

# $E_7$ , WIRTINGER INEQUALITIES, CAYLEY 4-FORM, AND HOMOTOPY

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ABSTRACT. We study optimal curvature-free inequalities of the type discovered by C. Loewner and M. Gromov, using a generalisation of the Wirtinger inequality for the comass. Using a model for the classifying space  $BS^3$  built inductively out of  $BS^1$ , we prove that the symmetric metrics of certain two-point homogeneous manifolds turn out not to be the systolically optimal metrics on those manifolds. We point out the unexpected role played by the exceptional Lie algebra  $E_7$  in systolic geometry, via the calculation of Wirtinger constants. Using a technique of pullback with controlled systolic ratio, we calculate the optimal systolic ratio of the quaternionic projective plane, modulo the existence of a Joyce manifold with  $\text{Spin}(7)$  holonomy and unit middle-dimensional Betti number.

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## 1. INEQUALITIES OF PU AND GROMOV

The present text deals with systolic inequalities for the projective spaces over the division algebras  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$ .

In 1952, P.M. Pu [Pu52] proved that the least length, denoted  $\text{sys}\pi_1$ , of a noncontractible loop of a Riemannian metric  $\mathcal{G}$  on the real projective plane  $\mathbb{R}\mathbb{P}^2$ , satisfies the optimal inequality

$$\text{sys}\pi_1(\mathbb{R}\mathbb{P}^2, \mathcal{G})^2 \leq \frac{\pi}{2} \text{area}(\mathbb{R}\mathbb{P}^2, \mathcal{G}).$$

Pu's bound is attained by a round metric, *i.e.* one of constant Gaussian curvature. This inequality extends the ideas of C. Loewner, who proved an analogous inequality for the torus in a graduate course at Syracuse University in 1949, thereby obtaining the first result in systolic geometry, *cf.* [Ka07].

Defining the optimal systolic ratio  $\text{SR}(\Sigma)$  of a surface  $\Sigma$  as the supremum

$$\text{SR}(\Sigma) = \sup_{\mathcal{G}} \left\{ \frac{\text{sys}\pi_1(\mathcal{G})^2}{\text{area}(\mathcal{G})} \mid \mathcal{G} \text{ Riemannian metric on } \Sigma \right\}, \quad (1.1)$$

we can restate Pu's inequality as the calculation of the value

$$\text{SR}(\mathbb{R}\mathbb{P}^2) = \frac{\pi}{2},$$

the supremum being attained by a round metric.

One similarly defines a homology systole, denoted  $\text{sysh}_1$ , by minimizing over loops in  $\Sigma$  which are not nullhomologous. One has  $\text{sys}\pi_1(\Sigma) \leq \text{sysh}_1(\Sigma)$ . For orientable surfaces, one has the identity

$$\text{sysh}_1(\Sigma) = \lambda_1(H_1(\Sigma, \mathbb{Z}), \|\cdot\|), \quad (1.2)$$

where  $\|\cdot\|$  is the stable norm in homology (see Section 4), while  $\lambda_1$  is the first successive minimum of the normed lattice. In other words, the homology systole and the stable 1-systole (see below) coincide in this case (and more generally in codimension 1). Thus, the homology 1-systole is the least stable norm of an integral 1-homology class of infinite order.

Therefore, either the homology  $k$ -systole or the stable  $k$ -systole can be thought of as a higher-dimensional generalisation of the 1-systole of surfaces. It has been known for over a decade that the homology systoles do not satisfy systolic inequalities; see [Ka95] where the case of the products of spheres  $S^k \times S^k$  was treated. Homology systoles will not be used in the present text.

For a higher dimensional manifold  $M^{2k}$ , the appropriate middle-dimensional invariant is therefore the stable  $k$ -systole  $\text{stsys}_k$ , defined as follows. Let  $H_k(M, \mathbb{Z})_{\mathbb{R}}$  be the image of the integral lattice in real  $k$ -dimensional homology of  $M$ . The  $k$ -Jacobi torus  $J_k M$  is the quotient

$$J_k M = H_k(M, \mathbb{R}) / H_k(M, \mathbb{Z})_{\mathbb{R}}. \tag{1.3}$$

We set

$$\text{stsys}_k(\mathcal{G}) = \lambda_1(H_k(M, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|), \tag{1.4}$$

where  $\|\cdot\|$  is the stable norm in homology, while  $\lambda_1$  is the first successive minimum of the normed lattice. In other words, the stable  $k$ -systole is the least stable norm of an integral  $k$ -homology class of infinite order. A detailed definition of the stable norm appears in Section 4.

By analogy with (1.1), one defines the optimal middle-dimensional stable systolic ratio,  $\text{SR}_k(M^{2k})$ , by setting

$$\text{SR}_k(M) = \sup_{\mathcal{G}} \frac{\text{stsys}_k(\mathcal{G})^2}{\text{vol}_{2k}(\mathcal{G})},$$

where the supremum is over all Riemannian metrics  $\mathcal{G}$  on  $M$ .

In 1981, M. Gromov [Gr81] proved an inequality analogous to Pu's, for the complex projective plane  $\mathbb{C}\mathbb{P}^2$ . Namely, he evaluated the optimal stable systolic ratio of  $\mathbb{C}\mathbb{P}^2$ , which turns out to be

$$\text{SR}_2(\mathbb{C}\mathbb{P}^2) = 2,$$

where, similarly to the real case, the implied optimal bound is attained by the symmetric metric, *i.e.* the Fubini-Study metric. In fact, Gromov proved a more general optimal inequality.

**Theorem 1.1** (M. Gromov). *Every metric  $\mathcal{G}$  on the complex projective space satisfies the inequality*

$$\text{stsys}_2(\mathbb{C}\mathbb{P}^n, \mathcal{G})^n \leq n! \text{vol}_{2n}(\mathbb{C}\mathbb{P}^n, \mathcal{G}). \tag{1.5}$$

Here  $\text{stsys}_2$  is still defined by formula (1.4) with  $k = 2$ , and we set  $M = \mathbb{C}\mathbb{P}^n$ .

A quaternionic analogue of the inequalities of Pu and Gromov was widely expected to hold. Namely, the symmetric metric on the quaternionic projective plane  $\mathbb{H}\mathbb{P}^2$  gives a ratio equal to  $\frac{10}{3}$ , calculated by a calibration argument in Section 5, following the approach of [Ber72]. It

was widely believed that the *optimal* systolic ratio  $\text{SR}_4(\mathbb{H}\mathbb{P}^2)$  equals  $\frac{10}{3}$ , as well. See also [Gr96, Section 4] and [Gr99, Remark 4.37, p. 262] or [Gr07]. Contrary to expectation, we prove the following theorem.

**Theorem 1.2.** *The quaternionic projective space  $\mathbb{H}\mathbb{P}^{2n}$  and the complex projective space  $\mathbb{C}\mathbb{P}^{4n}$  have a common optimal middle dimensional stable systolic ratio:  $\text{SR}_{4n}(\mathbb{H}\mathbb{P}^{2n}) = \text{SR}_{4n}(\mathbb{C}\mathbb{P}^{4n})$ .*

Theorem 1.2 is proved in Section 8. The Fubini-Study metric gives a middle-dimensional ratio equal to  $(4n)!/((2n)!)^2$  for the complex projective  $4n$ -space. For instance, the symmetric metric of  $\mathbb{C}\mathbb{P}^4$  gives a ratio of 6. The symmetric metric on  $\mathbb{H}\mathbb{P}^{2n}$  has a systolic ratio of  $(4n+1)!/((2n+1)!)^2$ , cf. [Ber72]. Since

$$(4n+1)!/((2n+1)!)^2 < (4n)!/((2n)!)^2,$$

we obtain the following corollary.

**Corollary 1.3.** *The symmetric metric on  $\mathbb{H}\mathbb{P}^{2n}$  is not systolically optimal.*

We also estimate the common value of the optimal systolic ratio in the first interesting case, as follows.

**Proposition 1.4.** *The common value of the optimal ratio for  $\mathbb{H}\mathbb{P}^2$  and  $\mathbb{C}\mathbb{P}^4$  lies in the following interval:*

$$6 \leq \text{SR}_4(\mathbb{H}\mathbb{P}^2) = \text{SR}_4(\mathbb{C}\mathbb{P}^4) \leq 14. \quad (1.6)$$

The constant 14 which appears above as the upper bound for the optimal ratio, is twice the dimension of the Cartan subalgebra of the exceptional Lie algebra  $E_7$ , reflected in our title. More specifically, the relevant ingredient is that every self-dual 4-form admits a decomposition into at most 14 decomposable (simple) terms with respect to a suitable orthonormal basis, cf. proof of Proposition 10.1.

Note that quaternion algebras and congruence subgroups of arithmetic groups were used in [KSV07] to study asymptotic behavior of the systole of Riemann surfaces. It was pointed out by a referee that for the first time in the history of systolic geometry, Lie algebra theory has been used in the field.

We don't know of any techniques for constructing metrics on  $\mathbb{C}\mathbb{P}^4$  with ratio greater than the value 6, attained by the Fubini-Study metric. Meanwhile, an analogue of Gromov's proof for  $\mathbb{C}\mathbb{P}^2$  only gives an upper bound of 14. This is due to the fact that the Cayley 4-form  $\omega_{\text{Ca}}$ , cf. [Ber72, HL82], has a higher Wirtinger constant than does the Kahler 4-form (*i.e.* the square of the standard symplectic 2-form). Nonetheless, we expect that the resulting inequality is optimal, *i.e.*

that the common value of the optimal systolic ratio of  $\mathbb{H}\mathbb{P}^2$  and  $\mathbb{C}\mathbb{P}^4$  is, in fact, equal to 14. The evidence for this is the following theorem, which should give an idea of the level of difficulty involved in evaluating the optimal ratio in the quaternionic case, as compared to Pu's and Gromov's calculations. Joyce manifolds [Jo00] are discussed in Section 11.

**Theorem 1.5.** *If there exists a compact Joyce manifold  $\mathcal{J}$  with  $\text{Spin}(7)$  holonomy and with  $b_4(\mathcal{J}) = 1$ , then the common value of the middle dimensional optimal systolic ratio of  $\mathbb{H}\mathbb{P}^2$  and  $\mathbb{C}\mathbb{P}^4$  equals 14.*

A smooth Joyce manifold with middle Betti number 1 would necessarily be rigid. Thus it cannot be obtained by any known techniques, relying as they do on deforming the manifold until it decays into something simpler. On the other hand, by relaxing the hypothesis of smoothness to, say, that of a PD(4) space, such a mildly singular Joyce space may be obtainable as a suitable quotient of an 8-torus, and may be sufficient for the purposes of calculating the systolic ratio in this dimension.

**Corollary 1.6.** *If there exists a compact Joyce manifold  $\mathcal{J}$  with  $\text{Spin}(7)$  holonomy and with  $b_4(\mathcal{J}) = 1$ , then the symmetric metric on  $\mathbb{C}\mathbb{P}^4$  is not systolically optimal.*

## 2. HISTORICAL REMARKS

If one were to give a synopsis of the history of the application of homotopy techniques in systolic geometry, one would have to start with D. Epstein's work [Ep66] on the degree of a map in the 1960's, continue with A. Wright's work [Wr74] on monotone mappings in the 1970's, then go on to developments in real semi-algebraic geometry which indicated that an arbitrary map can be homotoped to have good algebraic structure by M. Coste and others [BoCR98], in the 1980's.

M. Gromov, in his 1983 paper [Gr83], goes out of the category of manifolds in order to prove the main isoperimetric inequality relating the volume of a manifold, to its filling volume. Namely, the cutting and pasting constructions in the proof of the main isoperimetric inequality involve objects more general than manifolds.

In the 1992 paper in *Izvestia* by I. Babenko [Ba93], his Lemma 8.4 is perhaps the place where a specific homotopy theoretic technique was first applied to systoles. Namely, this technique derives systolically interesting consequences from the existence of maps from manifolds to simplicial complexes, by pullback of metrics. This work shows how the triangulation of a map  $f$ , based upon the earlier results mentioned

above, can help answer systolic questions, such as proving a converse to Gromov’s central result of 1983. What is involved, roughly, is the possibility of pulling back metrics by  $f$ , once the map has been deformed to be sufficiently nice (in particular, real semialgebraic).

In 1992-1993, Gromov realized that a suitable oblique  $\mathbb{Z}$  action on the product  $S^3 \times \mathbb{R}$  gives a counterexample to a  $(1, 3)$ -systolic inequality on the product  $S^1 \times S^3$ . This example was described by M. Berger [Ber93], who sketched also Gromov’s ideas toward constructing further examples of systolic freedom.

In 1995, metric simplicial complexes were used [Ka95] to prove the systolic freedom of the manifold  $S^n \times S^n$ . In this paper, a polyhedron  $P$  is defined in equation (3.1). It is exploited in an essential way in an argument in the last paragraph on page 202, in the proof of Proposition 3.3.

Thus, we will exploit a map of classifying spaces  $BS^1 \rightarrow BS^3$  so as to relate the systolic ratios of the quaternionic projective space and the complex projective space. We similarly relate the quaternionic projective space and a hypothetical Joyce manifold (with  $\text{Spin}_7$  holonomy) with  $b_4 = 1$ , relying upon a result by H. Shiga in rational homotopy theory.

An interesting related axiomatisation (in the case of 1-systoles) is proposed by M. Brunnbauer [Br08a], who proves that the optimal systolic constant only depends on the image of the fundamental class in the classifying space of the fundamental group, generalizing earlier results of I. Babenko.

Marcel Berger’s monograph [Ber03, pp. 325-353] contains a detailed exposition of the state of systolic affairs up to ’03. More recent developments are covered in [Ka07]. Recent publications in systolic geometry include [Ber08, DKR08, Br08a, Br08b, Br07, Ka08, KS08, RS08, Sa08].

The Cayley 4-form is an important example in the theory of calibrated geometries as defined by F. R. Harvey and H. B. Lawson in their landmark paper [HL82]. They remark that “the most fascinating and complex geometry discussed here is the geometry of Cayley 4-folds in  $\mathbb{R}^8 \cong \mathbb{O}$ ”. The Cayley 4-form is the calibrating form defining the Cayley 4-folds.

Research on calibrated geometries stimulated by [HL82] led to many new examples of spaces with exceptional holonomy. For example, the Cayley 4-form is the basic building block in the structure of 8-manifolds with exceptional  $\text{Spin}(7)$  holonomy, see [Jo00].

The wealth of new examples of  $\text{Spin}(7)$  and  $G_2$  manifolds constructed by R.L. Bryant, D. Joyce, S. Salamon have been used as vacua for string theories [Ac98, Be96, Le02, Sha95], and the Cayley 4-cycles on  $\text{Spin}(7)$

manifolds are candidates for the supersymmetric representatives of fundamental particles [Be96].

In Section 4, we present Gromov’s proof of the optimal stable 2-systolic inequality (1.5) for the complex projective space  $\mathbb{C}\mathbb{P}^n$ , *cf.* [Gr99, Theorem 4.36], based on the cup product decomposition of its fundamental class. The proof relies upon the Wirtinger inequality, proved in Section 3 following H. Federer [Fe69]. In Section 5, we analyze the symmetric metric on the quaternionic projective plane from the systolic viewpoint. A general framework for Wirtinger-type inequalities is proposed in Section 6.

A homotopy equivalence between  $\mathbb{H}\mathbb{P}^n$  and a suitable CW complex built out of  $\mathbb{C}\mathbb{P}^{2n}$  is constructed in Section 7 using a map  $BS^1 \rightarrow BS^3$ . Section 8 exploits such a homotopy equivalence to build systolically interesting metrics. Section 9 contains some explicit formulas in the context of the Kraines form and the Cayley form  $\omega_{\text{Ca}}$ . Section 10 presents a Lie-theoretic analysis of 4-forms on  $\mathbb{R}^8$ , using an idea of G. Hunt. Theorem 1.5 is proved in Section 11. Related results on the Hopf invariant and Whitehead products are discussed in Section 12.

### 3. FEDERER’S PROOF OF WIRTINGER INEQUALITY

Following H. Federer [Fe69, p. 40], we prove an optimal upper bound for the comass norm  $\| \cdot \|$ , *cf.* Definition 3.1, of the exterior powers of a 2-form.

Recall that an exterior form is called *simple* (or *decomposable*) if it can be expressed as a wedge product of 1-forms. The comass norm for a simple  $k$ -form coincides with the natural Euclidean norm on  $k$ -forms. In general, the comass is defined as follows.

**Definition 3.1.** The comass of an exterior  $k$ -form is its maximal value on a  $k$ -tuple of unit vectors.

Let  $V$  be a vector space over  $\mathbb{C}$ . Let  $H = H(v, w)$  be a Hermitian product on  $V$ , with real part  $v \cdot w$ , and imaginary part  $A = A(v, w)$ , where  $A \in \bigwedge^2 V$ , the second exterior power of  $V$ . Here we adopt the convention that  $H$  is complex linear in the *second* variable.

**Example 3.2.** Let  $Z_1, \dots, Z_\nu \in \bigwedge^1(\mathbb{C}^\nu, \mathbb{C})$  be the coordinate functions in  $\mathbb{C}^\nu$ . We then have the standard (symplectic) 2-form, denoted  $A \in \bigwedge^2(\mathbb{C}^\nu, \mathbb{C})$ , given by

$$A = \frac{i}{2} \sum_{j=1}^{\nu} Z_j \wedge \bar{Z}_j.$$

**Lemma 3.3.** *The comass of the standard symplectic form  $A$  satisfies  $\|A\| = 1$ .*

*Proof.* We can set  $\xi = v \wedge w$ , where  $v$  and  $w$  are orthonormal. We have  $H(v, w) = iA(v, w)$ , hence

$$\langle \xi, A \rangle = A(v, w) = H(iv, w) = (iv) \cdot w \leq 1 \quad (3.1)$$

by the Cauchy-Schwarz inequality; equality holds if and only if one has  $iv = w$ .  $\square$

**Remark 3.4.** R. Harvey and H. B. Lawson [HL82] provide a similar argument for the Cayley 4-form  $\omega_{\text{Ca}}$ . They realize  $\omega_{\text{Ca}}$  as the real part of a suitable multiple vector product on  $\mathbb{R}^8$ , defined in terms of the (non-associative) octonion multiplication, to calculate the comass of  $\omega_{\text{Ca}}$ , cf. Proposition 9.1.

**Proposition 3.5** (Wirtinger inequality). *Let  $\mu \geq 1$ . If  $\xi \in \bigwedge_{2\mu} V$  and  $\xi$  is simple, then*

$$\langle \xi, A^\mu \rangle \leq \mu! |\xi|;$$

*equality holds if and only if there exist elements  $v_1, \dots, v_\mu \in V$  such that*

$$\xi = v_1 \wedge (iv_1) \wedge \dots \wedge v_\mu \wedge (iv_\mu).$$

*Consequently,  $\|A^\mu\| = \mu!$*

*Proof.* The main idea is that in real dimension  $2\mu$ , every 2-form is either simple, or splits into a sum of at most  $\mu$  orthogonal simple pieces.

We assume that  $|\xi| = 1$ , where  $|\cdot|$  is the natural Euclidean norm in  $\bigwedge_{2\mu} V$ . The case  $\mu = 1$  was treated in Lemma 3.3.

In the general case  $\mu \geq 1$ , we consider the  $2\mu$  dimensional subspace  $T$  associated with  $\xi$ . Let  $f : T \rightarrow V$  be the inclusion map, and consider the pullback 2-form  $(\wedge^2 f)A \in \bigwedge^2 T$ . Next, we orthogonally diagonalize the skew-symmetric 2-form, *i.e.* decompose it into  $2 \times 2$  diagonal blocks. Thus, we can choose dual orthonormal bases  $e_1, \dots, e_{2\mu}$  of  $T$  and  $\omega_1, \dots, \omega_{2\mu}$  of  $\bigwedge^1 T$ , and nonnegative numbers  $\lambda_1, \dots, \lambda_\mu$ , so that

$$(\wedge^2 f)A = \sum_{j=1}^{\mu} \lambda_j (\omega_{2j-1} \wedge \omega_{2j}). \quad (3.2)$$

By Lemma 3.3, we have

$$\lambda_j = A(e_{2j-1}, e_{2j}) \leq \|A\| = 1 \quad (3.3)$$

for each  $j$ . Noting that  $\xi = \epsilon e_1 \wedge \dots \wedge e_{2\mu}$  with  $\epsilon = \pm 1$ , we compute

$$(\wedge^{2\mu} f)A^\mu = \mu! \lambda_1 \dots \lambda_\mu \omega_1 \wedge \dots \wedge \omega_{2\mu},$$



and therefore

$$\langle \xi, A^\mu \rangle = \epsilon \mu! \lambda_1 \dots \lambda_\mu \leq \mu! \quad (3.4)$$

Note that equality occurs in (3.4) if and only if  $\epsilon = 1$  and  $\lambda_j = 1$ . Applying the proof of Lemma 3.3, we conclude that  $e_{2j} = ie_{2j-1}$ , for each  $j$ .  $\square$

**Corollary 3.6.** *Every real 2-form  $A$  satisfies the comass bound*

$$\|A^\mu\| \leq \mu! \|A\|^\mu. \quad (3.5)$$

*Proof.* An inspection of the proof Proposition 3.5 reveals that the orthogonal diagonalisation argument, cf. (3.3), applies to an arbitrary 2-form  $A$  with comass  $\|A\| = 1$ .  $\square$

**Lemma 3.7.** *Given an orthonormal basis  $\omega_1, \dots, \omega_{2\mu}$  of  $\bigwedge^1 T$ , and real numbers  $\lambda_1, \dots, \lambda_\mu$ , the form*

$$f = \sum_{j=1}^{\mu} \lambda_j (\omega_{2j-1} \wedge \omega_{2j}) \quad (3.6)$$

has comass  $\|f\| = \max_j |\lambda_j|$ .

*Proof.* We can assume without loss of generality that each  $\lambda_j$  is non-negative. This can be attained in one of two ways. One can permute the coordinates, by applying the transposition flipping  $\omega_{2j-1}$  and  $\omega_{2j}$ . Alternatively, one can replace, say,  $\omega_{2j}$  by  $-\omega_{2j}$ .

Next, consider the hermitian inner product  $H_f$  obtained by polarizing the quadratic form

$$\sum_j \left( \lambda_j^{1/2} \omega_{2j} \right)^2 + \left( \lambda_j^{1/2} \omega_{2j+1} \right)^2.$$

Let  $\zeta = v \wedge w$  be an orthonormal pair such that  $\|f\| = f(\zeta)$ . As in (3.1), we have

$$f(\zeta) = -iH_f(\zeta) = H_f(iv, w) \leq \left( \max_j \lambda_j \right) (iv) \cdot w \leq \max_j \lambda_j,$$

proving the lemma.  $\square$

#### 4. GROMOV'S INEQUALITY FOR COMPLEX PROJECTIVE SPACE

First we recall the definition of the stable norm in the real  $k$ -homology of an  $n$ -dimensional polyhedron  $X$  with a piecewise Riemannian metric, following [BaK03, BaK04].

**Definition 4.1.** The stable norm  $\|h\|$  of  $h \in H_k(X, \mathbb{R})$  is the infimum of the volumes

$$\text{vol}_k(c) = \sum_i |r_i| \text{vol}_k(\sigma_i) \quad (4.1)$$

over all real Lipschitz cycles  $c = \sum_i r_i \sigma_i$  representing  $h$ .

Note that  $\|\cdot\|$  is indeed a norm, *cf.* [Fed74] and [Gr99, 4.C].

We denote by  $H_k(X, \mathbb{Z})_{\mathbb{R}}$  the image of  $H_k(X, \mathbb{Z})$  in  $H_k(X, \mathbb{R})$  and by  $h_{\mathbb{R}}$  the image of  $h \in H_k(X, \mathbb{Z})$  in  $H_k(X, \mathbb{R})$ . Recall that  $H_k(X, \mathbb{Z})_{\mathbb{R}}$  is a lattice in  $H_k(X, \mathbb{R})$ . Obviously

$$\|h_{\mathbb{R}}\| \leq \text{vol}_k(h) \quad (4.2)$$

for all  $h \in H_k(X, \mathbb{Z})$ , where  $\text{vol}_k(h)$  is the infimum of volumes of all integral  $k$ -cycles representing  $h$ . Moreover, one has  $\|h_{\mathbb{R}}\| = \text{vol}_n(h)$  if  $h \in H_n(X, \mathbb{Z})$ . H. Federer [Fed74, 4.10, 5.8, 5.10] (see also [Gr99, 4.18 and 4.35]) investigated the relations between  $\|h_{\mathbb{R}}\|$  and  $\text{vol}_k(h)$  and proved the following.

**Proposition 4.2.** *If  $h \in H_k(X, \mathbb{Z})$ ,  $1 \leq k < n$ , then*

$$\|h_{\mathbb{R}}\| = \lim_{i \rightarrow \infty} \frac{1}{i} \text{vol}_k(ih). \quad (4.3)$$

Equation (4.3) is the origin of the term *stable norm* for  $\|\cdot\|$ . Recall that the stable  $k$ -systole of a metric  $(X, \mathcal{G})$  is defined by setting

$$\text{stsys}_k(\mathcal{G}) = \lambda_1(H_k(X, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|), \quad (4.4)$$

*cf.* (1.2) and (1.4). Let us now return to systolic inequalities on projective spaces.

**Theorem 4.3** (M. Gromov). *Every Riemannian metric  $\mathcal{G}$  on complex projective space  $\mathbb{C}\mathbb{P}^n$  satisfies the inequality*

$$\text{stsys}_2(\mathcal{G})^n \leq n! \text{vol}_{2n}(\mathcal{G});$$

*equality holds for the Fubini-Study metric on  $\mathbb{C}\mathbb{P}^n$ .*

*Proof.* Following Gromov's notation in [Gr99, Theorem 4.36], we let

$$\alpha \in H_2(\mathbb{C}\mathbb{P}^n; \mathbb{Z}) = \mathbb{Z} \quad (4.5)$$

be the positive generator in homology, and let

$$\omega \in H^2(\mathbb{C}\mathbb{P}^n; \mathbb{Z}) = \mathbb{Z}$$

be the dual generator in cohomology. Then the cup power  $\omega^n$  is a generator of  $H^{2n}(\mathbb{C}\mathbb{P}^n; \mathbb{Z}) = \mathbb{Z}$ . Let  $\eta \in \omega$  be a closed differential 2-form. Since wedge product  $\wedge$  in  $\Omega^*(X)$  descends to cup product in  $H^*(X)$ , we have

$$1 = \int_{\mathbb{C}\mathbb{P}^n} \eta^{\wedge n}. \quad (4.6)$$

Now let  $\mathcal{G}$  be a metric on  $\mathbb{C}\mathbb{P}^n$ .

The comass norm of a differential  $k$ -form is, by definition, the supremum of the pointwise comass norms, *cf.* Definition 3.1. Then by the Wirtinger inequality and Corollary 3.6, we obtain

$$\begin{aligned} 1 &\leq \int_{\mathbb{C}\mathbb{P}^n} \|\eta^{\wedge n}\| \, d\text{vol} \\ &\leq n! (\|\eta\|_\infty)^n \text{vol}_{2n}(\mathbb{C}\mathbb{P}^n, \mathcal{G}) \end{aligned} \tag{4.7}$$

where  $\|\cdot\|_\infty$  is the comass norm on forms (see [Gr99, Remark 4.37] for a discussion of the constant in the context of the Wirtinger inequality). The infimum of (4.7) over all  $\eta \in \omega$  gives

$$1 \leq n! (\|\omega\|^*)^n \text{vol}_{2n}(\mathbb{C}\mathbb{P}^n, \mathcal{G}), \tag{4.8}$$

where  $\|\cdot\|^*$  is the comass norm in cohomology. Denote by  $\|\cdot\|$  the stable norm in homology. Recall that the normed lattices  $(H_2(M; \mathbb{Z}), \|\cdot\|)$  and  $(H^2(M; \mathbb{Z}), \|\cdot\|^*)$  are dual to each other [Fe69]. In our rank 1 setting, it follows that the class  $\alpha$  of (4.5) satisfies

$$\|\alpha\| = \frac{1}{\|\omega\|^*},$$

and hence

$$\text{stsys}_2(\mathcal{G})^n = \|\alpha\|^n \leq n! \text{vol}_{2n}(\mathcal{G}). \tag{4.9}$$

Equality is attained by the two-point homogeneous Fubini-Study metric, since the standard  $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^n$  is calibrated by the Fubini-Study Kahler 2-form, which satisfies equality in the Wirtinger inequality at every point.  $\square$

**Example 4.4.** Every metric  $\mathcal{G}$  on the complex projective plane satisfies the optimal inequality

$$\text{stsys}_2(\mathbb{C}\mathbb{P}^2, \mathcal{G})^2 \leq 2 \text{vol}_4(\mathbb{C}\mathbb{P}^2, \mathcal{G}).$$

This example generalizes to the manifold obtained as the connected sum of a finite number of copies of  $\mathbb{C}\mathbb{P}^2$  as follows.

**Proposition 4.5.** *Every Riemannian  $n\mathbb{C}\mathbb{P}^2$  satisfies the inequality*

$$\text{stsys}_2(n\mathbb{C}\mathbb{P}^2)^2 \leq 2 \text{vol}_4(n\mathbb{C}\mathbb{P}^2). \tag{4.10}$$

*Proof.* We define two varieties of conformal 2-systole of a manifold  $M$  as follows. The Euclidean norm  $|\cdot|$  and the comass norm  $\|\cdot\|$  on (linear) 2-forms define, by integration, a pair of  $L^2$  norms on  $\Omega^2(M)$ . Minimizing over representatives of a cohomology class, we obtain a pair of norms

in de Rham cohomology. The dual norms in homology will be denoted respectively  $|\cdot|_2$  and  $\|\cdot\|_2$ , cf. [Ka07, p. 122, 130]. We let

$$\text{Confsys}_2 = \lambda_1(H_2(M; \mathbb{Z}), \|\cdot\|_2)$$

and

$$\text{confsys}_2 = \lambda_1(H_2(M; \mathbb{Z}), |\cdot|_2).$$

Since every top dimensional form is simple (decomposable), by Corollary 3.6 we have an inequality

$$|x|^2 \leq \text{Wirt}_2 \|x\|^2 \tag{4.11}$$

where  $\text{Wirt}_2 = 2$ , between the pointwise Euclidean norm and the pointwise comass, for all  $x \in \bigwedge^2(n\mathbb{C}\mathbb{P}^2)$ . It follows that, dually, we have

$$\text{Confsys}_2^2 \leq 2 \text{confsys}_2^2. \tag{4.12}$$

For a metric of unit volume we have

$$\text{stsys}_k \leq \text{Confsys}_k. \tag{4.13}$$

Combining (4.12) and (4.13), we obtain

$$\text{stsys}_2^2(\mathcal{G}) \leq 2 \text{confsys}_2^2(\mathcal{G}) \text{vol}_4(\mathcal{G}).$$

Recall that the intersection form of  $n\mathbb{C}\mathbb{P}^2$  is given by the identity matrix. Every metric  $\mathcal{G}$  on a connected sum  $n\mathbb{C}\mathbb{P}^2$  satisfies the identity  $\text{confsys}_2(\mathcal{G}) = 1$  because of the identification of the  $L^2$  norm and the intersection form. We thus reprove Gromov's optimal inequality

$$\text{stsys}_2^2 \leq 2 \text{vol}_4,$$

but now it is valid for the connected sum of  $n$  copies of  $\mathbb{C}\mathbb{P}^2$ .  $\square$

In fact, the inequality can be stated in terms of the last successive minimum  $\lambda_n$  of the integer lattice in homology with respect to the stable norm  $\|\cdot\|$ .

**Corollary 4.6.** *The last successive minimum  $\lambda_n$  satisfies the inequality*

$$\lambda_n(H_2(n\mathbb{C}\mathbb{P}^2, \mathbb{Z}), \|\cdot\|)^2 \leq 2 \text{vol}_4(n\mathbb{C}\mathbb{P}^2)$$

The proof is the same as before. This inequality is in fact optimal for all  $n$ , though equality may not be attained.

**Question 4.7.** *What is the asymptotic behavior for the stable systole of  $n\mathbb{C}\mathbb{P}^2$  when  $n \rightarrow \infty$ ? Can the constant in (4.10) be replaced by a function which tends to zero as  $n \rightarrow \infty$ ?*

5. SYMMETRIC METRIC OF  $\mathbb{H}\mathbb{P}^2$  AND KRAINES 4-FORM

The quaternionic projective plane  $\mathbb{H}\mathbb{P}^2$  has volume  $\text{vol}_8(\mathbb{H}\mathbb{P}^2) = \frac{\pi^4}{5!}$  for the symmetric metric with sectional curvature  $1 \leq K \leq 4$ , while for the projective line with  $K \equiv 4$  we have  $\text{vol}_4(\mathbb{H}\mathbb{P}^1) = \frac{\pi^2}{3!}$ , cf. [Ber72, formula (3.10)]. Since the projective line is volume minimizing in its real homology class, we obtain  $\text{stsys}_4(\mathbb{H}\mathbb{P}^2) = \frac{\pi^2}{3!}$ , as well, resulting in a systolic ratio

$$\frac{\text{stsys}_4(\mathbb{H}\mathbb{P}^2)^2}{\text{vol}_8(\mathbb{H}\mathbb{P}^2)} = \frac{10}{3} \tag{5.1}$$

for the symmetric metric.

In more detail, we endow  $\mathbb{H}\mathbb{P}^n$  with the natural metric as the base space of the Riemannian submersion from the unit sphere

$$S^{4n+3} \subset \mathbb{H}^{n+1}.$$

A projective line  $\mathbb{H}\mathbb{P}^1 \subset \mathbb{H}\mathbb{P}^n$  is a round 4-sphere of (Riemannian) diameter  $\frac{\pi}{2}$  and sectional curvature  $+4$ , attaining the maximum of sectional curvatures of  $\mathbb{H}\mathbb{P}^n$ . The extension of scalars from  $\mathbb{R}$  to  $\mathbb{H}$  gives rise to an inclusion  $\mathbb{R}^3 \hookrightarrow \mathbb{H}^3$ , and thus an inclusion  $\mathbb{R}\mathbb{P}^2 \hookrightarrow \mathbb{H}\mathbb{P}^2$ . Then  $\mathbb{R}\mathbb{P}^2 \subset \mathbb{H}\mathbb{P}^2$  is a totally geodesic submanifold of diameter  $\frac{\pi}{2}$  and Gaussian curvature  $+1$ , attaining the minimum of the sectional curvatures of  $\mathbb{H}\mathbb{P}^2$ , cf. [CE75, p. 73].

The following proposition was essentially proved by V. Kraines [Kr66] and M. Berger [Ber72]. The invariant 4-form was briefly discussed in [HL82, p. 152].

**Proposition 5.1.** *There is a parallel 4-form  $\kappa_{\mathbb{H}\mathbb{P}} \in \Omega^4(\mathbb{H}\mathbb{P}^2)$  representing a generator of  $H^4(\mathbb{H}\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}$ , with*

$$|\kappa_{\mathbb{H}\mathbb{P}}^2| = \frac{10}{3} \|\kappa_{\mathbb{H}\mathbb{P}}\|^2 \tag{5.2}$$

and

$$|\kappa_{\mathbb{H}\mathbb{P}}|^2 = \frac{10}{3} \|\kappa_{\mathbb{H}\mathbb{P}}\|^2, \tag{5.3}$$

where  $|\cdot|$  and  $\|\cdot\|$  are, respectively, the Euclidean norm and the comass of the unit volume symmetric metric on  $\mathbb{H}\mathbb{P}^2$ .

*Proof.* The parallel differential 4-form  $\kappa_{\mathbb{H}\mathbb{P}}$  is obtained from an  $Sp(2)$ -invariant alternating 4-form on a tangent space at a point, by propagating it by parallel translation to all points of  $\mathbb{H}\mathbb{P}^2$ . The fact that parallel translation produces a well defined global 4-form results from the  $Sp(2)$  invariance of the alternating form.

In more detail, consider the quaternionic vector space  $\mathbb{H}^n = \mathbb{R}^{4n}$ . Each of the three quaternions  $i, j$ , and  $k$  defines a complex structure on  $\mathbb{H}^n$ , i.e. an identification  $\mathbb{H}^n \simeq \mathbb{C}^{2n}$ . The imaginary part of the

associated Hermitian inner product on  $\mathbb{C}^{2n}$  is the standard symplectic exterior 2-form, *cf.* Example 3.2. Let  $\omega_i, \omega_j,$  and  $\omega_k$  be the triple of 2-forms on  $\mathbb{H}^n$  defined by the three complex structures. We consider their wedge squares  $\omega_i^2, \omega_j^2,$  and  $\omega_k^2$ . We define an exterior 4-form  $\kappa_n$ , first written down explicitly by V. Kraines [Kr66], by setting

$$\kappa_n = \frac{1}{6} (\omega_i^2 + \omega_j^2 + \omega_k^2). \quad (5.4)$$

The coefficient  $\frac{1}{6}$  normalizes the form to unit comass, *cf.* Lemma 3.3. The form  $\kappa_n$  is invariant under transformations in  $Sp(n) \times Sp(1)$  [Kr66, Theorem 1.9] and thus defines a parallel differential 4-form in  $\Omega^4 \mathbb{H}\mathbb{P}^n$ , which is furthermore closed. We normalize the differential form in such a way as to represent a generator of integral cohomology, and denote the resulting form  $\kappa_{\mathbb{H}\mathbb{P}}$ , so that  $[\kappa_{\mathbb{H}\mathbb{P}}] \in H^4(\mathbb{H}\mathbb{P}^n, \mathbb{Z})_{\mathbb{R}} \simeq \mathbb{Z}$  is a generator.

In the case  $n = 2$ , explicit formulas appear in (9.1) and (9.2). Here  $\omega_i$  is the sum of 4 monomial terms, while  $\omega_i^2$  is twice the sum of 6 such terms.

The form  $3\kappa_2$  on  $\mathbb{H}^2$  decomposes into a sum of 18 simple 4-forms, *i.e.* monomials in the 8 coordinates. The 18 monomials are not all distinct. Two of them, denoted  $m_0$  and its Hodge star  $*m_0$ , occur with multiplicity 3. Thus, we obtain a decomposition as a linear combination of seven selfdual pairs

$$3\kappa_2 = 3(m_0 + *m_0) + \sum_{\ell=1}^6 (m_\ell + *m_\ell), \quad (5.5)$$

where  $*$  is the Hodge star operator. In Section 9, the explicit formulas for the three 2-forms will be used to write down the Cayley 4-form  $\omega_{\text{Ca}}$ .

Similarly to (4.7), we can write

$$\begin{aligned} 1 &= \int_{\mathbb{H}\mathbb{P}^2} \left| \kappa_{\mathbb{H}\mathbb{P}} \wedge \kappa_{\mathbb{H}\mathbb{P}} \right| d\text{vol} \\ &= \frac{10}{3} (\|\kappa_{\mathbb{H}\mathbb{P}}\|_{\infty})^2 \text{vol}_8(\mathbb{H}\mathbb{P}^2), \end{aligned} \quad (5.6)$$

thereby reproving (5.1) by the duality of comass and stable norm.

**Lemma 5.2.** *The Kraines form  $\kappa_2$  of (5.4) has unit comass:  $\|\kappa_2\| = 1$ .*

This was proved in [Ber72, DHM88]. Meanwhile, from (5.5) we have

$$(3\kappa_2)^2 = 2(9 \text{ vol} + 6 \text{ vol}),$$

where  $\text{vol} = e_1 \wedge e_2 \wedge \cdots \wedge e_8$  is the volume form of  $\mathbb{H}^2 = \mathbb{R}^8$ . Hence

$$|(3\kappa_2)^2| = 2 \cdot 15 = 30,$$

proving identity (5.2). Meanwhile,  $|3\kappa_2|^2 = 9 + 9 + 12 = 30$ , proving identity (5.3).  $\square$

**Remark 5.3.** There is a misprint in the calculation of the systolic constants in [Ber72, Theorem 6.3], as is evident from [Ber72, formula (6.14)]. Namely, in the last line on page [Ber72, p. 12], the formula for the coefficient  $s_{4,b}$  lacks the exponent  $b$  over the constant 6 appearing in the numerator. The formula should be

$$s_{4,b} = \frac{6^b}{(2b+1)!}.$$

## 6. GENERALIZED WIRTINGER INEQUALITIES

**Definition 6.1.** The *Wirtinger constant*  $\text{Wirt}_n$  of  $\mathbb{R}^{2n}$  is the maximal ratio  $\frac{|\omega^2|}{\|\omega\|^2}$  over all  $n$ -forms  $\omega \in \Lambda^n \mathbb{R}^{2n}$ . The *modified Wirtinger constant*  $\text{Wirt}'_n$  is the maximal ratio  $\frac{|\omega|^2}{\|\omega\|^2}$  over  $n$ -forms  $\omega$  on  $\mathbb{R}^{2n}$ .

The calculation of  $\text{Wirt}_n$  can thus be thought of as a generalisation of the Wirtinger inequality of Section 3.

In Section 10, we will deal in detail with the special case of selfdual 4-forms in the context of the Lie algebra  $E_7$ . We therefore gather here some elementary material pertaining to this case.

**Definition 6.2.** Let  $n$  be even. Let  $\text{Wirt}_{sd}$  be the maximal ratio  $\frac{|\omega^2|}{\|\omega\|^2}$  over all selfdual  $n$ -forms on  $\mathbb{R}^{2n}$ .

**Lemma 6.3.** *One has  $\text{Wirt}_n = \text{Wirt}_{sd} \leq \text{Wirt}'_n$  if  $n$  is even.*

*Proof.* In general for a skew-form  $\omega$  it may occur that  $|\omega^2| > |\omega|^2$ . This does not occur when  $\omega$  is middle-dimensional. If  $\omega$  is a middle-dimensional form, then

$$\|\omega^2\| = |\omega^2| = \langle \omega, *\omega \rangle \leq |\omega| |*\omega| = |\omega|^2, \quad (6.1)$$

proving that  $\text{Wirt}_n \leq \text{Wirt}'_n$ .

Let  $\eta$  be a form with nonnegative wedge-square (if it is negative, reverse the orientation of the ambient vector space  $\mathbb{R}^{2n}$  to make the square non-negative, without affecting the values of the relevant ratios). If  $n$  is even, the Hodge star is an involution. Let  $\eta = \eta_+ + \eta_-$  be the decomposition into selfdual and anti-selfdual parts under Hodge  $*$ . Then

$$\begin{aligned} \eta^2 &= (\eta_+ + \eta_-)^2 \\ &= \eta_+^2 + \eta_-^2 \end{aligned} \quad (6.2)$$

Thus

$$|\eta^2| = |\eta_+^2| - |\eta_-^2| \leq |\eta_+^2|. \quad (6.3)$$

Meanwhile,

$$\|\eta_+\| = \frac{1}{2} (\|\eta + *\eta\|) \leq \frac{1}{2} (\|\eta\| + \|\eta\|) = \|\eta\|$$

by the triangle inequality. Thus,  $\|\eta_+\| \leq \|\eta\|$  and we therefore conclude that

$$\frac{|\eta^2|}{\|\eta\|^2} \leq \frac{|\eta_+^2|}{\|\eta_+\|^2} \leq \text{Wirt}_{sd},$$

proving that  $\text{Wirt}_n = \text{Wirt}_{sd}$ .  $\square$

**Proposition 6.4.** *Let  $X$  be an orientable, closed manifold of dimension  $2n$ , with  $b_n(X) = 1$ . Then*

$$\text{SR}_n(X) \leq \text{Wirt}_n.$$

*Proof.* By Poincaré duality, the fundamental cohomology class in the group  $H^{2n}(X; \mathbb{Z}) \simeq \mathbb{Z}$  is the cup square of a generator of the cohomology group  $H^n(X; \mathbb{Z})_{\mathbb{R}} \simeq \mathbb{Z}$ . The inequality is now immediate by applying the method of proof of (4.7).  $\square$

Recall that the cohomology ring for  $\mathbb{C}\mathbb{P}^n$  is polynomial on a single 2-dimensional generator, truncated at the fundamental class. The cohomology ring for  $\mathbb{H}\mathbb{P}^n$  is the polynomial ring on a single 4-dimensional generator, similarly truncated. Thus the middle dimensional Betti number is 1 if  $n$  is even and 0 if  $n$  is odd.

**Corollary 6.5.** *Let  $n \in \mathbb{N}$ . We have the following bounds for the middle-dimensional stable systolic ratio:*

$$\text{SR}_{4n}(\mathbb{H}\mathbb{P}^{2n}) \leq \text{Wirt}_{4n}$$

$$\text{SR}_{2n}(\mathbb{C}\mathbb{P}^{2n}) \leq \text{Wirt}_{2n}$$

$$\text{SR}_8(M^{16}) \leq \text{Wirt}_8$$

where  $M^{16}$  is the Cayley projective plane.

**Remark 6.6.** The systolic ratio of the symmetric metric of  $\mathbb{C}\mathbb{P}^4$  is 6, while by Proposition 10.1 we have  $\text{Wirt}_4 = 14 > 6$ , so that Corollary 6.5 gives a weaker upper bound of 14 for the optimal systolic ratio of  $\mathbb{C}\mathbb{P}^4$ . Thus it is in principle impossible to calculate the optimal systolic ratio for either  $\mathbb{H}\mathbb{P}^2$  or  $\mathbb{C}\mathbb{P}^4$  by any direct generalisation of Gromov's calculation (4.7).

The detailed calculation of the Wirtinger constant  $\text{Wirt}_4$  appears in Section 10.

## 7. $BG$ SPACES AND A HOMOTOPY EQUIVALENCE

Systolically interesting metrics can be constructed as pullbacks by homotopy equivalences. A particularly useful one is described below.

**Proposition 7.1.** *The complex projective  $2n$ -space  $\mathbb{C}\mathbb{P}^{2n}$  admits a degree 1 map to the quaternionic projective space  $\mathbb{H}\mathbb{P}^n$ .*



*Proof.* Such a map can be defined in coordinates by including  $\mathbb{C}^{2n+1}$  in  $\mathbb{C}^{2n+2}$  as a hyperplane, identifying  $\mathbb{C}^{2n+2}$  with  $\mathbb{H}^{n+1}$ , and passing to the appropriate quotients. To verify the assertion concerning the degree in a conceptual fashion, we proceed as follows. We imbed  $\mathbb{C}\mathbb{P}^{2n}$  as the  $(4n)$ -skeleton of  $\mathbb{C}\mathbb{P}^\infty$ . The latter is a model for the classifying space  $BS^1$  of the circle. Similarly, we have

$$\mathbb{H}\mathbb{P}^n = (\mathbb{H}\mathbb{P}^\infty)^{(4n)} \subset \mathbb{H}\mathbb{P}^\infty \simeq BS^3,$$

where  $S^3$  is identified with the unit quaternions. Namely,  $BG$  can be characterized as the quotient of a contractible space  $S$  by a free  $G$  action. But  $\mathbb{H}\mathbb{P}^\infty$  is such a quotient for  $S = S^\infty$  and  $G = S^3$ . The inclusion of  $S^1$  as a subgroup of  $S^3$  defines a map  $\mathbb{C}\mathbb{P}^\infty \rightarrow \mathbb{H}\mathbb{P}^\infty$ . The composed map  $\mathbb{C}\mathbb{P}^{2n} \hookrightarrow \mathbb{C}\mathbb{P}^\infty \rightarrow \mathbb{H}\mathbb{P}^\infty$  is compressed, using the cellular approximation theorem, to the  $(4n)$ -skeleton. In matrix terms, an element  $u \in S^1$  goes to the element

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \in SU(2) = S^3. \tag{7.1}$$

The induced map on cohomology is computed for the infinite dimensional spaces, and then restricted to the  $(4n)$ -skeleta. By Proposition 7.2, the cohomology of  $BS^3$  is  $\mathbb{Z}[c_2]$ , *i.e.* a polynomial algebra on a 4-dimensional generator  $c_2$ , given by the second Chern class. Thus, to compute the induced homomorphism on  $H^4$ , we need to compute  $c_2$  of the sum of the tautological line bundle  $L$  on  $\mathbb{C}\mathbb{P}^\infty$  and its inverse, *cf.* (7.1). By the sum formula, it is

$$-c_1(L)^2,$$

but this is a generator of  $H^4(\mathbb{C}\mathbb{P}^\infty)$ . In other words, the map

$$H^4(BS^3) \rightarrow H^4(BS^1)$$

is an isomorphism. From the structure of the cohomology algebra, we see that the same is true for the induced homomorphism in  $H^{4n}$ . The inclusions of the  $(4n)$ -skeleta of these  $BG$  spaces are isomorphisms on cohomology  $H^{4n}$ , as well, in view of the absence of odd dimensional cells. Hence the conclusion follows for these finite-dimensional projective spaces.  $\square$

The lower bound of Theorem 1.2 for the optimal systolic ratio of  $\mathbb{H}\mathbb{P}^2$  follows from the two propositions below.

**Proposition 7.2.** *We have  $H^*(BS^3) = \mathbb{Z}[v]$ , where the element  $v$  is 4-dimensional. Meanwhile,  $H^*(BS^1) = \mathbb{Z}[c]$ , where  $c$  is 2-dimensional. Here  $i^*(v) = -c^2$  (with usual choices for basis),  $S^3 = SU(2)$ , and  $v$  is the second Chern class.*

$$\begin{array}{ccc}
H_6(\mathbb{C}\mathbb{P}^4 \cup e^3) & \longrightarrow & \pi_6(\mathbb{C}\mathbb{P}^4 \cup e^3) \\
\alpha \downarrow & & \uparrow \gamma \\
H_7(\mathbb{H}\mathbb{P}^2, \mathbb{C}\mathbb{P}^4 \cup e^3) & \xrightarrow{\beta^{-1}} & \pi_7(\mathbb{H}\mathbb{P}^2, \mathbb{C}\mathbb{P}^4 \cup e^3)
\end{array}$$

FIGURE 7.1. Commutation of boundary and Hurewicz homomorphisms

Now restrict attention to the  $4n$ -skeleta of these spaces. We obtain a map

$$\mathbb{C}\mathbb{P}^{2n} \rightarrow \mathbb{H}\mathbb{P}^n \quad (7.2)$$

which is degree one (from the cohomology algebra).

**Proposition 7.3.** *There exists a map  $\mathbb{H}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^{2n} \cup e^3 \cup e^7 \cup \dots \cup e^{4n-1}$  defining a homotopy equivalence.*

*Proof.* Coning off a copy of  $\mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^{2n}$ , we note that the map (7.2) factors through the CW complex  $\mathbb{C}\mathbb{P}^{2n} \cup e^3$ .

The map  $\mathbb{C}\mathbb{P}^4 \cup e^3 \rightarrow \mathbb{H}\mathbb{P}^2$  is an isomorphism on homology through dimension 5, and a surjection in dimension 6. We consider the pair

$$(\mathbb{H}\mathbb{P}^2, \mathbb{C}\mathbb{P}^4 \cup e^3).$$

Its homology vanishes through dimension 6 by the exact sequence of a pair. The relative group  $H_7(\mathbb{H}\mathbb{P}^2, \mathbb{C}\mathbb{P}^4 \cup e^3)$  is mapped by the boundary map to  $H_6(\mathbb{C}\mathbb{P}^4 \cup e^3) = \mathbb{Z}$ , generated by an element  $h \in H_6(\mathbb{C}\mathbb{P}^4 \cup e^3)$ . We therefore obtain an isomorphism

$$\alpha : H_6(\mathbb{C}\mathbb{P}^4 \cup e^3) \rightarrow H_7(\mathbb{H}\mathbb{P}^2, \mathbb{C}\mathbb{P}^4 \cup e^3),$$

*cf.* Figure 7.1.

Both spaces are simply connected and the pair is 6-connected as a pair. Applying the relative Hurewicz theorem, we obtain an isomorphism

$$\beta : \pi_7(\mathbb{H}\mathbb{P}^2, \mathbb{C}\mathbb{P}^4 \cup e^3) \rightarrow H_7(\mathbb{H}\mathbb{P}^2, \mathbb{C}\mathbb{P}^4 \cup e^3).$$

Applying the boundary homomorphism

$$\gamma : \pi_7(\mathbb{H}\mathbb{P}^2, \mathbb{C}\mathbb{P}^4 \cup e^3) \rightarrow \pi_6(\mathbb{C}\mathbb{P}^4 \cup e^3),$$

we obtain an element

$$h' = \gamma \circ \beta^{-1} \circ \alpha(h) \in \pi_6(\mathbb{C}\mathbb{P}^4 \cup e^3) \quad (7.3)$$

which generates  $H_6$  and is mapped to  $0 \in \pi_6(\mathbb{H}\mathbb{P}^2)$ .

We now attach a 7-cell to the complex  $\mathbb{C}\mathbb{P}^4 \cup e^3$  using the element  $h'$  of (7.3). We obtain a new CW complex

$$X = (\mathbb{C}\mathbb{P}^4 \cup e^3) \cup_{h'} e^7,$$

and a map  $X \rightarrow \mathbb{H}\mathbb{P}^2$ , by choosing a nullhomotopy of the composite map to  $\mathbb{H}\mathbb{P}^2$ . The new map is an isomorphism on all homology. Since both spaces are simply connected, the map is a homotopy equivalence. Reversing the arrow, we obtain a homotopy equivalence from  $\mathbb{H}\mathbb{P}^2$  to the union of  $\mathbb{C}\mathbb{P}^4$  with cells of dimension 3 and 7. A similar argument, applied inductively, establishes the general case.  $\square$

### 8. LOWER BOUND FOR QUATERNIONIC PROJECTIVE SPACE

In this section, we apply the homotopy equivalence constructed in Section 7, so as to obtain systolically interesting metrics.

**Proposition 8.1.** *One can homotope the map of Proposition 7.3 to a simplicial map, and choose a point in a cell of maximal dimension in*

$$\mathbb{C}\mathbb{P}^{2n} \subset \mathbb{C}\mathbb{P}^{2n} \cup e^3 \cup \dots \cup e^{4n-1} \tag{8.1}$$

*with a unique inverse image.*

*Proof.* To fix ideas, consider the case  $n = 2$ . The inverse image of a little ball around such a point is a union of balls mapping the obvious way to the ball in  $\mathbb{C}\mathbb{P}^4 \cup e^3 \cup e^7$ . We need to cancel balls occurring with opposite signs. Take an arc connecting the boundaries of two such balls where the end points are the same point of the sphere. Apply homotopy extension to make the map constant on a neighborhood of this arc ( $\pi_1$  of the target is 0). Then the union of these balls and fat arc is a bigger ball and we have a nullhomotopic map to the sphere on the boundary. We can homotope the map to the disc relative to the boundary to now lie in the sphere.  $\square$

**Corollary 8.2.** *The optimal middle dimensional stable systolic ratio of  $\mathbb{H}\mathbb{P}^{2n}$  equals that of  $\mathbb{C}\mathbb{P}^{4n}$ .*

*Proof.* We first prove the inequality  $\text{SR}_{4n}(\mathbb{C}\mathbb{P}^{4n}) \geq \text{SR}_{4n}(\mathbb{H}\mathbb{P}^{2n})$ . We exploit the degree one map (7.2). Recall that a map is called *monotone* if the preimage of every connected set is connected. By the work of A. Wright [Wr74], the map (7.2) can be homotoped to a simplicial monotone map. In particular, the preimage of every top-dimensional simplex is a single top-dimensional simplex. Thus the pull-back “metric” has the same total volume as the metric of the target. Pulling back metrics from  $\mathbb{H}\mathbb{P}^{2n}$  to  $\mathbb{C}\mathbb{P}^{4n}$  by the monotone simplicial map completes the proof in this direction.

Let us prove the opposite inequality. To fix ideas, we let  $n = 1$ . We need to show that  $\text{SR}_4(\mathbb{C}\mathbb{P}^4) \leq \text{SR}_4(\mathbb{H}\mathbb{P}^2)$ . Once the map

$$f : \mathbb{H}\mathbb{P}^2 \rightarrow \mathbb{C}\mathbb{P}^4 \cup e^3 \cup e^7 \tag{8.2}$$

is one-to-one on an 8-simplex

$$\Delta \subset \mathbb{C}\mathbb{P}^4 \cup e^3 \cup e^7$$

of the target (by Proposition 8.1), we argue as follows. The images of the attaching maps of  $e^3$  and  $e^7$  may be assumed to lie in a hyperplane  $\mathbb{C}\mathbb{P}^3 \subset \mathbb{C}\mathbb{P}^4$ . Take a self-diffeomorphism

$$\phi : \mathbb{C}\mathbb{P}^4 \rightarrow \mathbb{C}\mathbb{P}^4 \tag{8.3}$$

preserving the hyperplane, and sending the 8-simplex  $\Delta$  to the complement of a thin neighborhood of the hyperplane, so that most of the volume of the symmetric metric of  $\mathbb{C}\mathbb{P}^4$  is contained in the image of  $\Delta$ .

Now pull back the metric of the target by the composition  $\phi \circ f$  of the maps (8.2) and (8.3). The resulting “metric” on  $\mathbb{H}\mathbb{P}^2$  is degenerate on certain simplices. The metric can be inflated slightly to make the quadratic form nondegenerate everywhere, without affecting the total volume significantly. The proof is completed by the following proposition.  $\square$

**Proposition 8.3.** *Fix any background metric on  $\mathbb{C}\mathbb{P}^{4n}$ , e.g. the Fubini-Study. Then the metric can be extended to the 3-cell, the 7-cell,  $\dots$ , the  $(8n - 1)$ -cell, as in (8.1), in such a way as to decrease the stable systole by an arbitrarily small amount.*

*Proof.* We work in the category of simplicial polyhedra  $X$ , cf. [Ba06]. Here volumes and systoles are defined, as usual, simplex by simplex. When attaching a cell along its boundary, the attaching map is always assumed to be simplicial, so that all systolic notions are defined on the new space, as well.

The metric on the attached cells needs to be chosen in such a way as to contain a long cylinder capped off by a hemisphere.

To make sure the attachment of a cell  $e^p$  does not significantly decrease the stable systole, we argue as follows.

To fix ideas, let  $n = 1$ . Normalize  $X$  to unit stable 4-systole. Let  $W = X \cup e^p$ , and consider a metric on  $e^p$  which includes a cylinder of length  $L \gg 0$ , based on a sphere  $S^{p-1}$ , of radius  $R$  chosen in such a way that the attaching map  $\partial e^p \rightarrow X$  is distance-decreasing. Here  $R$  is fixed throughout the argument (and in particular is independent of  $L$ ).

Now consider an  $n$ -fold multiple of the generator  $g \in H_4(W)$ , well approximating the stable norm in the sense of (4.3). Consider a simplicial 4-cycle  $M$  with integral coefficients, in the class  $ng \in H_4(W)$ . We are looking for a lower bound for the stable norm  $\|g\|$  in  $W$ . Here we have to deal with the possibility that the 4-cycle  $M$  might “spill” into the cell  $e^p$ . Applying the coarea inequality  $\text{vol}_4(M) \geq \int_0^L \text{vol}_3(M_t) dt$

along the cylinder, we obtain a 3-dimensional section  $S = M_{t_0}$  of  $M$  of 3-volume at most

$$\text{vol}_3(S) = \frac{n\|g\|}{L}, \tag{8.4}$$

*i.e.* as small as one wishes compared to the 4-volume of  $M$  itself. Here  $M$  decomposes along  $S$  as the union

$$M = M_+ \cup M_-$$

where  $M_+$  admits a distance decreasing projection to the polyhedron  $X$ , while  $M_-$  is entirely contained in  $e^p$ . For any 4-chain  $C \subset S^{p-1}$  filling  $S$ , the new 4-cycle

$$M' = M_+ \cup C$$

represents the same homology class  $ng \in H_4(W)$ , since the difference 4-cycle  $M - M'$  is contained in a  $p$ -ball whose homology is trivial. Now we apply the linear (without the exponent  $\frac{n+1}{n}$ ) isoperimetric inequality in  $S^{p-1}$ . This allows us to fill the section  $S = \partial M_+$  by a suitable 4-chain  $C \subset S^{p-1}$  of volume at most

$$\text{vol}_4(C) \leq f(R)n\|g\|L^{-1}$$

by (8.4), where  $f(R)$  is a suitable function of  $R$ . The corresponding cycle  $M'$  has volume at most

$$\left(n + \frac{n}{L}\right) \|g\| = n\|g\|(1 + f(R)L^{-1}).$$

Since  $M'$  admits a short map to  $X$ , its volume is bounded below by  $n$ . Thus,  $\frac{1}{n}M'$  is a cycle in  $X$  representing the class  $g$ , whose mass exceeds the mass of  $\frac{1}{n}M$  at most by an arbitrarily small amount. This yields a lower bound for  $\|g\|$  which is arbitrarily close to 1. Note that similar arguments have appeared in the work of I. Babenko and his students [Ba93, Ba02, Ba04, BB05, Ba06], as well as the recent work of M. Brunnbauer [Br08a, Br07].  $\square$

## 9. THE CAYLEY FORM AND THE KRAINES FORM

The proof of the upper bound (1.6) for the optimal stable 4-systolic ratio depends on the calculation of the Wirtinger constant  $\text{Wirt}_4$  of  $\mathbb{R}^8$ , *cf.* Corollary 6.5.

This section contains an explicit description (9.3) of the Cayley 4-form  $\omega_{\text{Ca}}$  in terms of a Euclidean basis. The seven self-dual forms appearing in the decomposition of  $\omega_{\text{Ca}}$  turn out to have Lie-theoretic significance as a basis for a Cartan subalgebra of the Lie algebra  $E_7$ , discussed in detail in Section 10. The fact that  $\omega_{\text{Ca}}$  has unit comass

constitutes the lower bound part of the evaluation of the Wirtinger constant of  $\mathbb{R}^8$ . The upper bound follows from the Lie-theoretic analysis of Section 10.

In more detail, let  $\{dx_1, dx_2, dx_3, dx_4\}$  denote the dual basis to the standard real basis  $\{1, i, j, k\}$  for the quaternion algebra  $\mathbb{H}$ . Furthermore, let  $\{dx_\ell, dx_{\ell'}\}$ , where  $\ell = 1, \dots, 4$ , be the dual basis for  $\mathbb{H}^2$ . The three symplectic forms  $\omega_i, \omega_j$ , and  $\omega_k$  on  $\mathbb{H}^2$  defined by the three complex structures  $i, j, k$  are

$$\begin{aligned}\omega_i &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_{1'} \wedge dx_{2'} + dx_{3'} \wedge dx_{4'}, \\ \omega_j &= dx_1 \wedge dx_3 - dx_2 \wedge dx_4 + dx_{1'} \wedge dx_{3'} - dx_{2'} \wedge dx_{4'}, \\ \omega_k &= dx_1 \wedge dx_4 + dx_2 \wedge dx_3 + dx_{1'} \wedge dx_{4'} + dx_{2'} \wedge dx_{3'}.\end{aligned}\tag{9.1}$$

Let

$$dx_{abcd} := dx_a \wedge dx_b \wedge dx_c \wedge dx_d,$$

where  $\{a, b, c, d\} \subset \{1, \dots, 4, 1', \dots, 4'\}$ . The corresponding wedge squares satisfy

$$\begin{aligned}\frac{1}{2}\omega_i^2 &= (dx_{1234} + dx_{1'2'3'4'}) + (dx_{121'2'} + dx_{343'4'}) + (dx_{123'4'} + dx_{341'2'}), \\ \frac{1}{2}\omega_j^2 &= (dx_{1234} + dx_{1'2'3'4'}) + (dx_{131'3'} + dx_{242'4'}) - (dx_{132'4'} + dx_{241'3'}), \\ \frac{1}{2}\omega_k^2 &= (dx_{1234} + dx_{1'2'3'4'}) + (dx_{141'4'} + dx_{232'3'}) + (dx_{142'3'} + dx_{231'4'})\end{aligned}\tag{9.2}$$

The seven distinct self-dual 4-forms appearing in decomposition (5.5) of the Kraines form, which are also displayed in parentheses in (9.2), form a basis of a 7-dimensional abelian subalgebra  $\mathfrak{h}$  of the exceptional real Lie algebra  $E_7$ . In fact, the subalgebra that they generate is a maximal abelian subalgebra of  $E_7$ , as explained in Section 10. The Cayley form

$$\omega_{Ca} = \frac{1}{2} (\omega_i^2 + \omega_j^2 - \omega_k^2)$$

is the sum of the seven selfdual forms, with suitable signs, and without multiplicities:

$$\omega_{Ca} = e^{1234} + e^{1256} + e^{1278} + e^{1357} - e^{1467} - e^{1368} - e^{1458},\tag{9.3}$$

where  $e^{abcd} = dx_{abcd} + *dx_{abcd}$ , while indices  $1' \dots, 4'$  are relabeled as  $5, \dots, 8$ .

**Proposition 9.1.** *The Cayley form has unit comass.*

*Proof.* R. Harvey and H. B. Lawson [HL82] clarify the nature of the Cayley form, as follows. They realize the Cayley form as the real part of a suitable multiple vector product on  $\mathbb{R}^8$  [HL82, Lemma B.9(3), p. 147].

One can then calculate the comass of the Cayley form, denoted  $\Phi$  in [HL82], as follows. Let  $\zeta = x \wedge y \wedge z \wedge w$  be a 4-tuple. Then

$$\Phi(\zeta) = \Re(x \times y \times z \times w) \leq |x \times y \times z \times w| = |x \wedge y \wedge z \wedge w|,$$

and therefore  $\|\Phi\| = 1$ . See also [KS08] for an alternative proof.  $\square$

By way of comparison, note that the square  $\eta = \tau^2$  of the Kahler form  $\tau$  on  $\mathbb{C}^4$  satisfies  $\frac{\|\eta\|^2}{\|\eta\|^2} = 6$ . Meanwhile, the Cayley form yields a higher ratio, namely 14, by Proposition 9.1. The Cayley form, denoted  $\omega_1$  in [DHM88, p. 14], has unit comass, satisfies  $|\omega_1|^2 = 14$ , and is shown there to have the maximal ratio among all selfdual forms on  $\mathbb{R}^8$ .

The  $E_7$  viewpoint was not clarified in [HL82, DHM88]. Thus, the “very nice seven-dimensional cross-section” referred to in [DHM88, p. 3, line 8] and [DHM88, p. 12, line 5], is in fact a Cartan subalgebra of  $E_7$ , cf. Lemma 10.6.

The calculation of  $\text{Wirt}_4$  results from combining Lemma 6.3 and [DHM88]. We will give a more transparent proof, using  $E_7$ , in the next section.

### 10. $E_7$ , HUNT’S TRICK, AND WIRTINGER CONSTANT OF $\mathbb{R}^8$

To prove the upper bound of (1.6), by Proposition 6.4, we need to calculate the Wirtinger constant of  $\mathbb{R}^8$ .

**Proposition 10.1.** *We have  $\text{Wirt}_2 = 2$ , while  $\text{Wirt}_4 = 14$ .*

*Proof.* By the Wirtinger inequality and Corollary 3.6, we obtain the value  $\text{Wirt}_2 = 2$ .

To calculate the value of  $\text{Wirt}_4$ , it remains to show that no 4-form  $\omega$  on  $\mathbb{R}^8$  has a ratio  $|\omega|^2/\|\omega\|^2$  higher than 14. By Lemma 6.3, we can restrict attention to selfdual forms. We will decompose every such 4-form into the sum of at most 14 simple (decomposable) forms with the aid of a particular representation of a self-dual 4-form, stemming from an analysis of the exceptional Lie algebra  $E_7$ . Such a representation of a self-dual 4-form was apparently first described explicitly by L. Antonyan [An81], in the context of the study of  $\theta$ -groups by V. Kac and E. Vinberg [GV78] and E. Vinberg and A. Elashvili [VE78].

We first recall the structure of the Lie algebra  $E_7$ , following the approach of J. Adams [Ad96]. The Lie algebra  $E_7$  can be decomposed as a direct sum

$$E_7 = sl(8) \oplus \Lambda^4(8), \tag{10.1}$$

cf. [Ad96, p. 76]. The Lie bracket on  $sl(8) \subset E_7$  is the standard one. The Lie bracket  $[a, x]$  of an element  $a \in sl(8)$  with an element  $x \in \Lambda^4(8)$  is given by the standard action of  $sl(8)$  on  $\Lambda^4(8)$ . The Lie bracket of

a pair of elements  $x, y \in \Lambda^4(8)$  is defined by the formula guaranteeing  $ad$ -invariance, *cf.* [Ad96, p. 76, line 9]:

$$(a, [x, y])_{sl} = ([a, x], y)_\Lambda.$$

**Proposition 10.2.** *In coordinates, the Lie bracket on  $\Lambda^4(8) \subset E_7$  can be written as follows. Let  $e_1, \dots, e_8$  be a basis of determinant 1. Then*

$$\begin{aligned} [e_{r_1} e_{r_2} e_{r_3} e_{r_4}, e_{s_1} e_{s_2} e_{s_3} e_{s_4}] &= 0 \quad \text{if two or more } r\text{'s equal } s\text{'s,} \\ [e_1 e_2 e_3 e_4, e_4 e_5 e_6 e_7] &= e_4 \otimes e_8^*, \\ [e_1 e_2 e_3 e_4, e_5 e_6 e_7 e_8] &= \frac{1}{2}((e_1 \otimes e_1^* + e_2 \otimes e_2^* + e_3 \otimes e_3^* + e_4 \otimes e_4^*) \\ &\quad - (e_5 \otimes e_5^* + e_6 \otimes e_6^* + e_7 \otimes e_7^* + e_8 \otimes e_8^*)). \end{aligned} \tag{10.2}$$

This is proved in [Ad96, p. 76].

One can define a non-degenerate, indefinite, inner product on  $sl(8)$  by the formula

$$(a, b)_{sl} = \text{trace } ab.$$

and a non-degenerate, indefinite, inner product on  $\Lambda^4(8)$  by the formula

$$(\alpha, \beta)_\Lambda \, d\text{vol} = \alpha \wedge \beta,$$

where  $d\text{vol}$  is the volume form. If this definition is extended to an inner product on  $E_7$  in which  $sl(8)$  and  $\Lambda^4(8)$  are orthogonal, then the result is an  $ad$ -invariant, non-degenerate, indefinite inner product  $(, )$  on  $E_7$ .

**Lemma 10.3.** *The Killing form on  $E_7$  is  $36(, )$ .*

*Proof.* See [Ad96, p. 78, ‘‘Addendum’’]. □

Thus the Killing form of  $E_7$  is positive definite on the symmetric traceless matrices in  $sl(8)$ , and the self-dual 4-forms  $\Lambda_+^4$ , and negative definite on  $so(8)$  and the anti-self-dual 4-forms  $\Lambda_-^4$ .

The decomposition in (10.1) can be refined to a Cartan decomposition. Recall that, in general, a Cartan decomposition of a real Lie algebra consists of a maximal compact subalgebra, on which the restriction of the Killing form is negative definite, and an orthogonal positive definite complement. The Cartan decomposition of  $sl(8)$  is

$$sl(8) = so(8) \oplus sym_0(8),$$

where  $sym_0(8)$  is the set of  $8 \times 8$  traceless symmetric matrices. The  $so(8)$  representation  $\Lambda^4(8)$  is a direct sum

$$\Lambda^4(8) = \Lambda_+^4(8) \oplus \Lambda_-^4(8).$$



From the properties of the Killing form we see that the Cartan decomposition of  $E_7$  is given by

$$\begin{aligned} E_7 &= \mathfrak{k} \oplus \mathfrak{p} \\ \mathfrak{k} &= so(8) \oplus \Lambda_-^4(8) \\ \mathfrak{p} &= sym_0(8) \oplus \Lambda_+^4(8). \end{aligned}$$

Let  $K$  be the subgroup of the adjoint group corresponding to the maximal compact subalgebra  $\mathfrak{k}$ . One of the standard results in the theory of real reductive Lie groups, Theorem 10.4 below, establishes the  $K$ -conjugacy of maximal abelian subalgebras of the noncompact component  $\mathfrak{p}$  of the Cartan decomposition. Here the term ‘‘maximal abelian subalgebra’’ refers to a subalgebra of  $\mathfrak{p}$  which is maximal with respect to the condition of being an abelian subalgebra of the full Lie algebra  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , see [Wa88, §2.1.6, §2.3.4].

For the convenience of the reader we present a partial proof of the conjugacy theorem.

**Theorem 10.4.** [Wo67, Theorem 8.6.1] *Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the Cartan decomposition associated to a Riemannian symmetric space  $G/K$ . Let  $\mathfrak{a}$  and  $\mathfrak{a}'$  be two maximal subalgebras of  $\mathfrak{p}$ . Then*

- (1) *there exist an element  $X \in \mathfrak{a}$  whose centralizer in  $\mathfrak{p}$  is just  $\mathfrak{a}$ ,*
- (2) *there is an element  $k \in K$  such that  $Ad(k)\mathfrak{a}' = \mathfrak{a}$ ,*
- (3)  $\mathfrak{p} = \bigcup_{k \in K} Ad(k)\mathfrak{a}$ .

*Partial proof of Theorem 10.4.* The proof of item (1) makes use of the compact dual symmetric space. In the compact model the desired element of the algebra is such that the associated one parameter subgroup is dense in a maximal torus. For details of the proof of (1) see [Wo67, page 253]. We will prove (2) and (3), beginning with (3). The proof uses an idea of G. Hunt [Hu56].

Let  $X \in \mathfrak{a}$  be the element whose existence is established in (1):

$$\mathfrak{a} = \{Y \in \mathfrak{p} \mid [Y, X] = 0\}.$$

Let  $Z \in \mathfrak{p}$  be arbitrary. Consider the following function  $f$  on  $SO(8)$ :

$$f(k) = B(Ad(k)Z, X),$$

where  $B(-, -)$  is the Killing form on  $\mathfrak{g}$ . Since  $SO(8)$  is compact, the function attains a minimum at some point  $k$ . For all  $W \in so(8)$ , we have

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} B(Ad(\exp(tW)k)Z, X) \\ &= B([W, Ad(k)Z], X) \\ &= B(W, [Ad(k)Z, X]) \end{aligned}$$

by the  $ad$ -invariance of the Killing form. Since the Killing form on  $so(8)$  is negative definite, it follows that  $[Ad(k)Z, X] = 0$ . Thus  $Ad(k)Z \in \mathfrak{a}$ , and  $Z \in Ad(k^{-1})(\mathfrak{a})$ , proving (3).

To prove (2) let  $X'$  be an element whose centralizer in  $\mathfrak{p}$  is  $\mathfrak{a}'$ :

$$\mathfrak{a}' = \{Y \in \mathfrak{p} \mid [Y, X'] = 0\}.$$

We have just proved that there exists an element  $k \in K$  such that

$$[Ad(k)(X'), X] = 0;$$

therefore,  $Ad(k)(X') \in \mathfrak{a}$ . Thus  $\mathfrak{a}$  centralizes  $Ad(k)(X')$  and  $Ad(k^{-1})\mathfrak{a}$  centralizes  $X'$ ; so  $Ad(k^{-1})\mathfrak{a} \subset \mathfrak{a}'$ . Similarly,  $[Ad(k^{-1})(X), X'] = 0$  and  $Ad(k)\mathfrak{a}' \subset \mathfrak{a}$ . Thus  $Ad(k)\mathfrak{a}' = \mathfrak{a}$ , concluding the proof of (2).  $\square$

For our purposes, we do not want to apply the conjugacy property to the Lie algebra  $E_7$ , since we are interested in  $SO(8)$ -conjugacy in  $\Lambda_+^4(8)$ . The maximal compact subgroup of  $E_7$  has Lie algebra  $so(8) \oplus \Lambda_+^4(8)$ , and the action of the corresponding compact subgroup of the adjoint group would mix  $\Lambda_+^4(8)$  and  $sym_0(8)$ . Rather we restrict to a subalgebra of  $E_7$ , denoted  $\mathfrak{g}_0$ , defined by

$$\mathfrak{g}_0 := so(8) \oplus \Lambda_+^4(8) = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \quad (10.3)$$

That this defines a subalgebra follows from the definition of the Lie bracket in  $E_7$ , in particular equation (10.2) implies that the Lie bracket of two self-dual forms is a skew symmetric matrix and thus is in  $so(8)$ .

**Definition 10.5.** Define the subspace  $\mathfrak{h}$  of  $\Lambda_+^4(8)$  as the span of the self-dual 4-forms of (9.3), namely

$$\mathfrak{h} = \mathbb{R}e^{1234} \oplus \mathbb{R}e^{1256} \oplus \mathbb{R}e^{1278} \oplus \mathbb{R}e^{1357} \oplus \mathbb{R}e^{1467} \oplus \mathbb{R}e^{1368} \oplus \mathbb{R}e^{1458}.$$

**Lemma 10.6.** *The subspace  $\mathfrak{h}$  is a maximal abelian subalgebra of*

$$\Lambda_+^4(8) = \mathfrak{p}_0 \subset \mathfrak{g}_0.$$

*Proof.* The bracket on  $\mathfrak{g}_0$  is the restriction of the  $E_7$  Lie bracket described in [Ad96, p. 76] and Proposition 10.2. The bracket of two simple 4-forms vanishes whenever the forms have a common  $dx_i \wedge dx_j$  factor, and it is easy to see that this condition is satisfied for all the Lie brackets of pairs of simple forms which occur among the seven self-dual forms. Since  $E_7$  is of rank 7, the maximal dimension of an abelian subalgebra of  $\mathfrak{p} \supset \mathfrak{p}_0$  is 7, which gives an upper bound on the dimension of an abelian subalgebra of  $\Lambda_+^4(8)$ .  $\square$

The subalgebra  $\mathfrak{h}$  contains the Cayley form  $\omega_{Ca}$ , see [Jo00, Definition 10.5.1]. The Cayley form is the signed sum of the 7 self-dual 4-forms defining the basis of  $\mathfrak{h}$  in Definition 10.5. The exact expression for  $\omega_{Ca}$  is given in (9.3), see [Br87] and [Jo00, equation 10.19].

The conjugacy theorem, Theorem 10.4, shows that any self-dual 4-form is  $SO(8)$ -conjugate to an element of  $\mathfrak{h}$ . Note that the coefficients of any comass 1 self-dual 4-form relative to the basis in Definition 10.5 have absolute value at most 1. It follows that the Cayley form, which is a signed sum of all 7 forms, has maximal Euclidean norm 14, completing the proof of Proposition 10.1.  $\square$

### 11. $b_4$ -CONTROLLED SURGERY AND SYSTOLIC RATIO

We will refer to an 8-manifold with exceptional  $\text{Spin}(7)$  holonomy as a Joyce manifold, *cf.* [Jo00]. Known examples of Joyce manifolds have middle dimensional Betti number ranging from 84 into the tens of thousands. It is unknown whether or not a Joyce manifold with  $b_4 = 1$  exists. Yet no restrictions on  $b_4$  other than  $b_4 \geq 1$  are known. The obligatory cohomology class in question is represented by a parallel Cayley 4-form  $\omega_{\text{Ca}}^{\parallel}$ , *cf.* (9.3), representing a generator in the image of integer cohomology.

**Proposition 11.1.** *A hypothetical Joyce manifold  $\mathcal{J}$  with unit middle Betti number would necessarily have a systolic ratio of 14.*

*Proof.* A generator of  $H^4(\mathcal{J}, \mathbb{Z})_{\mathbb{R}} = \mathbb{Z}$  is represented by  $\omega_{\text{Ca}}^{\parallel}$ . By Poincaré duality, the square of the generator is the fundamental cohomology class of  $\mathcal{J}$ . Thus, similarly to (4.7) and (5.6), we can write

$$\begin{aligned} 1 &= \int_{\mathcal{J}} \left| \omega_{\text{Ca}}^{\parallel} \wedge \omega_{\text{Ca}}^{\parallel} \right|^2 d\text{vol} \\ &= 14 \left( \|\omega_{\text{Ca}}^{\parallel}\|_{\infty} \right)^2 \text{vol}_8(\mathcal{J}), \end{aligned} \tag{11.1}$$

and the proposition follows by duality of comass and stable norm, as in Gromov’s calculation.  $\square$

The theorem below may give an idea of the difficulty involved in evaluating the optimal ratio in the quaternionic case, as compared to Pu’s and Gromov’s calculations.

**Theorem 11.2.** *If there exists a Joyce manifold with  $b_4 = 1$ , then the common value of the middle dimensional optimal systolic ratio of  $\mathbb{H}\mathbb{P}^2$  and  $\mathbb{C}\mathbb{P}^4$  equals 14. In particular, in neither case is the symmetric metric optimal for the systolic ratio.*

We introduce a convenient term in the context of surgery on an 8-dimensional manifold  $M$ .

**Definition 11.3.** A  $b_4$ -controlled surgery is a surgery which induces an isomorphism of the 4-Jacobi torus (1.3).

In particular, such a surgery does not alter the middle dimensional Betti number  $b_4(M)$ . It was shown in Section 8 that such a surgery does not alter the stable 4-systolic ratio.

**Proposition 11.4.** *Every simply connected spin 8-manifold  $M$  satisfying  $b_4(M) = 1$  admits a sequence of  $b_4$ -controlled surgeries, resulting in a 2-connected manifold, denoted  $\mathcal{P}$ , with the rational cohomology ring of the quaternionic projective plane:  $H^*(\mathcal{P}, \mathbb{Q}) = H^*(\mathbb{H}\mathbb{P}^2, \mathbb{Q})$ .*

*Proof.* We choose a system of generators  $(g_i)$  for  $H_2(M, \mathbb{Z})$ . By the Hurewicz theorem, each  $g_i$  can be represented by an imbedded 2-sphere  $S_i \subset M$ . The spin condition implies the triviality of the normal bundle of each  $S_i$ . We can therefore perform successive surgeries along each  $S_i$  to remove 2-dimensional homology, resulting in a 2-connected manifold  $M'$ . Clearly,  $b_4(M') = 1$ , while the third Betti number may have changed during the surgeries.

Similarly, we choose a system of 3-spheres representing a basis for  $H_3(M', \mathbb{Q})$ . The normal bundles are automatically trivial, and surgeries along the 3-spheres reduce the  $b_3$  to zero without altering  $b_4$ , resulting in a manifold  $\mathcal{P}$  with the rational cohomology of the quaternionic projective plane by Poincaré duality.  $\square$

**Corollary 11.5.** *A Joyce manifold with  $b_4 = 1$  admits a sequence of  $b_4$ -controlled surgeries which produce a manifold  $\mathcal{P}$  which has the rational cohomology of  $\mathbb{H}\mathbb{P}^2$ .*

*Proof.* Manifolds with Spin(7) holonomy are simply connected and spin by [Jo00, Theorem 10.6.8, p. 261], and we apply Proposition 11.4.  $\square$

Note that the “cylinder” of a surgery transforming  $X$  to  $Y$  is homotopy equivalent to a complex  $W$  obtained from  $X$  by attaching a cell. Thus the inclusion of  $Y$  as the other end of the cylinder defines a map  $Y \rightarrow W$  inducing an isomorphism of the Jacobi torus  $J_4$ . Applying the pullback techniques of Section 8, we conclude that  $\text{SR}_4(X) = \text{SR}_4(Y)$ . An interesting related axiomatisation (in the case of 1-systoles) is proposed in [Br08a].

**Proposition 11.6.** *A manifold  $\mathcal{P}$  with the rational cohomology of the quaternionic projective plane admits a nonzero degree map  $\mathbb{H}\mathbb{P}^2 \rightarrow \mathcal{P}$  from  $\mathbb{H}\mathbb{P}^2$ .*

*Proof.* The fact that  $\mathbb{H}\mathbb{P}^2$  has a map of nonzero degree to a manifold with its rational cohomology algebra, follows from the formality of the space combined with the theorem of H. Shiga [Shi79]. Namely, the theorem gives enough self maps of any formal space to build its

rational homotopy type by iterated mapping cylinders. Hence  $\mathbb{H}\mathbb{P}^2$  admits a map to the rationalisation of  $\mathcal{P}$ . By compactness, the image of the map lies in a finite piece of the iterated space. The finite piece admits a retraction to  $\mathcal{P}$  itself. This produces the desired map.  $\square$

**Corollary 11.7.** *A manifold  $\mathcal{P}$  with the rational cohomology of the quaternionic projective plane satisfies  $\text{SR}_4(\mathbb{H}\mathbb{P}^2) \geq \text{SR}_4(\mathcal{P})$ .*

*Proof.* Let  $d^2$  be the degree of the map. We then construct suitable metrics on the quaternionic projective plane by pullback. The argument is similar to that of Section 8 and relies upon the existence of  $d$ -monotone maps, *i.e.* maps such that the preimage of a path-connected set has at most  $d$  path connected components, see [Wr74, Br08a, Br07]. In more detail, we have  $\text{vol}(\mathbb{H}\mathbb{P}^2) = d^2 \text{vol}(\mathcal{P})$ . Meanwhile, the induced homomorphism in  $H_4$  is multiplication by  $d$ . Since the stable norm is by definition multiplicative. Hence  $\text{stsys}_2(\mathbb{H}\mathbb{P}^2) \geq d \text{stsys}_2(\mathcal{P})$ , proving the corollary.  $\square$

**Remark 11.8.** A referee asked whether the map in Proposition 11.6 can be taken to be of degree 1. Whereas in general this is not the case, it turns out that in the absence of torsion, degree 576 is sufficient, as shown in Section 12.

*Proof of Theorem 11.2.* A Joyce manifold has systolic ratio of 14 by Proposition 11.1. By Corollary 11.5, the manifold  $\mathcal{P}$  must also satisfy  $\text{SR}_4(\mathcal{P}) = 14$ . Finally, Corollary 11.7 implies that  $\text{SR}_4(\mathbb{H}\mathbb{P}^2) = 14$ , as well.  $\square$

## 12. HOPF INVARIANT, WHITEHEAD PRODUCT, AND SYSTOLIC RATIO

This section answers a question referred to in Remark 11.8. S. Smale as well as J. Eells and N. Kuiper [EK62] proved that every manifold which is a homology  $\mathbb{H}\mathbb{P}^2$ , is homotopy equivalent to  $S^4 \cup_h e^8$ , where the attaching map lies in a class

$$[h] \in \pi_7(S^4) = \mathbb{Z} + \mathbb{Z}_{12} \tag{12.1}$$

which is an infinite generator.

Let  $m \geq 2$  be an even integer. Let  $e \in \pi_m(S^m)$  be the fundamental class. Let  $q \geq 1$ , and consider a self map of  $S^m$  of degree  $q$ . Let

$$\phi_q : \pi_{2m-1}(S^m) \rightarrow \pi_{2m-1}(S^m) \tag{12.2}$$

be the induced homomorphism. The following result is immediate from standard properties of Whitehead products [ , ].

**Lemma 12.1.** *The class  $[e, e] \in \pi_{2m-1}(S^m)$  satisfies  $\phi_q([e, e]) = q^2[e, e]$ .*

Given an element  $x \in \pi_{2m-1}(S^m)$ , we can write

$$2x = s + H(x)[e, e], \quad (12.3)$$

where  $s$  is torsion, and  $H(x)$  is the Hopf invariant of  $x$ . Note that if  $x$  is the class represented by the Hopf fibration, then  $s$  is a generator of the torsion subgroup. In particular, the class  $[e, e]$  is primitive (*i.e.* not twice another class) in the quaternionic case, unlike the complex case.

We have the following formula for the map (12.2), *cf.* B. Eckmann [Ec42] and G. Whitehead [Wh78, p. 537]:

$$\phi_q(x) = qx + \binom{q}{2} H(x)[e, e]. \quad (12.4)$$

**Lemma 12.2.** *For all  $x \in \pi_7(S^4)$ , if  $q$  is a multiple of 24, then*

$$\phi_q(x) = q^2x = \frac{q^2}{2}[e, e].$$

*Proof.* Let  $a$  be the attaching map of the true  $\mathbb{H}\mathbb{P}^2$ . By (12.3) and (12.1), the multiple  $qa$  (and hence  $q^2a$ ) is proportional to  $[e, e]$ . Therefore by (12.4), the image  $\phi_q(a)$  is also proportional to  $[e, e]$ . Thus,  $\phi_q(a)$  is proportional to every infinite generator  $x$  by Lemma 12.1, proving the lemma.  $\square$

**Theorem 12.3.** *Any homology  $\mathbb{H}\mathbb{P}^2$  admits a continuous map of degree 576 from the true  $\mathbb{H}\mathbb{P}^2$ .*

*Proof.* By Lemma 12.2, a self-map of  $S^4$  of degree a multiple of 24, necessarily sends the attaching map of the true  $\mathbb{H}\mathbb{P}^2$ , to a class proportional to the attaching map of the homology one. Hence the map can be extended over the entire 8-manifold.  $\square$

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