# FINAL REVIEW SOLUTIONS

MATH 151 SECTION 35, FALL 2005

Note that there are frequently several possible solutions to a problem, so just because your solution is not the same as the one given here doesn't mean it is incorrect.

*Problem* 1. Solve the following inequalities.

(a) $\sqrt{x^2+1} > 3$
(b) $|13x^2 - 52| < 0$

*Solution* 1. (a) Each inequality in the following sequence is equivalent to the previous one.

$$\begin{aligned} \sqrt{x^2+1} &> 3 \\ x^2+1 &> 9 \\ x^2 &> 8 \\ x > 2\sqrt{2} \quad \text{or} \quad x &< -2\sqrt{2}. \end{aligned}$$

(b) An absolute value can never be less than zero, so there are *no solutions*.

*Problem* 2. Prove that for all integers $n > 0$, we have

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

*Solution* 2. Proof by induction.

*Base Case:* For $n = 1$, the left hand side is just $2^0 = 1$, and the right hand side is $2^1 - 1 = 1$, so they are equal.

*Induction Step:* Assume that

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1; \tag{1}$$

we want to show that

$$2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

Add $2^n$ to both sides of (1) to get

$$\begin{aligned} 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} + 2^n &= 2^n - 1 + 2^n \\ &= 2 \cdot 2^n - 1 \\ &= 2^{n+1} - 1 \end{aligned}$$

which is what we wanted to show.

Therefore, by induction, the desired statement is true for all $n > 0$.

*Problem* 3. Evaluate the following limits.

(a) $\lim_{x \to 1} \dfrac{x^2+3}{x}$

(b) $\lim_{x\to 0}\left[x\left(1-\frac{1}{x}\right)\right]$

(c) $\lim_{x\to \pi}\left(\frac{\sin x}{\tan x}\right)$

*Solution* 3. (a) We have $\lim_{x\to 1} x = 1$ and $\lim_{x\to 1}(x^2+3) = 4$, since both are polynomials. Both limits are nonzero, so by the rule for limits of quotients, the limit of the given function is $\frac{4}{1} = 4$.

(b) We have $x(1-\frac{1}{x}) = x-1$ for $x \neq 0$, so the given limit is equal to $\lim_{x\to 0}(x-1) = -1$.

(c) $\lim_{x\to\pi}\sin x = 0$ and $\lim_{x\to\pi}\tan x = 0$, so we have an indeterminate form $\frac{0}{0}$ and must do something else to evaluate the limit. We have $\tan x = \frac{\sin x}{\cos x}$, and therefore when $\sin x \neq 0$, we have

$$\frac{\sin x}{\tan x} = \frac{\sin x}{\frac{\sin x}{\cos x}} = \cos x.$$

Thus the given limit is equal to $\lim_{x\to\pi}\cos x = -1$.

*Problem* 4. Give a formal (i.e. $\varepsilon - \delta$) proof that

$$\lim_{x\to 1}(x^2+3) = 4.$$

*Solution* 4. Let $\varepsilon > 0$. We want to show that there exists a $\delta > 0$ such that whenever $|x-1| < \delta$, then $|x^2+3-4| < \varepsilon$. Since

$$\begin{aligned}|x^2+3-4| &= |x^2-1| \\ &= |x-1|\cdot|x+1|\end{aligned}$$

and we can control $|x-1|$, we want to control $|x+1|$. Suppose that $\delta \leq 1$; then we will have $|x-1| < 1$, hence $|x+1| < 3$. So we take $\delta = \min\{1, \frac{\varepsilon}{3}\}$. Then if $|x-1| < \delta$, we have $|x-1| < 1$ and so $|x+1| < 3$, while

$$\begin{aligned}|x^2-1| &= |x-1|\cdot|x+1| \\ &< \frac{\varepsilon}{3}\cdot 3 = \varepsilon\end{aligned}$$

as desired.

*Problem* 5.

(a) Prove that the sum of two rational numbers is a rational number.
(b) Prove that the sum of a rational number and an irrational number is irrational.
(c) Give an example of two irrational numbers whose sum is rational.

*Solution* 5. (a) Let $\frac{p}{q}$ and $\frac{r}{s}$ be two rational numbers, where $p$, $q$, $r$, and $s$ are integers. Then their sum is

$$\frac{ps+rq}{qs}$$

which is a quotient of integers, hence rational.

(b) Proof by contradiction. Suppose that $x$ is rational and $y$ is irrational, but $x+y$ is rational. Then $y = (x+y)-x$ would be the difference of two rational numbers, hence (by part (a)) $y$ would be rational, contradicting the assumption that $y$ is irrational. Thus it must be that $x+y$ is irrational.

(c) One such pair is $\sqrt{2}$ and $-\sqrt{2}$, which are both are irrational, but their sum is the rational number 0. There are many others.

*Problem* 6. Compute $\frac{dy}{dx}$ in each case.

(a) $y = (\tan x)^2$

(b) $y = (x-1)^{1/2}(x+1)^{1/3}$

(c) $(x-y)^2 - y = 0$

*Solution* 6. (a) By the chain rule,

$$\frac{dy}{dx} = 2(\tan x)(\sec x)^2.$$

(b) By the product rule and the chain rule,

$$\frac{dy}{dx} = \frac{1}{2}(x-1)^{-1/2}(x+1)^{1/3} + \frac{1}{3}(x-1)^{1/2}(x+1)^{-2/3}.$$

(c) We differentiate the given equation implicitly, using the chain rule on $(x-y)^2$, to get

$$2(x-y)\left(1 - \frac{dy}{dx}\right) - \frac{dy}{dx} = 0.$$

We then solve for $\frac{dy}{dx}$ as follows.

$$\begin{aligned} 2(x-y) - 2(x-y)\frac{dy}{dx} - \frac{dy}{dx} &= 0 \\ 2(x-y) &= (2x-2y+1)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{2x-2y}{2x-2y+1} \end{aligned}$$

*Problem* 7. Use the formal limit definition of the derivative to show the derivatives of the following function.

$$f(x) = \frac{1}{2}x^2 + x$$

*Solution* 7. The derivative of $f$ is given by the following limit.

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\frac{1}{2}(x+h)^2 + (x+h) - \frac{1}{2}x^2 - x}{h} \\ &= \lim_{h\to 0} \frac{xh + \frac{1}{2}h^2 + h}{h} \\ &= \lim_{h\to 0} (x + \frac{1}{2}h + 1) \\ &= x + 1. \end{aligned}$$

*Problem* 8. Consider the following function.

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}.$$

Find all its critical points and determine if each is a relative maximum, a relative minimum, or neither.

*Solution* 8. The derivative of $f$ is

$$f'(x) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x > 0 \\ -2x & \text{if } x < 0 \end{cases}.$$

At 0, $f$ has a cusp and so is not differentiable there. Since $f'$ is never zero where it is defined, the only critical point is at $x = 0$. Since $f'(x) > 0$ for $x > 0$ and also for $x < 0$, by the first-derivative test the critical point $x = 0$ is neither a relative maximum nor a relative minimum.

*Problem* 9. Let $f(x) = x^3 + ax^2 + bx + c$. Under what conditions on $a$, $b$, and $c$ does $f$ have exactly one relative maximum and one relative minimum?

*Solution* 9. We have $f'(x) = 3x^2 + 2ax + b$. In order for $f$ to have exactly two relative extrema, $f'$ must have at least two distinct zeros. Since $f'$ is a quadratic, it has two distinct zeros precisely when its discriminant $4a^2 - 12b$ is positive, in other words when $a^2 > 3b$. In this case, $f'$ changes sign at each zero, and therefore the first critical point is a relative maximum and the second is a relative minimum.

*Problem* 10. Let $f(x) = 6x^5 + 13x - 1$.

(a) Show that there exists a number $a$ such that $f(a) = 0$. *Hint: Use the Intermediate Value Theorem.*
(b) Show that there do not exist numbers $a$ and $b$ with $a \neq b$ such that $f(a) = 0$ and $f(b) = 0$. *Hint: Use Rolle's Theorem or the Mean Value Theorem.*

*Solution* 10. (a) Consider the interval $[-1, 1]$, on which $f$ is defined and continuous. We have $f(-1) = -20$ and $f(1) = 18$, so since $-20 < 0 < 18$, by the intermediate value theorem there is a number $a$ in $[-1, 1]$ such that $f(a) = 0$.
(b) Suppose, for contradiction, that there did exist two such numbers $a$ and $b$ with $a \neq b$ and $f(a) = 0$ and $f(b) = 0$. Then since $f$ is differentiable everywhere, by Rolle's Theorem there would exist a number $c$ in $[a, b]$ such that $f'(c) = 0$. But $f'(x) = 30x^4 + 13$ which is never equal to zero, so this is a contradiction; hence there cannot exist such $a$ and $b$.

*Problem* 11. *Boyle's law* for a gas held at constant temperature is that $PV = \text{const}$ where $P$ is the pressure and $V$ is the volume. Suppose that a tank of natural gas is held at constant temperature and its pressure decreases by 0.05 pounds per square inch per hour. At the moment when its pressure is 5 pounds per square inch and its volume is 1000 cubic feet, how fast is its volume increasing?

*Solution* 11. We differentiate the given equation implicitly with respect to $t$ (time), remembering that the derivative of a constant is always zero:

$$P\frac{dV}{dt} + \frac{dP}{dt}V = 0.$$

At the given moment, $P = 5\,lb/in^2$, $V = 1000\,ft^3$, and $\frac{dP}{dt} = -0.05\,lb/in^2/hr$, so

$$\frac{dV}{dt} = \frac{-\frac{dP}{dt} \cdot V}{P} = \frac{-(-0.05)(1000)}{5} = 10ft^3/hr.$$

*Problem* 12. Find all relative maxima and minima of the function

$$f(x) = \frac{1}{|x| - 3}.$$

*Solution* 12. We write $f$ as a piecewise function in order to differentiate it:

$$f(x) = \begin{cases} \frac{1}{x-3} & \text{if } x \geq 0 \\ \frac{1}{-x-3} & \text{if } x < 0 \end{cases}$$

Then we get

$$f'(x) = \begin{cases} \frac{-1}{(x-3)^2} & \text{if } x > 0 \\ \frac{1}{(-x-3)^2} & \text{if } x < 0 \end{cases}$$

Note that $f$ has a cusp at $x = 0$ and is not differentiable there, and $f'$ is also undefined at $x = 3$ and $x = -3$ where $f$ has asymptotes. Since $f'$ is never zero where it is defined, these are the only critical points. Since $f'$ is always negative for $x > 0$ and always positive for $x < 0$, by the first-derivative test there is one relative maximum at $x = 0$ and no relative minima.

*Problem* 13. Find the absolute maximum and minimum values of the function $f(x) = (x-1)^2(x-2)^2$ on the interval $[0, 4]$ and say at which $x$-values they occur.

*Solution* 13. We have

$$\begin{aligned} f'(x) &= 2(x-1)(x-2)^2 + 2(x-1)^2(x-2) \\ &= 2(x-1)(x-2)(x-2+x-1) \\ &= 2(x-1)(x-2)(2x-3) \end{aligned}$$

Thus the critical points are at $x = 1$, $x = 2$, and $x = \frac{3}{2}$. We evaluate

$$\begin{aligned} f(0) &= 4 \\ f(1) &= 0 \\ f\left(\frac{3}{2}\right) &= \frac{1}{16} \\ f(2) &= 0 \\ f(4) &= 36 \end{aligned}$$

Therefore, the absolute maximum value is 36, which occurs at $x = 4$, and the absolute minumum value is 0, which occurs at $x = 1$ and $x = 2$.

*Problem* 14. True or False?

(a) If $f$ is a continuous function on $[a, b]$ and $f(a) = f(b)$, then $f$ has at least one critical point in $(a, b)$.
(b) If $f$ is continuous on the interval $(-1, 4]$ then $f$ attains an absolute maximum on that interval.
(c) There exists a continuous function which has infinitely many relative maxima.
(d) There exists a continuous function on the interval $(0, 1)$ which has infinitely many relative maxima on that interval.
(e) Suppose $f$ is a function such that whenever $|x - 3| < 0.2$, we have $|f(x) - 16| < 0.001$. Then $\lim_{x \to 3} f(x) = 16$.
(f) If a differentiable function $f$ has a relative maximum at every value of $x$, then $f$ is a constant function.
(g) If $f$ is continuous at all values of $x$, then $f$ is differentiable at all values of $x$.

(h) If $\lim_{x\to 0} |f(x)| = |L|$, then $\lim_{x\to 0} f(x) = L$.

*Solution* 14. (a) True. If $f$ is differentiable on $(a, b)$, then by Rolle's theorem there is a point where $f'(c) = 0$ which is therefore a critical point, while if $f$ is not differentiable, then a point where it is not differentiable is a critical point.

(b) False. The limit value of $f$ from the right at $-1$ might be greater than any value of $f$ in $(-1, 4]$.

(c) True; consider a constant function. A more interesting example is $f(x) = \sin x$.

(d) True; again a constant function works, as does $f(x) = \sin(\frac{1}{x})$.

(e) False. The limit statement makes a claim about *all* $\varepsilon > 0$, while we are given a fact only for $\varepsilon = 0.001$.

(f) True. At every relative maximum, $f'(x) = 0$, so $f' = 0$ everywhere, which implies $f$ must be constant.

(g) False. A function with a cusp is continuous but not differentiable.

(h) False. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Then $\lim_{x\to 0} f(x)$ does not exist, but $\lim_{x\to 0} |f(x)| = 1$.

*Problem* 15.

(a) Prove that if $\lim_{x\to 0} f(x) = 0$, and there exists a positive real number $M$ such that $|g(x)| < M$ for all $x$, then $\lim_{x\to 0} f(x)g(x) = 0$.

(b) Use the result of part (a) to show, using the formal limit definition of the derivative, that the function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x = 0$, and its derivative is 0.

*Solution* 15. (a) Let $\varepsilon > 0$; we want to show that there exists a $\delta > 0$ such that if $|x| < \delta$, then $|f(x)g(x)| < \varepsilon$. By assumption, since $\lim_{x\to 0} f(x) = 0$, we know that for any $\varepsilon_1 > 0$, there exists a $\delta > 0$ such that if $|x| < \delta$, then $|f(x)| < \varepsilon_1$. If we pick $\varepsilon_1 = \frac{\varepsilon}{M}$, then for the resulting $\delta$, we have that whenever $|x| < \delta$,

$$\begin{aligned} |f(x)g(x)| &= |f(x)| \cdot |g(x)| \\ &< \frac{\varepsilon}{M} \cdot M = \varepsilon \end{aligned}$$

as desired.

(b) The derivative of the given $f$ is defined to be the following limit, if it exists:

$$\lim_{h\to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = \lim_{h\to 0} \left( h \sin\left(\frac{1}{h}\right)\right)$$

But we have $\lim_{h\to 0} h = 0$ and $|\sin(\frac{1}{h})| < 2$ for all $h \neq 0$; therefore by part (a), the above limit is equal to 0. Thus $f$ is differentiable at 0 and $f'(0) = 0$.