REVIEW SHEET #1 — SOLUTIONS

MATH 152 SECTION 35, WINTER 2006

Solution 1. (a)

$$\lim_{x\to\infty}\frac{1}{x^2-1} = \lim_{x\to\infty}\frac{\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{0}{1-0} = 0$$

(b)

$$\lim_{x\to-\infty}\frac{4x^3+x-5}{x^3+3} = \lim_{x\to-\infty}\frac{4+\frac{1}{x^2}-\frac{5}{x^3}}{1+\frac{3}{x^3}} = \frac{4+0-0}{1+0} = 4$$

(c)

$$\lim_{x\to 4}\frac{1}{x^2-8x+16} = \lim_{x\to 4}\frac{1}{(x-4)^2} = \text{“}\frac{1}{+0}\text{”} \to \infty$$

(Here we notice that the denominator $(x-4)^2$ goes to 0 through positive values only, while the numerator 1 stays constant; thus the quotient becomes arbitrarily large positive.)

(d)

$$\lim_{x\to 0^-}\frac{x}{x^3+10x} = \lim_{x\to 0^-}\frac{1}{x^2+10} = \frac{1}{0+10} = \frac{1}{10}$$

Solution 2. (a)

$$\int (x^2-2x+4)\,dx = \frac{1}{3}x^3 - x^2 + 4x + C$$

(b)

$$\int \left(\frac{x^4-1}{x^2}\right)\,dx = \int \left(x^2 - x^{-2}\right)\,dx$$
$$= \frac{1}{3}x^3 + x^{-1} + C$$

Solution 3. (a) Since $f'(x) = 1 - x^2$, we must have $f(x) = x - \frac{1}{3}x^3 + C$ for some constant C. Since $f(3) = 2$, we must have $2 = 3 - \frac{1}{3}(3)^3 + C$ which implies $C = 8$; thus

$$f(x) = x - \frac{1}{3}x^3 + 8.$$

(b) Since $f''(x) = \sin x$, we must have $f'(x) = -\cos x + C$, and since $f'(0) = 0$, we have $0 = -\cos 0 + C = -1 + C$, so $C = 1$. Thus $f'(x) = 1 - \cos x$, so $f(x) = x - \sin x + D$ for some other constant D. Since $f(0) = 0$, we have $0 = 0 - \sin 0 + D = 0 - 0 + D$, so $D = 0$. Thus

$$f(x) = x - \sin x.$$

Solution 4. Let $N > 0$. We must show there exists a $\delta > 0$ such that whenever $0 < |x+1| < \delta$, we have $\frac{1}{(x+1)^2} > N$. Since $(x+1)^2$ is always positive for $x \neq -1$, the latter inequality is equivalent to $1 > N(x+1)^2$, and thus to $\frac{1}{N} > (x+1)^2$. Thus we are motivated to pick $\delta = \frac{1}{\sqrt{N}}$. With this choice, whenever $0 < |x+1| < \delta$, we have $(x+1)^2 = |x+1|^2 < \delta^2 = \frac{1}{N}$ and thus $\frac{1}{(x+1)^2} > N$, as desired.

Solution 5. The given function $f(x) = (x - \sqrt{x})^2$ is defined on the domain $[0, \infty)$. We have

$$f'(x) = 2(x - \sqrt{x})\left(1 - \frac{1}{2}x^{-1/2}\right)$$

on $(0, \infty)$, which equals zero either when $x = \sqrt{x}$ or when $x^{-1/2} = 2$. The former happens (for $x > 0$) only at $x = 1$, while the latter happens at $x = \frac{1}{4}$. Thus the critical points of f are at $x = \frac{1}{4}$ and at $x = 1$. We make a table of signs for the derivative f':

$(0, \frac{1}{4})$	$+$
$(\frac{1}{4}, 1)$	$-$
$(1, \infty)$	$+$

Thus, f has a local maximum at $x = \frac{1}{4}$ and a local minimum at $x = 1$. It has an endpoint minimum at $x = 0$, since the derivative is positive to the right of the endpoint $x = 0$. Finally, to find the absolute extrema we evaluate f at all these points and at its infinite limit:

$$f(0) = 0$$
$$f(\tfrac{1}{4}) = \tfrac{1}{16}$$
$$f(1) = 0$$
$$\lim_{x\to\infty} f(x) \longrightarrow \infty$$

Thus, f has an absolute minimum value of 0, which is achieved at $x = 0$ and $x = 1$, and no absolute maximum.

Solution 6. (a) For $f(x) = \frac{2x^2}{x+1}$, the only x-intercept is at $x = 0$ and the y-intercept is also $f(0) = 0$. There is a potential vertical asymptote at $x = -1$, and we calculate

$$\lim_{x \to -1^+} \frac{2x^2}{x+1} = \infty$$
$$\lim_{x \to -1^-} \frac{2x^2}{x+1} = -\infty$$
$$\lim_{x \to \infty} \frac{2x^2}{x+1} \longrightarrow \infty$$
$$\lim_{x \to -\infty} \frac{2x^2}{x+1} \longrightarrow -\infty$$

so $x = -1$ is a vertical asymptote, and there are no horizontal asymptotes. The derivative is

$$f'(x) = \frac{4x(x+1) - 2x^2}{(x+1)^2} = \frac{2x(x+2)}{(x+1)^2}$$

which equals zero at $x = 0$ and $x = -2$, so those are the critical points. We apply the first-derivative test on the intervals between them and also the asymptote, making the following table:

$(-\infty, -2)$	$+$
$(-2, -1)$	$-$
$(-1, 0)$	$-$
$(0, \infty)$	$+$

Thus, f has a local maximum at $x = -2$ and a local minumum at $x = 0$. There are no vertical tangents, since f' never goes to $\pm\infty$ except at the asymptote. The second derivative is

$$\begin{aligned} f''(x) &= \frac{(4x+4)(x+1)^2 - 2(x+1)(2x)(x+2)}{(x+1)^4} \\ &= \frac{4}{(x+1)^3} \end{aligned}$$

This is negative for $x < -1$ and positive for $x > -1$, so f is concave down on the first interval and concave up on the second, and there are no inflection points.

(b) For $f(x) = x - x^{1/3}$, the x-intercepts are where $x = x^{1/3}$, so either $x = 0$ or $x = \pm 1$. The y-intercept is $f(0) = 0$. There are no vertical asymptotes, since f is defined everywhere, and we have

$$\lim_{x\to\infty}(x - x^{1/3}) = \infty$$
$$\lim_{x\to-\infty}(x - x^{1/3}) = -\infty$$

so there are no horizontal asymptotes. The derivative is

$$f'(x) = 1 - \frac{1}{3}x^{-2/3}$$

which equals zero at $x = \pm\frac{1}{3\sqrt{3}}$, and does not exist at $x = 0$. We calculate

$$\lim_{x\to 0+}(1 - \frac{1}{3}x^{-2/3}) = -\infty$$
$$\lim_{x\to 0-}(1 - \frac{1}{3}x^{-2/3}) = -\infty$$

so f has a vertical tangent line at $x = 0$. To find the extrema, we evaluate the sign of f' on the following intervals:

$(-\infty, -\frac{1}{3\sqrt{3}})$	$+$
$(-\frac{1}{3\sqrt{3}}, 0)$	$-$
$(0, \frac{1}{3\sqrt{3}})$	$-$
$(\frac{1}{3\sqrt{3}}, \infty)$	$+$

So $x = -\frac{1}{3\sqrt{3}}$ is a local maximum and $x = \frac{1}{3\sqrt{3}}$ is a local minimum. The second derivative is

$$f''(x) = \frac{2}{9}x^{-5/3}$$

which is negative for $x < 0$ and positive for $x > 0$, so f is concave down on the former interval and concave up on the latter, and has an inflection point at $x = 0$.

Solution 7. (a)

(b)

(c)

(d)

Solution 8. By the chain rule, we have $(f \circ g)'(x) = f'(g(x))g'(x)$. Since f is increasing, we have $f'(g(x)) > 0$, so $(f \circ g)'(x) = 0$ at the same values of x for which $g'(x) = 0$. Thus, $f \circ g$ and g have the same critical points. Similarly, the sign of $(f \circ g)'(x)$ is always the same as that of $g'(x)$, so the first-derivative test tells us that they have the same local maxima and minima.

Solution 9. Let θ be the angle subtended by the tapestry at the observer's eye, and let α and β be the angle formed by the observer's sightline to the top and bottom of the tapestry, respectively, as shown. Then $\theta = \alpha - \beta$. Let x denote the distance the observer stands from the wall; then α, β, and θ are functions of x.

As hinted, by Problem 8, since tan is an increasing function, the extrema of θ are at the same x-values as the extrema of $\tan\theta$. The trigonometric angle formula for tangent gives us

$$\tan\theta = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

But inspecting the right angles in the picture, we have

$$\tan\alpha = \frac{16}{x}$$
$$\tan\beta = \frac{9}{x}$$

so that this becomes

$$\begin{aligned}\tan\theta &= \frac{\frac{16}{x} - \frac{9}{x}}{1 + \frac{16}{x}\cdot\frac{9}{x}} \\ &= \frac{16x - 9x}{x^2 + 16\cdot 9} \\ &= \frac{7x}{x^2 + 144}\end{aligned}$$

We now apply our calculus techniques to maximize this function of x. Its derivative is

$$\frac{7(x^2 + 144) - 2x(7x)}{(x^2 + 144)^2} = \frac{7(144 - x^2)}{(x^2 + 144)^2}$$

which equals 0 precisely when $x = \pm 12$. The value $x = -12$ makes no sense, so we are only interested in $x = 12$. Since the derivative changes sign from positive to negative there, it is a local maximum.

Finally, to check that it is an absolute maximum, we must evaluate f and its limits at the endpoints. The domain is $[0, \infty)$. At $x = 0$, the angle is $\theta = 0$, while as $x \to \infty$ the angle also goes to 0. Thus, since at $x = 12$, the angle is positive, that is where the absolute maximum occurs.

(We can actually evaluate that $\tan\theta = \frac{7}{24}$, but this gives us no simple expression for the angle θ itself.)

Solution 10. As hinted, we notice that the initial horizontal velocity will be $v_0 \cos\theta$ and the initial vertical velocity will be $v_0 \sin\theta$. Thus, the height of the ball as a function of time is

$$y = (v_0 \sin\theta)t - \frac{1}{2}gt^2.$$

Setting $y = 0$, we solve for t, getting $t = 0$ and

$$t = \frac{2v_0 \sin\theta}{g}.$$

Since $t = 0$ is when the ball is thrown, the other value is the time when the ball falls back to the ground.

The horizontal distance traveled is given by

$$x = v_0 \cos\theta t$$

since there is no horizontal acceleration (gravity acts only downward). When the ball falls back down, it will thus have traveled a total distance of

$$\frac{2v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g}.$$

We now want to maximize this as a function of θ. Taking its derivative, we obtain

$$\frac{2v_0^2 \cos(2\theta)}{g}$$

which equals zero when $\cos(2\theta) = 0$. The relevant domain of θ is $[0, \frac{\pi}{2}]$, so the only critical point is $\theta = \frac{\pi}{4}$. Checking the sign of the derivative, we see that this is a local maximum.

Finally, we observe that at $\theta = 0$ and $\theta = \frac{\pi}{2}$, the ball travels no horizontal distance, so the maximum distance traveled will be at $\theta = \frac{\pi}{4} = 45^o$.