FINAL TOPICS AND REVIEW QUESTIONS

MATH 152 SECTION 35, WINTER 2006

The final exam will be on Monday, March 13, from 10:30 to 12:30. No books, notes, or calculators will be allowed. Below is a list of topics we have covered and a large collection of questions designed to help you review for the final. Many of the questions on the test will be modified version of these or problems from the homework or the previous two tests.

Since no calculators are allowed, answers involving natural logarithms or exponentials should be left in exact form, e.g. write $\ln 2$ rather than $0.693\ldots$.

1. Topics

- Infinite limits, asymptotes, and vertical tangents (§4.7)
- Curve sketching (§4.8)
- Antiderivatives and indefinite integrals (§5.6)
- Definite integrals (§5.1-5.2)
- The Fundamental Theorem of Calculus (§5.3-5.4)
- u-substitution (§5.7)
- Finding areas with integration (§5.5 and §6.1)
- Finding volumes with disc, washer, and shell methods (§6.2-6.3)
- One-to-one functions and inverses (§7.1)
- Natural logarithms and exponentials (§7.2-7.4)
- Logarithms and exponentials with other bases (§7.5)
- Exponential growth and decay (§7.6)

2. Review Questions

Problem 1. Calculate the following limits. For limits that do not exist, say if the function becomes arbitrarily large positive ($\to \infty$) or arbitrarily large negative ($\to -\infty$).

(a) $\displaystyle\lim_{x\to-\infty} \frac{x^2+3x}{2x^2-1}$

(b) $\displaystyle\lim_{x\to 1^+} \frac{1}{x^2-1}$

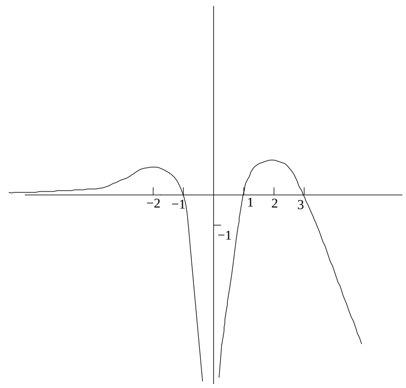
(c) $\displaystyle\lim_{x\to 0^+} e^{1/x}$

Problem 2. Recall that the definition of $\displaystyle\lim_{x\to\infty} f(x) = L$ is that for all $\varepsilon > 0$, there exists a $K > 0$ such that whenever $x > K$, then $|f(x) - L| < \varepsilon$. Prove, using this definition, that

$$\lim_{x\to\infty} \frac{x}{x+1} = 1$$

Problem 3. Find all the interesting points on the graph of the function $f(x) = e^{-x^2}$, including intercepts, asymptotes, extrema, regions where it is increasing and decreasing, vertical tangents, concavity, and inflection points, and sketch a graph of it.

Problem 4. A function f is continuous on $(-\infty, \infty)$, differentiable for all $x \neq 0$, and $f(0) = 0$. The graph of the derivative of f (that is, the graph of f') is shown below.



(a) Where does f (not f') increase and decrease? Where are its local extrema?
(b) Sketch a possible graph of f''. Where is f concave up and concave down, and what are its inflection points?
(c) Sketch a possible graph of f.

Problem 5. Evaluate the following integrals.

(a) $\int x\sqrt{2x^2+4}\,dx$

(b) $\int_0^{\pi} \cos^5 x \sin x dx$

(c) $\int \tan x dx$

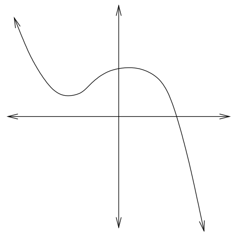
(d) $\int \frac{x^2}{x^3-1} dx$

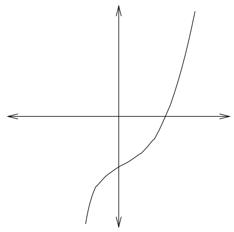
(e) $\int_0^1 4^x dx$

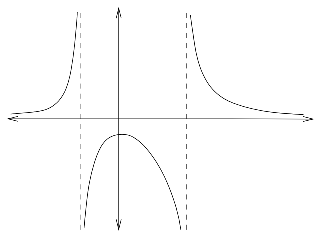
Problem 6. Draw possible graphs of functions $f(x)$ satisfying the following conditions.

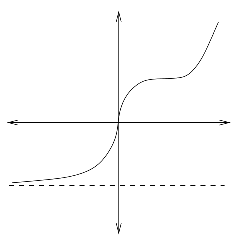
(a) f has two different vertical asymptotes but no horizontal asymptotes.
(b) f has two vertical tangent lines and no absolute extrema.
(c) f has horizontal asymptotes at $y = 0$ and at $y = 1$ and is concave up on the intervals $(-\infty, -1)$ and $(1, \infty)$.
(d) f is increasing on the intervals $(-\infty, 0)$ and $(0, \infty)$ but is not one-to-one.
(e) f is the same function as its inverse f^{-1}.

Problem 7. For each graph of a function f shown below, say whether f is one-to-one. If it is, draw a graph of its inverse f^{-1}.

(a)


(b)


(c)


(d)


Problem 8. Let $f(x) = x^3 + ax^2 + bx + c$ be a cubic polynomial. Prove that if the graph of f has both a relative maximum and a relative minimum, then it has a point of inflection halfway between the two extrema.

Problem 9. Find the area of the region bounded by the graphs of $y = x^2 - 6$ and $y = |x|$.

Problem 10. Consider the region under the graph of $f(x) = \frac{1}{x^2}$ between $x = 1$ and $x = 5$; call this region Ω. Calculate the volume resulting from rotating Ω around the y-axis.

Problem 11. Consider the circle given by the equation $(x-a)^2+y^2=r^2$, for positive constants a and r. Revolve the area inside this circle around the y-axis and consider the resulting solid.

(a) Write down a definite integral in which you would integrate with respect to x to calculate the volume of this solid.
(b) Write down a definite integral in which you would integrate with respect to y to calculate the volume of this solid.

(You don't have to actually evaluate either integral.)

Problem 12. Consider the function

$$f(x)=\begin{cases}0 & x\neq 0\\ 40 & x=0.\end{cases}$$

on the interval $[-2,2]$ and consider the partition $P_n=\{-2,\frac{-1}{n},\frac{1}{n},2\}$.

(a) Calculate $L_f(P_n)$.
(b) Calculate $U_f(P_n)$.
(c) Show, using parts (a) and (b) and the definition of the integral, that f is integrable on $[-2,2]$.

Problem 13. Consider the function $f(x)=x$ on the interval $[0,1]$ and the partition $P_n=\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n},1\}$.

(a) Calculate $L_f(P_n)$.
(b) Calculate $U_f(P_n)$.
(c) Show, using parts (a) and (b) and the definition of the integral, that f is integrable on $[-2,2]$.

Hint: you will probably want to use the equality $1+2+3+\cdots+n=\frac{n(n+1)}{2}$.

Problem 14. Suppose that $f(x)$ is one-to-one and thus has an inverse f^{-1}. Define $g(x)=f(x+3)$.

(a) Show that g is one-to-one.
(b) Express g^{-1} in terms of f^{-1}.

Problem 15. Consider the function

$$f(x)=\int_2^x \sqrt{1+t^2}dt.$$

(a) Show that f is one-to-one.
(b) Calculate $f^{-1}(0)$. *(Hint: you cannot calculate the whole function $f^{-1}(x)$.)*
(c) Calculate $(f^{-1})'(0)$.

Problem 16. Prove that for for a,b,c all real numbers greater than 1, we have

$$\log_a c=(\log_a b)(\log_b c)$$

Problem 17. Suppose that radioactive substance X has the property that given a certain amount of it, after 5 years only 2/3 of that amount is left. What is the half-life of substance X?