TOPICS AND FINAL REVIEW QUESTIONS

MATH 151 SECTION 35, FALL 2005

The final exam will be from 10:30 a.m. to 12:30 p.m. on Friday, December 9th, in our usual room (SS 107). Following is a list of topics we have covered in class and a selection of questions to help you review for the final. Many of the final exam questions will be modified versions of those on this list; some may also be taken from the homework. You don't need to turn in solutions to these problems, but we will discuss them in a special review problem session at 7:00 on Wednesday, December 7th, in Ryerson 358.

1. List of Topics

- Real Numbers (§1.2)
- Inequalities (§1.3)
- Functions (§1.5-1.7)
- Mathematical Induction (§1.8)
- $\varepsilon - \delta$ Proofs of Limits (§2.2)
- Evaluating Limits with Rules (§2.3)
- Continuous Functions (§2.4)
- The Intermediate Value Theorem and Extreme Value Theorem (§2.6)
- Proofs of Derivatives using Limits (§3.1)
- Computing Derivatives with Rules and Tricks (§3.2, 3.3, 3.5-3.7)
- Rates of Change and Particle Motion (§3.4, 3.8)
- The Mean Value Theorem (§4.1)
- Relative Extrema and Increasing/Decreasing Functions (§4.2, 4.3)
- Absolute Extrema on Closed Intervals (§4.4)

2. Review Questions

Problem 1. Solve the following inequalities.

(a) $\sqrt{x^2+1} > 3$
(b) $|13x^2 - 52| < 0$

Problem 2. Prove that for all integers $n > 0$, we have

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

Problem 3. Evaluate the following limits.

(a) $\lim_{x\to 1} \frac{x^2+3}{x}$

(b) $\lim_{x\to 0}\left[x\left(1-\frac{1}{x}\right)\right]$

(c) $\lim_{x\to \pi}\left(\frac{\sin x}{\tan x}\right)$

Problem 4. Give a formal (i.e. $\varepsilon-\delta$) proof that

$$\lim_{x\to 1}(x^2+3)=4.$$

Problem 5.

(a) Prove that the sum of two rational numbers is a rational number.
(b) Prove that the sum of a rational number and an irrational number is irrational.
(c) Give an example of two irrational numbers whose sum is rational.

Problem 6. Compute $\frac{dy}{dx}$ in each case.

(a) $y=(\tan x)^2$

(b) $y=(x-1)^{1/2}(x+1)^{1/3}$

(c) $(x-y)^2-y=0$

Problem 7. Use the formal limit definition of the derivative to show the derivatives of the following function.

$$f(x)=\frac{1}{2}x^2+x$$

Problem 8. Consider the following function.

$$f(x)=\begin{cases}\sqrt{x} & \text{if } x\geq 0\\ -x^2 & \text{if } x<0\end{cases}.$$

Find all its critical points and determine if each is a relative maximum, a relative minimum, or neither.

Problem 9. Let $f(x)=x^3+ax^2+bx+c$. Under what conditions on a, b, and c does f have exactly one relative maximum and one relative minimum?

Problem 10. Let $f(x)=6x^5+13x-1$.

(a) Show that there exists a number a such that $f(a)=0$. *Hint: Use the Intermediate Value Theorem.*
(b) Show that there do not exist numbers a and b with $a\neq b$ such that $f(a)=0$ and $f(b)=0$. *Hint: Use Rolle's Theorem or the Mean Value Theorem.*

Problem 11. *Boyle's law* for a gas held at constant temperature is that $PV=\text{const}$ where P is the pressure and V is the volume. Suppose that a tank of natural gas is held at constant temperature and its pressure decreases by 0.05 pounds per square inch per hour. At the moment when its pressure is 5 pounds per square inch and its volume is 1000 cubic feet, how fast is its volume increasing?

Problem 12. Find all relative maxima and minima of the function

$$f(x) = \frac{1}{|x| - 3}.$$

Problem 13. Find the absolute maximum and minimum values of the function $f(x) = (x-1)^2(x-2)^2$ on the interval $[0, 4]$ and say at which x-values they occur.

Problem 14. True or False?

(a) If f is a continuous function on $[a, b]$ and $f(a) = f(b)$, then f has at least one critical point in (a, b).
(b) If f is continuous on the interval $(-1, 4]$ then f attains an absolute maximum on that interval.
(c) There exists a continuous function which has infinitely many relative maxima.
(d) There exists a continuous function on the interval $(0, 1)$ which has infinitely many relative maxima on that interval.
(e) Suppose f is a function such that whenever $|x - 3| < 0.2$, we have $|f(x) - 16| < 0.001$. Then $\lim_{x \to 3} f(x) = 16$.
(f) If a differentiable function f has a relative maximum at every value of x, then f is a constant function.
(g) If f is continuous at all values of x, then f is differentiable at all values of x.
(h) If $\lim_{x \to 0} |f(x)| = |L|$, then $\lim_{x \to 0} f(x) = L$.

Problem 15.

(a) Prove that if $\lim_{x \to 0} f(x) = 0$, and there exists a positive real number M such that $|g(x)| < M$ for all x, then $\lim_{x \to 0} f(x)g(x) = 0$.
(b) Use the result of part (a) to show, using the formal limit definition of the derivative, that the function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x = 0$, and its derivative is 0.