RELATIVE, ENDPOINT, AND ABSOLUTE EXTREMA

Here is a summary of our “official” terminology regarding various kinds of extrema. Unfortunately the book is a little bit vague about some of these definitions, and so was I in class last quarter.

Let f be a function whose domain is some union of intervals. The intervals can be open, closed, or half-open, and finite, infinite, or half-infinite. A point c is an ***endpoint*** of the domain of f if it is a closed endpoint of some interval on which f is defined. All points in the domain of f which are not endpoints are called ***interior points***.

1. Local Extrema

Definition 1.1. A number c is a ***critical point*** of f if the following two conditions hold:

- c is an interior point of the domain of f, and
- Either $f'(c) = 0$ or $f'(c)$ does not exist.

Note that we never consider endpoints to be critical points, even if the derivative vanishes or does not exist there.

Definition 1.2. f has a ***local maximum*** at a number c if the following two conditions hold:

- c is an interior point of the domain of f, and
- For all x sufficiently close to c in the domain of f, we have $f(x) \leq f(c)$.

Definition 1.3. f has a ***local minimum*** at a number c if the following two conditions hold:

- c is an interior point of the domain of f, and
- For all x sufficiently close to c in the domain of f, we have $f(x) \geq f(c)$.

We proved last quarter that if f has a local maximum or local minimum at c, then c is a critical point of f (but not every critical point is a local extremum). Thus, to look for local extrema, we first find the critical points. Then we can apply one of the following two tests.

Test 1.1 (The First-Derivative Test). Let c be a critical point of f such that f is continuous at c. If the following conditions hold:

- $f'(x) > 0$ for x sufficiently close to, but less than, c, and
- $f'(x) < 0$ for x sufficiently close to, but greater than, c,

then f has a local maximum at c.

Similarly, if the following conditions hold:

- $f'(x) < 0$ for x sufficiently close to, but less than, c, and
- $f'(x) > 0$ for x sufficiently close to, but greater than, c,

then f has a local minimum at c.

And if f' has constant sign for x sufficiently close to, but not equal to, c, then f has no local extremum at c.

Test 1.2 (**The Second-Derivative Test**). Let $f'(c) = 0$ (so that, in particular, c is a critical point of f), and assume that $f''(c)$ exists. Then:

- If $f''(c) < 0$, then f has a local maximum at c, and
- If $f''(c) > 0$, then f has a local minimum at c.

Note that the second-derivative test tells us nothing if the second derivative is zero, or if the function is not twice differentiable at c.

2. Endpoint Extrema

Definition 2.1. f has an ***endpoint maximum*** at a number c if the following two conditions hold:

- c is an endpoint of the domain of f, and
- For all x in the domain of f sufficiently close to c, we have $f(x) \leq f(c)$.

Definition 2.2. f has an ***endpoint minimum*** at a number c if the following two conditions hold:

- c is an endpoint of the domain of f, and
- For all x in the domain of f sufficiently close to c, we have $f(x) \geq f(c)$.

Note that endpoint extrema are again a "relative" concept—the notion refers only to the behavior of f nearby. No point can be both a local extremum and an endpoint extremum, since local extrema occur only at interior points and endpoint extrema occur only at endpoints. To find the endpoint extrema we can use the following test.

Test 2.1 (**The First-Derivative Test for Endpoints**). Let c be a right endpoint of the domain of f and suppose f is continuous from the left at c. Then

- If $f'(x) > 0$ for all x sufficiently close to, but less than, c, then f has an endpoint maximum at c, while
- If $f'(x) < 0$ for all x sufficiently close to, but less than, c, then f has an endpoint minimum at c.

Similarly, if c is a left endpoint of the domain of f and f is continuous from the right at c, then

- If $f'(x) < 0$ for all x sufficiently close to, but less than, c, then f has an endpoint maximum at c, while
- If $f'(x) > 0$ for all x sufficiently close to, but less than, c, then f has an endpoint minimum at c.

3. Absolute Extrema

Definition 3.1. f has an ***absolute maximum*** at a number c if the following two conditions hold:

- c is in the domain of f, and
- For all x in the domain of f, we have $f(x) \leq f(c)$.

Definition 3.2. f has an ***absolute minimum*** at a number c if the following two conditions hold:

- c is in the domain of f, and
- For all x in the domain of f, we have $f(x) \geq f(c)$.

Note that the function f can have absolute extrema at multiple values of x, if the function takes on the same greatest value there. Every absolute extremum is either a local extremum or an endpoint extremum, depending on whether it is an interior point or an endpoint of the domain of f. However, not every local or endpoint extremum is an absolute extremum. To find the absolute extrema we must compare the values and limit values of f.

Test 3.1 (Test for Absolute Extrema). Suppose that f has either a local maximum or an endpoint maximum at c. If the following two conditions hold:

- For all points a where f has a local maximum or endpoint maximum, $f(c) \geq f(a)$, and
- For all points b which are open or infinite endpoints of the domain of f (thus here we allow $b = \pm\infty$), we have $f(c) \geq \lim_{x \to b} f(x)$,

then f has an absolute maximum at c.

Similarly, suppose that f has either a local minimum or an endpoint minimum at c. If the following two conditions hold:

- For all points a where f has a local minimum or endpoint minimum, $f(c) \leq f(a)$, and
- For all points b which are open or infinite endpoints of the domain of f (so again, we allow $b = \pm\infty$), we have $f(c) \leq \lim_{x \to b} f(x)$,

then f has an absolute minimum at c.