - \langle v_b, [f] \rangle \omega;$$

and finally using Lemma 3.4 to ensure that

$$\|\varepsilon(w_b)\|_\infty = O(L^{11}); \quad \|\varepsilon(v_c)\|_\infty = O(L^{11}); \quad \|\varepsilon(w_c)\|_\infty = O(L^{13}).$$

This construction gives us $\|\varepsilon(dz)\|_\infty = O(L^{16})$, hence so is its integral over the disk. □

## 7. Proof of Theorem A

In this section we prove Theorem A together with Theorem B(a). First, we restate these results:

**Theorem.** *The following are equivalent for a simply connected finite complex $Y$:*

*(i) There is a DGA homomorphism $i : H^*(Y) \to \Omega^*_\flat Y$ which sends each cohomology class to a representative of that class.*
*(ii) There is a constant $C(Y)$ and infinitely many (indeed, a logarithmically dense set of) $p \in \mathbb{N}$ such that there is a $C(Y)(p+1)$-Lipschitz self-map which induces multiplication by $p^n$ on $H^n(Y;\mathbb{R})$.*
*(iii) $Y$ is formal, and for all finite simplicial complexes $X$, nullhomotopic $L$-Lipschitz maps $X \to Y$ have $C(X,Y)(L+1)$-Lipschitz nullhomotopies.*
*(iv) $Y$ is formal, and for all $n < \dim Y$, nullhomotopic $L$-Lipschitz maps $S^n \to Y$ have $C(X,Y)(L+1)$-Lipschitz homotopies.*

*Moreover, this property is a rational homotopy invariant.*

*Proof.* We start by proving the equivalence of (i) and (ii), followed by rational invariance; the statements on homotopies are the most involved and are deferred to the end.

*(i) ⇒ (ii).* We start by showing:

**Lemma 7.1.** *A space satisfying (i) is formal.*

*Proof.* A basic property of minimal models (see e.g. [GM81, Thm. 10.8]) is that any minimal model $\mathcal{M}^*_Y \to \Omega^*_\flat Y$ lifts through a quasi-isomorphism $H^*(Y) \to \Omega^*_\flat Y$, giving a quasi-isomorphism $h_Y : \mathcal{M}^*_Y \to H^*(Y)$. Moreover, we can pick this quasi-isomorphism so that it restricts to a quasi-isomorphism with rational coefficients. □

Composing this quasi-isomorphism with the map of (i), we get a new minimal model $m_Y : \mathcal{M}^*_Y \to \Omega^* Y$ which sends all homologically trivial elements to 0.

On the other hand, let $\rho_t : H^*(Y) \to H^*(Y)$ be the grading automorphism which multiplies $H^k$ by $t^k$. Then by Lemma 5.4, for every $q \in \mathbb{Q}$, there is some $p$ such that $\rho_{pq} \circ h_Y$ is realized by a genuine map $Y \to Y$. This map is in the rational homotopy class of the map $i\rho_{pq} h_Y : \mathcal{M}^*_Y \to \Omega^* Y$, whose dilatation is $O(pq)$; therefore, by the shadowing principle 5.2, we can build an $O(pq)$-Lipschitz map in this homotopy class. Such a map exists at least for powers of any particular $pq$, therefore the maps are at least logarithmically dense.

*(ii) ⇒ (i).* Suppose that there is an infinite sequence of $p \in \mathbb{N}$ and $C(Y)(p+1)$-Lipschitz maps $r_p$ as given. Let $m_Y : \mathcal{M}_Y^* \to \Omega^* Y$ be a minimal model, and

$$\mathcal{M}_Y^* = \bigwedge_{\ell=1}^{\infty} W_\ell$$

a bigrading as described in §5.3, such that there is an automorphism $r_p$ of $\mathcal{M}_Y^*$ extending the grading automorphism on $H^*(Y)$ which sends $a \in W_\ell$ to $p^{\deg a + \ell} a$.

Now for each $p$ the map $\varphi_p = r_p^* m_Y \rho_{1/p} : \mathcal{M}_Y^* \to \Omega_\flat^*(Y)$ sends

$$w \mapsto \frac{1}{p^{\ell + \deg w}} r_p^* m_Y w, \qquad w \in W_\ell.$$

This sequence of maps is uniformly bounded, and therefore has a subsequence which weak$^\flat$-converges to some $\varphi_\infty$.

**Lemma 7.2.** *For indecomposables $w$, $\varphi_\infty(w) = 0$ if and only if $w \in \oplus_{\ell=1}^{\infty} W_\ell$.*

*Proof.* If $w \in \oplus_{\ell=1}^{\infty} W_\ell$, then its image is zero since $\|r_p^* m_Y w\|_\infty \leq [C(Y)(p+1)]^{\deg w}$. On the other hand, if $w \in W_0$, then it is cohomologically nontrivial, and thus there is a flat cycle $A$ and a $C_A > 0$ such that $\int_A r_p^* m_Y w = C_A p^{\deg w}$ for every $p$. Thus $\int_A \varphi_\infty(w) = C_A$ and so $\varphi_\infty(w) \neq 0$. □

Now, if an element $w \in \bigwedge W_0$ is zero in $H^*(Y; \mathbb{R})$, then it is the differential of some element of $W_1$ and therefore again $\varphi_\infty(w) = 0$. Thus $\varphi_\infty : \mathcal{M}_Y^* \to \Omega_\flat^*(Y)$ factors through $H^*(Y; \mathbb{R})$, showing (i).

*Rational homotopy invariance of (ii).* Suppose that $Y$ has property (ii) and $Z$ is a rationally equivalent finite complex. By results of [BMSS98], since $Y$ has positive weights, there are maps $Z \xrightarrow{f} Y \xrightarrow{g} Z$ inducing rational homotopy equivalences, and in particular $g \circ f$ induces the automorphism $\rho_q$ for some $q$. Then we can get a sequence of maps verifying (ii) for $Z$ by composing

$$Z \xrightarrow{f} Y \xrightarrow{r_p} Y \xrightarrow{g} Z$$

for each $p$ in the sequence verifying (ii) for $Y$.

*(iii) ⇒ (iv).* This is clear.

*(iv) ⇒ (ii).* Suppose that $Y$ is formal and admits linear nullhomotopies of maps from $S^n$. Lemma 5.4 gives a way of realizing the grading automorphism $\rho_t$ of $Y$ by a map $r_t : Y \to Y$ for some infinite, logarithmically dense sequence of $t$, but without geometric constraints. It thus remains to construct homotopic maps with Lipschitz constant $O(L)$. We defer the details to the next section as they require some additional technical machinery from [Manarb].

In fact, our construction will give a more general result, which may be thought of as a strengthening of the shadowing principle for scalable spaces:

**Lemma 7.3.** *Suppose $Y$ is formal and admits linear nullhomotopies of maps from $S^k$, $k \leq n-1$. Let $X$ be an $n$-dimensional simplicial complex, and let $\varphi : \mathcal{M}_Y^* \to \Omega_\flat^*(X)$ be a homomorphism which satisfies*

$$\operatorname{Dil}^U(\varphi) \leq L,$$

*and which is formally homotopic to $f^* m_Y$ for some $f : X \to Y$. Then there is a $g : X \to Y$ which is $C(n,Y)(L+1)$-Lipschitz and homotopic to $f$, where $C(n,Y)$ depends on the choices of norms on $V_k$.*

As a special case, in combination with Lemma 5.4, we see that such a $Y$ satisfies (ii). Formally, this lemma also implies Gromov's distortion conjecture for $Y$, Theorem B(d). In fact, though, we will prove this separately and use it in the proof.

*(ii) ⇒ (iii).* Let $X$ be a finite simplicial complex and $f : X \to Y$ a nullhomotopic $L$-Lipschitz map. Choose a natural number $p > 1$ such that there is an automorphism $r_p : Y \to Y$.

We will define a nullhomotopy of $f$ by homotoping through a series of maps which are more and more "locally organized". Specifically, for $1 \leq k \leq s = \lceil \log_p L \rceil$, we build a $C(X,Y)(L/p^k + 1)$-Lipschitz map $f_k : X \to Y$ by applying the shadowing principle 5.2 to the map

$$f^* m_Y \rho_{p^{-k}} : \mathcal{M}_Y^* \to \Omega^* X.$$

We will build a nullhomotopy of $f$ through the sequence of maps

$$\begin{tikzcd}
f \arrow[r, dash] & r_p \circ f_1 \arrow[dr, dash] & r_{p^2} \circ f_2 \arrow[d, dash] \arrow[dr, dash] & \cdots \arrow[dr, dash] & r_{p^s} \circ f_s \arrow[d, dash] \arrow[r, dash] & \text{const.} \\
& & r_p \circ r_p \circ f_2 & \cdots & r_{p^{s-1}} \circ r_p \circ f_s &
\end{tikzcd}$$

As we go right, the *length* (Lipschitz constant in the time direction) of the $k$th intermediate homotopy increases—it is $O(p^k)$—while the *thickness* (Lipschitz constant in the space direction) remains $O(L)$. Thus all together, these homotopies can be glued into an $O(L)$-Lipschitz nullhomotopy of $f$.

Informally, the intermediate maps $r_{p^k} \circ f_k$ look at scale $p^k/L$ like thickness-$p^k$ "bundles" or "cables" of identical standard maps at scale $1/L$. This structure makes them essentially as easy to nullhomotope as $L/p^k$-Lipschitz maps.

We now build the aforementioned homotopies:

**Lemma 7.4.** *There is an $O(p^k)$-Lipschitz homotopy $F_k : Y \times [0,1] \to Y$ between $r_{p^k}$ and $r_{p^{k-1}} \circ r_p$.*

**Lemma 7.5.** *There is a linear thickness, constant length homotopy $G_k : X \times [0,1] \to Y$ between $f_k$ and $r_p \circ f_{k+1}$.*

This induces homotopies of thickness $O(L)$ and length $O(p^k)$:

- $F_k \circ (f_k \times \mathrm{id})$ from $r_{p^{k-1}} \circ r_p \circ f_k$ to $r_{p^k} \circ f_k$;
- $r_{p^k} \circ G_k$ from $r_{p^k} \circ f_k$ to $r_{p^k} \circ r_p \circ f_{k+1}$.

Finally, the map $f_s$ is $C(X,Y)$-Lipschitz and therefore has a short homotopy to one of a finite set of nullhomotopic simplicial maps $X \to Y$. For each map in this finite set, we can pick a fixed nullhomotopy, giving a constant bound for the Lipschitz constant of a nullhomotopy of $f_s$ and therefore a linear one for $r_{p^s} \circ f_s$.

Adding up the lengths of all these homotopies gives a geometric series which sums to $O(L)$, completing the proof of the theorem modulo the two lemmas above. □

*Proof of Lemma 7.4.* We use the fact that the maps $r_{p^i}$ were built using the shadowing principle. Thus there are formal homotopies $\Phi_i$ of length $C(X,Y)$ between $m_Y \rho_{p^i}$ and $r_{p^i}^* m_Y$. This allows us to construct the following formal homotopies:

- $\Phi_k$, time-reversed, between $r_{p^k}^* m_Y$ and $m_Y \rho_{p^k}$, of length $C(X,Y)$;
- $\Phi_1 \rho_{p^{k-1}}$ between $m_Y \rho_{p^k}$ and $r_p^* m_Y \rho_{p^{k-1}}$, of length $C(X,Y)p^{k-1}$;
- and $(r_{p^{k-1}}^* \otimes \mathrm{id})\Phi_{k-1}$ between $r_p^* m_Y \rho_{p^{k-1}}$ and $r_p^* r_{p^{k-1}}^* m_Y$, of length $C(X,Y)$.

Concatenating these three homotopies and applying the relative shadowing principle 5.3 to the resulting map $\mathcal{M}_Y^* \to \Omega^*(Y \times [0,1])$ rel ends, we get a linear thickness homotopy of length $O(p^{k-1})$ between the two maps. $\square$

*Proof of Lemma 7.5.* We use the fact that the maps $f_k$ and $f_{k+1}$ were built using the shadowing principle. Thus there are formal homotopies $\Psi_i$ of length $C(X,Y)$ between $f^* m_Y \rho_{p^{-i}}$ and $f_i$. This allows us to construct the following formal homotopies:

- $\Psi_k$, time-reversed, between $f_k$ and $f^* m_Y \rho_{p^{-k}}$, of length $C(X,Y)$;
- $\Psi_{k+1}\rho_p$ between $f^* m_Y \rho_{p^{-k}}$ and $f_{k+1}^* m_Y \rho_p$, of length $C(X,Y)p$;
- and $(f_{k+1}^* \otimes \mathrm{id})\Phi_1$ between $f_{k+1}^* m_Y \rho_p$ and $r_p^* f_{k+1}^* m_Y$, of length $C(X,Y)$.

Concatenating these three homotopies and applying the relative shadowing principle 5.3 to the resulting map $\mathcal{M}_Y^* \to \Omega^*(X \times [0,1])$ rel ends, we get a linear thickness homotopy of length $O(p)$ between the two maps. $\square$

## 8. Maps to scalable spaces

The purpose of this section is to prove Lemma 7.3, which we restate here:

**Lemma.** *Suppose $Y$ is formal and admits linear nullhomotopies of maps from $S^k$, $k \leq n-1$. Let $X$ be an $n$-dimensional simplicial complex, and let $\varphi : \mathcal{M}_Y^* \to \Omega_\flat^*(X)$ be a homomorphism which satisfies*

$$\mathrm{Dil}^U(\varphi) \leq L,$$

*and which is formally homotopic to $f^* m_Y$ for some $f : X \to Y$. Then there is a $g : X \to Y$ which is $C(n,Y)(L+1)$-Lipschitz and homotopic to $f$, where $C(n,Y)$ depends on the choices of norms on $V_k$.*

Both scalability and Gromov's distortion conjecture follow as corollaries of this lemma. These should be thought of as instances of a wider principle that the lemma facilitates the construction of maximally efficient maps. While the original shadowing principle gives a close relationship between the (usual) dilatation of the "most efficient" homomorphism $\mathcal{M}_Y^* \to \Omega^* X$ and the best Lipschitz constant of a map $X \to Y$ in a given homotopy class, the homomorphisms involved can be as difficult to construct as the maps. On the other hand, homomorphisms with the smallest possible $U$-dilatation can often be constructed by factoring through maps between minimal models; that is, they can be described using a finite amount of data. Although in more general situations obstruction theory will often complicate this picture, this seems like a compelling reason to study $U$-dilatation.

We prove Lemma 7.3 by induction using the following statements:

($a_n$) Lemma 7.3 holds through dimension $n$ (we make this more precise during the proof, but in particular it holds for $n$-dimensional $X$).

($b_n$) If $Z$ is an $n$-complex which is formal and admits linear nullhomotopies of maps from $S^k$, $k < n$, then it satisfies (ii).

Clearly, ($a_n$) implies ($b_n$). In particular, since any skeleton of a formal space is formal [Shi79, Lemma 3.1], $Y^{(n)}$ satisfies (ii). Therefore it also satisfies Gromov's distortion conjecture, as shown below. We will use this in the proof of ($a_{n+1}$).

**Theorem 8.1.** *Gromov's distortion conjecture holds for all spaces $Y$ satisfying (i) or (ii). That is, all elements $\alpha \in \pi_k(Y) \cap \Lambda_\ell$ outside $\Lambda_{\ell+1}$ have distortion $\Theta(L^{k+\ell})$.*

This is a mild generalization of [Manarb, Theorem 5–4].

*Proof.* Let $r_p$ be maps realizing (ii) for any $p$ for which they exist.

It is not hard to see, using Gromov's method from [Gro98, Ch. 7] as described in [Manarb, §3.3], that the distortion of an element $\alpha \in \pi_k(Y)$ which pairs trivially with indecomposables in $U_0, \ldots, U_{\ell-1}$ and nontrivially with $U_\ell$ is $O(L^{k+\ell})$.

Now suppose $\alpha$ is contained in $\Lambda_\ell$. We will show that its distortion is $\Omega(L^{k+\ell})$. Let $f : S^k \to Y$ be a representative of $\alpha$. Then $r_p f$ is an $O(L)$-Lipschitz representative of $q^{k+\ell}\alpha$. Such a map $r_p$ exists for at least a logarithmically dense set of integers $p$, so all other multiples can also be represented with a similar Lipschitz constant. □

In addition, we need the following easy extension of this result.

**Lemma 8.2.** *Suppose that $Y$ is an $n$-complex and the distortion conjecture holds for $Y^{(n-1)}$. Then it holds for $\pi_n(Y)$.*

*Proof.* This follows from the exact sequence

$$\cdots \to \pi_n(Y^{(n-1)}) \xrightarrow{i} \pi_n(Y) \xrightarrow{j} \pi_n(Y, Y^{(n-1)}) \to \cdots .$$

Since $\operatorname{im} j \cong H_n(Y)$, all elements of $\pi_n(Y)$ not in $\ker j$ are undistorted. Conversely, elements in the image of $i$ are at least as distorted as their preimages. To show that this is consistent with the distortion conjecture, we must analyze the induced map $\mathcal{M}_Y^* \to \mathcal{M}_{Y^{(n-1)}}^*$. In fact, this map is injective in degrees $\leq n-1$ (and hence preserves the filtration by the $U_j$) and all extra $n$-dimensional generators of $\mathcal{M}_Y^*$ have zero differential; see [FHT12, §13(d)]. This completes the proof of the lemma. □

We now proceed with the proof of the inductive step.

*Proof of Lemma 7.3.* The structure of this proof is very similar to the original proof of the shadowing principle in [Manarb, §4]. That is, we pull $f$ to a map with small Lipschitz constant skeleton by skeleton, all the while using $\varphi$ as a model to ensure that we don't end up with overly large obstructions at the next stage (as might occur if we pulled in an arbitrary way.)

The details follow. Suppose, as an inductive hypothesis, that we have constructed the following data:

- A map $g_k : X \to Y$, homotopic to $f$, whose restriction to $X^{(k)}$ is $C(k, Y)(L+1)$-Lipschitz.
- A homotopy $\Phi_k : \mathcal{M}_Y^* \to \Omega_\flat^*(X) \otimes \mathbb{R}(t, dt)$ from $g_k^* m_Y$ to $\varphi$ such that
$$\operatorname{Dil}^U_{1/L}((\Phi_k|_{\mathcal{M}_Y^*(k)})|_{X^{(k)}}) \leq C(k, Y)(L+1).$$

We write $\beta_k = \int_0^1 \Phi_k$; note that for $v \in V_i$, $d\beta_k(v) = \varphi(v) - g_k^* m_Y(v) - \int_0^1 \Phi_k(dv)$ and $\beta_k(v)|_A = 0$.

We then construct the analogues one dimension higher, starting with $g_{k+1}$. Let $b \in C^k(X; \pi_{k+1}(Y))$ be the simplicial cochain obtained by integrating $\beta_k|_{V_{k+1}}$ over $k$-simplices and choosing an element of $\pi_{k+1}(Y)$ whose image in $V_{k+1}$ is as close as possible in norm (but otherwise arbitrary.) Note that the values of $b$ are not a priori bounded in any way. We use $b$ to specify a homotopy $H_{k+1} : X \times [0,1] \to Y$ from $g_k$ to a new map $g_{k+1}$.

We start by setting $H_{k+1}$ to be constant on $X^{(k-1)}$. On each $k$-simplex $q$, we set $H_{k+1}|_q$ to be a map such that

$$g_{k+1}|_q = H_{k+1}|_{q \times \{1\}} = H_{k+1}|_{q \times \{0\}} = g_k|_q,$$

but such that on the cell $q \times [0,1]$, the map traces out the element $\langle b, q\rangle \in \pi_{k+1}(Y)$. This is well-defined since $H_{k+1}|_{\partial(q\times[0,1])}$ is canonically nullhomotopic by precomposition with a linear contraction of the simplex.

Now, given that $g_{k+1} = g_k$ on the $k$-skeleton, the possible relative homotopy classes of the restriction of $g_{k+1}$ to a $(k+1)$-simplex $p$ form a torsor for $\pi_{k+1}(Y)$. No matter how we extend $H_{k+1}$ over $p \times [0,1]$, we will get $g_{k+1}|_p - g_k|_p = \langle \delta b, p\rangle$ in this torsor. We would like to show that we can do so in such a way that $g_{k+1}|_p$ is $C(k+1,Y)(L+1)$-Lipschitz.

Note first that by assumption we can extend $g_{k+1}|_{\partial p}$ to $p$ via a $C(k+1,Y)(L+1)$-Lipschitz map $D^{k+1} \to Y$. However, this map may be in the wrong homotopy class. To build the extension we want, we first estimate the size of this obstruction in $\pi_{k+1}(Y)$; Lemma 8.2 applied to $Y^{(k+1)}$ then implies that it is represented by a $C(k+1,Y)(L+1)$-Lipschitz map $S^{k+1} \to Y$ which we then glue into the original extension to define $g_{k+1}|_p$.

**Lemma 8.3.** *The obstruction above can be written as $\alpha = \sum_i \alpha_i$ where $\alpha_i \in \pi_{k+1}(Y) \cap \Lambda_i$ and its coefficients in terms of a generating set for this subgroup are $O(L^{k+1+i})$.*

In other words, it is contained in a subset of $\pi_{k+1}(Y)$ whose elements, by Lemma 8.2, can be represented by $C(k,Y)(L+1)$-Lipschitz map. The proof is exactly like that of Lemma 4-2 in [Manarb], with Prop. 5.5 as an input.

After fixing $g_{k+1}|_p$ for each $(k+1)$-cell $p$, we can extend $H_{k+1}$ to higher-dimensional cells arbitrarily. The final task is to build a second-order homotopy from $\Phi_k$ to a homotopy $\Phi_{k+1}$ from $\varphi$ to $g_{k+1}$ such that

$$\mathrm{Dil}^U_{1/L}((\Phi_{k+1}|_{\mathcal{M}^*_Y(k+1)})|_{X^{(k+1)}}) \leq C(k+1,Y)(L+1).$$

Intuitively, this can be done since $\Phi_k|_{V_{k+1}}$ and $H^*_{k+1}|_{V_{k+1}}$ have, by construction, very similar integrals over $(k+1)$-cells; hence the obstruction to constructing such a homotopy is easy to kill. The details are once again the same as in the proof of the shadowing principle in [Manarb]. □

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