FINAL REVIEW SOLUTIONS

MATH 152 SECTION 35, WINTER 2006

Solution 1. (a)

$$\begin{aligned}\lim_{x\to-\infty}\frac{x^2+3x}{2x^2-1} &= \lim_{x\to-\infty}\frac{1+\frac{3}{x}}{2-\frac{1}{x^2}}\\ &= \frac{1+0}{2-0}\\ &= \frac{1}{2}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to1^+}\frac{1}{x^2-1} &\to \frac{1}{+0}\\ &= \infty\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to0^+} e^{1/x} &\to e^{\infty}\\ &= \infty\end{aligned}$$

Solution 2. Let $\varepsilon > 0$. We want to show that there is a $K > 0$ such that whenever $x > K$, then $|\frac{x}{x+1} - 1| < \varepsilon$. We have

$$\begin{aligned}\left|\frac{x}{x+1}-1\right| &= \left|\frac{x}{x+1}-\frac{x+1}{x+1}\right|\\ &= \left|\frac{-1}{x+1}\right|\\ &= \frac{1}{|x+1|}\end{aligned}$$

so what we want to ensure is that $\frac{1}{|x+1|} < \varepsilon$. This is equivalent to $|x+1| > \frac{1}{\varepsilon}$, and since we will eventually choose $x > K > 0$, we know that $x > 0$ and so $|x+1| = x+1$. Thus we choose $K = \frac{1}{\varepsilon} - 1$. Now assume $x > K$. Then $x+1 > \frac{1}{\varepsilon}$, so $\frac{1}{x+1} < \varepsilon$ and thus $|\frac{x}{x+1} - 1| < \varepsilon$, as desired. Q.E.D.

Solution 3. Find all the interesting points on the graph of the function $f(x) = e^{-x^2}$, including intercepts, asymptotes, extrema, regions where it is increasing and decreasing, vertical tangents, concavity, and inflection points, and sketch a graph of it.

We have $f(0) = e^0 = 1$, so the y-intercept is at $y = 1$. An exponential is never zero, so there is no x-intercept. The function is always defined, so there are no vertical asymptotes. We have

$$\lim_{x\to\infty} e^{-x^2} = e^{-\infty} = 0$$
$$\lim_{x\to-\infty} e^{-x^2} = e^{-\infty} = 0$$

so there is a horizontal asymptote at $x = 0$.

The first derivative is

$$f'(x) = -2xe^{-x^2}$$

and we have $f'(x) = 0$ only at $x = 0$, so this is the only critical point. The sign of f' is as follows:

$(-\infty, 0)$	$+$
$(0, \infty)$	$-$

so f is increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$ and has a local maximum at $x = 0$. This local maximum is therefore also the absolute maximum, and $f(0) = 1$ is the maximum value. There is no absolute minimum, since f gets arbitrarily small but never reaches 0.

Since $f'(x)$ is defined everywhere, there are no vertical tangents.

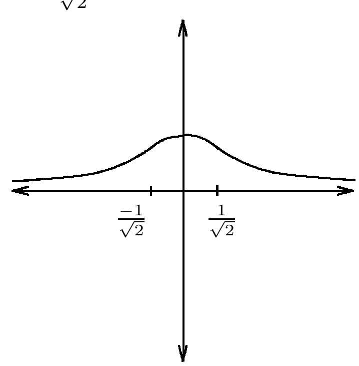
The second derivative is

$$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$$

and $f''(x) = 0$ when $4x^2 - 2 = 0$, or $x = \pm\frac{1}{\sqrt{2}}$. The sign of f'' is

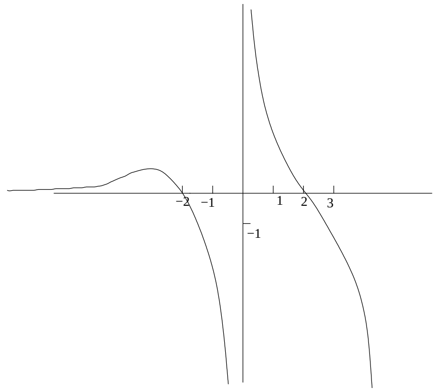
$(-\infty, -\frac{1}{\sqrt{2}})$	$+$
$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$-$
$(\frac{1}{\sqrt{2}}, \infty)$	$+$

so f is concave up on $(-\infty, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \infty)$, concave down on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and has inflection points at $x = \pm\frac{1}{\sqrt{2}}$.



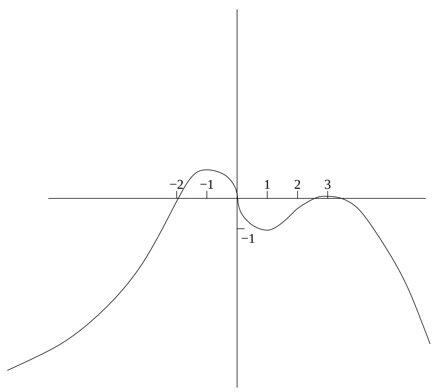
Solution 4. (a) f increases where f' is positive and decreases where f' is negative. Thus, f increases on $(-\infty, -1)$ and $(1, 3)$ and decreases on $(-1, 0)$, $(0, 1)$, and $(3, \infty)$. The local extrema of f are the places where it changes from increasing to decreasing or vice versa. Thus f has local maxima at $x = -1$ and $x = 3$ and a local minimum at $x = 1$.

(b) We can tell where f'' should be positive, negative, or zero by seeing where f' is increasing, decreasing, or flat, and similarly for where it goes to $\pm\infty$. A possible graph of f'' is shown below.



The sign of f'' tells us about the concavity of f; thus we know that f is concave up on $(-\infty, -2)$ and $(0, 2)$ and concave down on $(-2, 0)$ and $(2, \infty)$, and thus has inflection points at -2, 0, and 2. (We are given that f is continuous at 0, even though f' is not.)

(c) A possible graph of f is shown below. Note the vertical tangent line at $x = 0$, which we know must be there because $f'(x) \to -\infty$ as $x \to 0$, but f is known to be continuous at 0.



Solution 5. Evaluate the following integrals.

(a) We do a u-substitution with

$$u = 2x^2 + 4$$
$$du = 4x\,dx$$

Then the integral becomes

$$\int x\sqrt{2x^2+4}\,dx = \frac{1}{4}\int \sqrt{u}\,du$$
$$= \frac{1}{6}u^{3/2} + C$$
$$= \frac{1}{6}(2x^2+4)^{3/2} + C$$

(b) We do a u-substitution with

$$u = \cos x$$
$$du = -\sin x\,dx$$

When $x = 0$, $u = 1$, and when $x = \pi$, $u = -1$. Thus the integral becomes

$$\begin{aligned}\int_0^\pi \cos^5 x \sin x \, dx &= -\int_1^{-1} u^5 \, du \\ &= -\frac{1}{6} u^6 \Big|_1^{-1} \\ &= -\frac{1}{6}\left[(-1)^6 - 1^6\right] \\ &= 0\end{aligned}$$

(c) We know that $\int \tan x dx = \ln|\sec x| + C$. We could also re-derive this the way we did in class: write $\tan x = \frac{\sin x}{\cos x}$ and let $u = \cos x$, $du = -\sin x \, dx$, so the integral becomes

$$\begin{aligned}\int \tan x \, dx &= \int \frac{-1}{u} du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C.\end{aligned}$$

(d) We do a u-substitution with

$$\begin{aligned}u &= x^3 - 1 \\ du &= 3x^2 \, dx\end{aligned}$$

Then the integral becomes

$$\begin{aligned}\int \frac{x^2}{x^3 - 1} dx &= \frac{1}{3} \int \frac{1}{u} \, du \\ &= \frac{1}{3} \ln|u| + C \\ &= \frac{1}{3} \ln|x^3 - 1| + C\end{aligned}$$

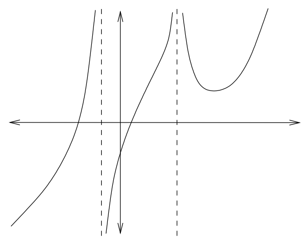
(e) We recall that $4^x = e^{x \ln 4}$, so we do a u-substitution with

$$\begin{aligned}u &= x \ln 4 \\ du &= \ln 4 \, dx\end{aligned}$$

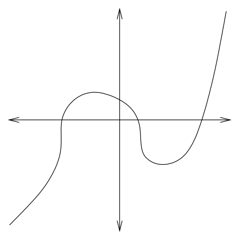
When $x = 0$, $u = 0$, and when $x = 1$, $u = \ln 4$. Thus the integral becomes

$$\begin{aligned}\int_0^1 4^x dx &= \frac{1}{\ln 4} \int_0^{\ln 4} e^u \, du \\ &= \frac{1}{\ln 4}\left[e^u\right]_0^{\ln 4} \\ &= \frac{1}{\ln 4}\left[e^{\ln 4} - e^0\right] \\ &= \frac{1}{\ln 4}(4 - 1) \\ &= \frac{3}{\ln 4}.\end{aligned}$$

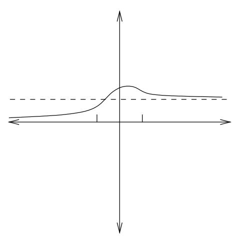
Solution 6. In each case, there are many different possible graphs, so yours may not look like these sample solutions.



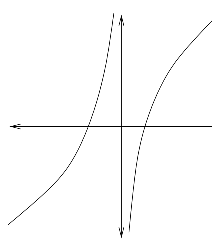
(a)



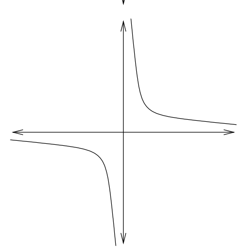
(b)



(c)



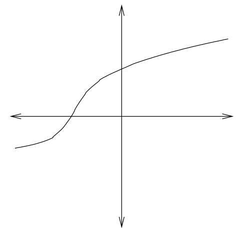
(d)



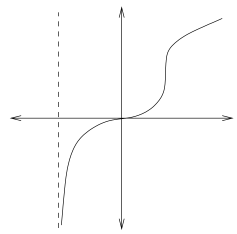
(e)

Solution 7. (a) Not one-to-one.

(b) One-to-one. The inverse is shown below.



(c) Not one-to-one.
(d) One-to-one. The inverse is shown below.

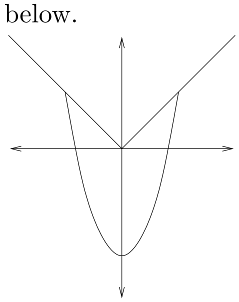


Solution 8. We have $f(x) = x^3 + ax^2 + bx + c$ and thus $f'(x) = 3x^2 + 2ax + b$. Thus the critical points are the roots of this quadratic, namely

$$x = \frac{-2a \pm \sqrt{4a^2 - 12b}}{6} = \frac{-a \pm \sqrt{a^2 - 3b}}{3}$$

If f has both a local maximum and a local minimum, one must occur at one of these two values of x, and the other at the other of these two values of x. The point halfway between these two values is $x = -\frac{a}{3}$, so we want to show f has a point of inflection at $x = -\frac{a}{3}$. To do this, we calculate $f''(x) = 6x + 2a$. Thus $f''(x)$ is zero at $x = -\frac{a}{3}$, negative for $x < -\frac{a}{3}$ and positive for $x > -\frac{a}{3}$, so f has an inflection point at $x = -\frac{a}{3}$, as desired. Q.E.D.

Solution 9. The area is shown below.



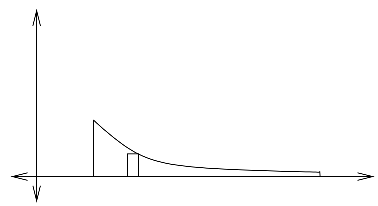
To find the area, we have to split it into two regions, since the absolute value is essentially a piecewise function. First we find the intersection points.

If $x \geq 0$, we have to solve $x = x^2 - 6$, which factors to $(x - 3)(x + 2) = 0$. Since this equation is only valid when $x \geq 0$, the solution $x = -2$ is extraneous; thus $x = 3$ is one intersection point. Similarly, when $x \leq 0$, we have $-x = x^2 - 6$, which factors to $(x + 3)(x - 2) = 0$. In this case the solution $x = 2$ is extraneous, so we get only $x = -3$. Thus the intersection points are $(3, 3)$ and $(-3, 3)$.

Now we can do the integral, splitting it at $x = 0$.

$$\begin{aligned} A &= \int_{-3}^{0} [(-x) - (x^2 - 6)]\,dx + \int_{0}^{3} [(x) - (x^2 - 6)]\,dx \\ &= \Big[-\frac{x^2}{2} - \frac{x^3}{3} + 6x \Big]_{-3}^{0} + \Big[\frac{x^2}{2} - \frac{x^3}{3} + 6x \Big]_{0}^{3} \\ &= \Big[(-0 - 0 + 0) - \Big(-\frac{9}{2} - \frac{-27}{3} - 18 \Big) \Big] + \Big[\Big(\frac{9}{2} - \frac{27}{3} + 18 \Big) - (0 - 0 + 0) \Big] \\ &= 27 \end{aligned}$$

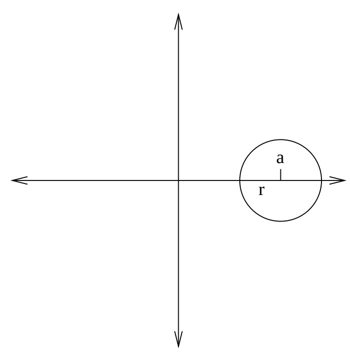
Solution 10. The region Ω is shown below.



The resulting volume is probably easiest to calculate using the shell method, rotating the slices such as the one shown about the y-axis. Thus we have

$$\begin{aligned} V &= \int_{1}^{5} 2\pi x \frac{1}{x^2}\,dx \\ &= 2\pi \int_{1}^{5} \frac{1}{x}\,dx \\ &= 2\pi \Big[\ln|x| \Big]_{1}^{5} \\ &= 2\pi(\ln 5 - \ln 1) \\ &= 2\pi \ln 5 \end{aligned}$$

Solution 11. The circle is shown below. The radius is r, and the center is $(a, 0)$.



(a) To integrate with respect to x, we would use the shell method. The distance to the axis of rotation is x and the height of each shell is

$$\sqrt{r^2 - (x-a)^2} - (-\sqrt{r^2 - (x-a)^2}) = 2\sqrt{r^2 - (x-a)^2}.$$

Thus, the volume would be given by

$$V = 4\pi \int_{a-r}^{a+r} x\sqrt{r^2 - (x-a)^2}\,dx.$$

(b) To integrate with respect to y, we would use the washer method. The outer radius of each washer is $a + \sqrt{r^2 - y^2}$ and the inner radius is $a - \sqrt{r^2 - y^2}$, so the volume would be given by

$$V = \pi \int_{-r}^{r} \left[(a + \sqrt{r^2 - y^2})^2 - (a - \sqrt{r^2 - y^2})^2\right] dy.$$

Solution 12. There are three intervals in question, and the minimum and maximum values of f on each interval is as follows:

$[x_i, x_{i+1}]$	m_i	M_i
$[-2, \frac{1}{n}]$	0	0
$[-\frac{1}{n}, \frac{1}{n}]$	0	40
$[\frac{1}{n}, 2]$	0	0

(a) Using the above values of m_i, we have

$$\begin{aligned} L_f(P_n) &= (0)\left(-\frac{1}{n} - (-2)\right) + (0)\left(\frac{1}{n} - \left(-\frac{1}{n}\right)\right) + (0)\left(2 - \frac{1}{n}\right) \\ &= 0 + 0 + 0 = 0 \end{aligned}$$

(b) Using the above values of M_i, we have

$$\begin{aligned} U_f(P_n) &= (0)\left(-\frac{1}{n} - (-2)\right) + (40)\left(\frac{1}{n} - \left(-\frac{1}{n}\right)\right) + (0)\left(2 - \frac{1}{n}\right) \\ &= \frac{80}{n} \end{aligned}$$

(c) To show that f is integrable, we must show that there is a unique I such that $L_f(P) \le I \le U_f(P)$ for all P. Taking $P = P_n$, this says that

$$0 \le I \le \frac{80}{n}$$

for all $n > 0$. Thus, if we take the limit as $n \to \infty$, we see that I is squeezed between 0 and 0, so the only possible value for I is $I = 0$. Thus, $I = 0$ is the *unique* number such that $L_f(P) \le I \le U_f(P)$ for all P, so f is integrable on $[-2, 2]$ and $\int_{-2}^{2} f(x)\,dx = 0$.

Solution 13. There are n intervals in P_n, each of the form $[\frac{i}{n}, \frac{i+1}{n}]$, for $0 \le i \le n-1$. The minimum value m_i of $f(x)$ on the interval $[\frac{i}{n}, \frac{i+1}{n}]$ is $\frac{i}{n}$ and the maximum value M_i is $\frac{i+1}{n}$. The width of each interval is $\frac{1}{n}$.

(a) Using the above values of m_i, we have

$$\begin{aligned} L_f(P_n) &= 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \cdots + \frac{n-1}{n} \cdot \frac{1}{n} \\ &= \frac{1}{n^2}\left(0 + 1 + 2 + \cdots + (n-1)\right) \\ &= \frac{1}{n^2} \cdot \frac{(n-1)n}{2} \\ &= \frac{n^2 - n}{2n^2} \\ &= \frac{1}{2} - \frac{1}{2n}. \end{aligned}$$

(b) Using the above values of M_i, we have

$$\begin{aligned} L_f(P_n) &= \frac{1}{n}\cdot\frac{1}{n}+\frac{2}{n}\cdot\frac{1}{n}+\cdots+\frac{n-1}{n}\cdot\frac{1}{n}+1\cdot\frac{1}{n} \\ &= \frac{1}{n^2}\left(1+2+\cdots+(n-1)+n\right) \\ &= \frac{1}{n^2}\cdot\frac{n(n+1)}{2} \\ &= \frac{n^2+n}{2n^2} \\ &= \frac{1}{2}+\frac{1}{2n}. \end{aligned}$$

(c) To show that f is integrable, we must show that there is a unique I such that $L_f(P)\le I\le U_f(P)$ for all P. Taking $P=P_n$, this says that

$$\frac{1}{2}-\frac{1}{2n}\le I\le\frac{1}{2}+\frac{1}{2n}$$

for all $n>0$. Thus, if we take the limit as $n\to\infty$, we see that I is squeezed between $\frac{1}{2}$ and $\frac{1}{2}$, so the only possible value for I is $I=\frac{1}{2}$. Thus, $I=\frac{1}{2}$ is the *unique* number such that $L_f(P)\le I\le U_f(P)$ for all P, so f is integrable on $[0,1]$ and $\int_0^1 f(x)\,dx=\frac{1}{2}$.

Solution 14. (a) We want to show that if $g(x_1)=g(x_2)$, then $x_1=x_2$, since that is the definition of what it means for g to be one-to-one. Suppose that $g(x_1)=g(x_2)$. Then by definition of g, we have $f(x_1+3)=f(x_2+3)$. Since f is one-to-one, this implies $x_1+3=x_2+3$, and therefore, subtracting 3, we have $x_1=x_2$. This shows that g is one-to-one.

(b) Write $y=g(x)=f(x+3)$. Then we have

$$\begin{aligned} y &= f(x+3) \\ f^{-1}(y) &= x+3 \\ f^{-1}(y)-3 &= x \end{aligned}$$

Therefore, $g^{-1}(y)=f^{-1}(y)-3$.

Solution 15. (a) By the fundamental theorem of calculus, we have $f'(x)=\sqrt{1+x^2}$ which is everywhere positive; thus f is increasing everywhere and hence one-to-one.

(b) We have $f(2)=\int_2^2\sqrt{1+t^2}dt=0$, so $f^{-1}(0)=2$.

(c) By the formula for the derivative of an inverse function,

$$\begin{aligned} (f^{-1})'(0) &= \frac{1}{f'(f^{-1}(0))} \\ &= \frac{1}{f'(2)}. \end{aligned}$$

Since we have $f'(2)=\sqrt{1+2^2}=\sqrt{5}$, it follows that

$$(f^{-1})'(0)=\frac{1}{\sqrt{5}}.$$

Solution 16. By definition, we have

$$\begin{aligned}(\log_a b)(\log_b c) &= \left(\frac{\ln b}{\ln a}\right)\left(\frac{\ln c}{\ln b}\right)\\ &= \frac{\ln c}{\ln a} = \log_a c.\end{aligned}$$

Solution 17. Since radioactive decay is exponential, we have $X = X_0 e^{kt}$ where X_0 is the initial amount of substance X, k is the as-yet-unknown decay constant, and t is the time (measured in years). The given information tells us that at $t = 5$, we have $X = \frac{2}{3}X_0$. Thus

$$\begin{aligned}X_0 e^{k\cdot 5} &= \frac{2}{3}X_0\\ e^{5k} &= \frac{2}{3}\\ 5k &= \ln\left(\frac{2}{3}\right)\\ k &= \frac{1}{5}\big(\ln 2 - \ln 3\big)\end{aligned}$$

We now want to calculate the half-life. The half-life is the time T after which $X = \frac{1}{2}X_0$, or

$$\begin{aligned}X_0 e^{k\cdot T} &= \frac{1}{2}X_0\\ e^{kT} &= \frac{1}{2}\\ kT &= \ln\left(\frac{1}{2}\right)\\ &= -\ln 2\end{aligned}$$

(We could also take the equation $kT = -\ln 2$ as a known characterization of the half-life T.) Putting in our value of k, we obtain

$$\begin{aligned}T &= \frac{-5\ln 2}{\ln 2 - \ln 3}\\ &\approx 8.547\ldots\end{aligned}$$