3. Differentiation

Definition X. Let $A \subset \mathbb{R}$, let $f\colon A \to \mathbb{R}$ be a function, let $a \in \overline{A}$, and let $L \in \mathbb{R}$. Define a function $g\colon A \cup \{a\} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & x \neq a \\ L & x = a. \end{cases}$$

If g is continuous at a, we say that L is the **limit of f as x approaches a**, and write $\lim_{x \to a} f(x) = L$.

Theorem 22. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$.

Definition XI. If $f\colon A \to \mathbb{R}$, $a \in A$, and for some $L \in \mathbb{R}$ we have

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L,$$

we say that f is **differentiable at** a and that L is the **derivative of f at** a, and we write $f'(a) = L$.

Definition XII. Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$. Then we define $f + g\colon A \cap B \to \mathbb{R}$ and $fg\colon A \cap B \to \mathbb{R}$ by

$$(f + g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x).$$

Theorem 23. If f and g are differentiable at a, then so is $f + g$. What is $(f + g)'(a)$ in terms of f and g?

Theorem 24. If f and g are differentiable at a, then so is fg. What is $(fg)'(a)$ in terms of f and g?

Theorem 25. If f is differentiable at a, then f is continuous at a.

Question. Is the converse of Theorem 25 true?

Theorem 26 (Rolle's Theorem)**.** Let $f\colon [a, b] \to \mathbb{R}$ be continuous at every point of $[a, b]$ and differentiable at every point of (a, b), and suppose that $f(a) = f(b) = 0$. Then there exists a point $p \in (a, b)$ such that $f'(p) = 0$.

Corollary 26.1 (Mean Value Theorem)**.** Let $f\colon [a, b] \to \mathbb{R}$ be continuous at every point of $[a, b]$ and differentiable at every point of (a, b). Then there exists a point $p \in (a, b)$ such that $f'(p) = \frac{f(b) - f(a)}{b - a}$.

Theorem 27. Let $f\colon [a, b] \to \mathbb{R}$ be continuous at every point of $[a, b]$ and differentiable at every point of (a, b), and suppose that $f'(x) = 0$ for all $x \in (a, b)$. Then there exists $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in [a, b]$.

Corollary 27.1. Let f and g both be continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in [a, b]$.

Definition XIII. If $g\colon A \to \mathbb{R}$ and $f\colon B \to \mathbb{R}$, we define $f \circ g\colon D \to \mathbb{R}$ by

$$(f \circ g)(x) = f(g(x)),$$

where $D = \{x \in A : g(x) \in B\}$.

Theorem 28 (Chain rule)**.** If g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a, and

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Theorem 29 (Inverse function theorem)**.** Let R be a region containing p, let f be differentiable on R and f' be continuous on R, and suppose $f'(p) \neq 0$. Then there exists a region S with $p \in S \subset R$, a region $T \ni f(p)$, and a continuous function $g\colon T \to \mathbb{R}$ such that for all $x \in S$, we have $g(f(x)) = x$.

Theorem 30. The function g in Theorem 29 is differentiable at $f(p)$, and we have $g'(f(p)) = \frac{1}{f'(p)}$.