MATH 162 SECTION 50
FINAL SOLUTIONS

Note: The amount of detail that makes a proof correct is not uniquely determined. Some people gave more detail and some people gave less, and both were usually acceptable.

Problem 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined as follows:*

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \pm\frac{p}{q},\ p,q \in \mathbb{N},\ \gcd(p,q) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

At what points is f continuous and at what points is it discontinuous? Prove your answer. You may assume that any region in $\mathbb{R}$ contains both rational and irrational numbers.

Solution to Problem 1. f is continuous at all irrationals and discontinuous at all rationals.

(a) First, let $x \in \mathbb{Q}$. Then $0 < f(x) \leq 1$, so if we let $R = (0,2)$, then R is a region containing $f(x)$. However, any region S containing x must also contain some irrational number y. Since y is irrational, we have $f(y) = 0 \notin R$. Thus $f(S) \not\subset R$, so there does not exist any region $S \ni x$ with $f(S) \subset R$. Thus, f is not continuous at x.

(b) Second, let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then $f(x) = 0$. Thus, to show that f is continuous at x, we must show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $y \in (x-\delta, x+\delta)$, then $f(y) \in (-\varepsilon, \varepsilon)$. Since $f(y) \geq 0$ for all y, this last is equivalent to saying $f(y) < \varepsilon$. Now fix some $\varepsilon > 0$; we must find a δ.

By the Archimedean property, there exists a natural number n with $\frac{1}{n} < \varepsilon$. Then there are only finitely many natural numbers q with $q < n$. Let a be an integer such that $a < x < a+1$. Then for any natural number q, there are finitely many integers p such that $\frac{p}{q} \in (a, a+1)$ (in fact, there are at most q of them). Therefore, there are finitely many rational numbers $\frac{p}{q}$ such that $\frac{p}{q} \in (a, a+1)$ and $q < n$. So we can choose a $\delta > 0$ such that $(x-\delta, x+\delta) \subset (a, a+1)$ and contains no rational numbers $\frac{p}{q}$ with $q < n$.

With this choice of δ, for all rational $y \in (x-\delta, x+\delta)$, we have $y = \frac{p}{q}$ in lowest terms with $q \geq n$, and therefore $f(y) = \frac{1}{q} \leq \frac{1}{n} < \varepsilon$. And for all irrational $y \in (x-\delta, x+\delta)$, we have $f(y) = 0 < \varepsilon$. Thus, for all $y \in (x-\delta, x+\delta)$, we have $f(y) < \varepsilon$. This shows that f is continuous at x.

Problem 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and let (x_n) be a bounded sequence. Is it necessarily true that $f\left(\overline{\lim_{n\to\infty}}\, x_n\right) = \overline{\lim_{n\to\infty}}\, f(x_n)$? If so, prove it; if not, give a counterexample.*

Solution to Problem 2. Not necessarily. Consider the sequence $x_n = (-1)^{n+1}$ and the function $f(x) = -x$. Then $\overline{\lim}_{n\to\infty} x_n = 1$, so $f\left(\overline{\lim}_{n\to\infty} x_n\right) = -1$. However, $f(x_n) = (-1)^n$, so $\overline{\lim}_{n\to\infty} f(x_n) = 1$, and $1 \neq -1$.

Problem 3. *Let $A \subset \mathbb{R}$ and let (x_n) and (y_n) be sequences such that for all n, $x_n \in A$ and y_n is an upper bound of A. Prove that if (x_n) and (y_n) converge to the same limit z, then $z = \sup A$.*

Solution to Problem 3. Proof by contradiction: suppose that z is not the supremum of A. Then there are two cases.

(a) z is not an upper bound of A. Then there exists $a \in A$ with $a > z$. Let $\varepsilon = a - z > 0$; then since $\lim y_n = z$, there exists an $N \in \mathbb{N}$ such that if $n > N$ then $|y_n - z| < \varepsilon = a - z$, and therefore $y_n < a$. This is a contradiction because y_n is an upper bound of A.

(b) There exists an upper bound w of A such that $w < z$. Now let $\varepsilon = z - w$; then since $\lim x_n = z$, there exists $N \in \mathbb{N}$ such that if $n > N$ then $|x_n - z| < \varepsilon = z - w$, and therefore $x_n > w$. This is a contradiction because $x_n \in A$ and w is an upper bound of A.

Since both cases lead to a contradiction, z must be the supremum of A.

Problem 4. *Let F be an ordered field with the following two properties:*

(a) *The Archimedean property (for all $x \in F$ with $x > 0$ there are $n, m \in \mathbb{N}$ such that $\frac{1}{n} < x < m$).*

(b) *Any Cauchy sequence in F converges to an element of F.*

Prove that F satisfies axiom 3, and therefore is the real numbers. (Hint: use the idea of problem 3 to construct a supremum for any subset of F which is bounded above.)

Solution to Problem 4. We proved on a homework set that if C is a set satisfying axioms 1 and 2, betweenness, and the least-upper-bound property, then C necessarily satisfies axiom 3 as well. We know that any ordered field satisfies axioms 1 and 2 and betweenness, so it remains to show that any nonempty bounded subset of F has a least upper bound.

Let $A \subset F$ be bounded and nonempty; let $x_1 \in A$ and y_1 be an upper bound of A which is not an element of A, and hence $y_1 > x_1$. Define $z_1 = \frac{x_1+y_1}{2}$. If z is an upper bound of A, define $y_2 = z_1$ and $x_2 = x_1$. Otherwise, there is some $a \in A$ with $a > z_1$; in this case define $y_2 = y_1$ and $x_2 = a$. Now define $z_2 = \frac{x_2+y_2}{2}$ and repeat to define x_3, y_3, and so on. In this way we define two infinite sequences (x_n) and (y_n) such that for all n, $x_n \in A$ and y_n is an upper bound of A. It is clear (by induction) that $x_n < y_n$ for all n. We also have $(y_n - x_n) \leq \frac{1}{2}(y_{n-1} - x_{n-1})$, and thus by induction

$$(y_n - x_n) \leq \frac{1}{2^{n-1}}(y_1 - x_1).$$

Since $y_1 > x_1$, we have $y_1 - x_1 > 0$ and thus for any $\varepsilon > 0$ in F, we have $\frac{\varepsilon}{y_1 - x_1} > 0$. Thus, by the Archimedean property, there exists an $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{\varepsilon}{y_1 - x_1}$, and therefore

$$\frac{1}{2^{n-1}} \leq \frac{1}{n} < \frac{\varepsilon}{y_1 - x_1}.$$

This is also true for any $k > n$; hence for any $\varepsilon > 0$ there exists n such that if $k > n$ then

$$|y_k - x_k| = y_k - x_k \leq \frac{1}{2^{n-1}}(y_1 - x_1) < \varepsilon.$$

Thus, by definition we have

$$\lim_{n \to \infty} (y_n - x_n) = 0.$$

Now we prove that (x_n) is Cauchy. Let $\varepsilon > 0$ in F; then there exists $N \in \mathbb{N}$ such that if $n > N$, then $y_n - x_n < \varepsilon$. For any $m > n$, we have $x_n < x_m \leq y_n$, since on the one hand (x_n) is an increasing sequence, and on the other hand y_n is an upper bound for A and $x_m \in A$. Therefore, for any $m > n > N$, we have $x_m - x_n < \varepsilon$, so (x_n) is Cauchy. By the assumption on F, it therefore converges to some $\alpha \in F$. Similarly, we show that (y_n) is Cauchy, and thus it converges to some $\beta \in F$.

Suppose that $\alpha \neq \beta$. Then if we let $\varepsilon < \frac{|\beta-\alpha|}{3}$, there is some N such that if $n > N$, then $|x_n - \alpha| < \varepsilon$, and some M such that if $n > M$, then $|y_n - \beta| < \varepsilon$. But then for all $n > \max(N, M)$, we have $|x_n - y_n| \geq \varepsilon$, which contradicts $\lim_{n\to\infty}(y_n - x_n) = 0$. Therefore, $\alpha = \beta$.

We cannot apply Problem 3 as written, because it is a statement about $\mathbb{R}$ and we are working in a general ordered field F. However, the *proof* given above for Problem 3 works verbatim in F, without assuming *a priori* that the set A has a supremum. Thus we can conclude that $\alpha = \beta$ is the supremum of A, and in particular, that A has a supremum. Thus any bounded nonempty set has a least upper bound, so F satisfies axiom 3.

Remark. Every single person simply referred to Problem 3 to complete the proof, without noticing that it is a statement about $\mathbb{R}$ rather than a general ordered field F. I did not take off points for this, but to be correct you should really mention that a more general statement is needed.

Problem 5. *Let $f\colon A \to B$ be a function, and let x be a point in A which is not a limit point of the set A. Prove that f is continuous at x.*

Solution to Problem 5. Since x is not a limit point of A, there is a region $S \ni x$ such that $S \cap (A \setminus \{x\}) = \emptyset$. Since $x \in A$, this implies $S \cap A = \{x\}$. Thus, if R is any region containing $f(x)$, we have $f(S \cap A) = f(\{x\}) = \{f(x)\} \subset R$; so f is continuous at x.

Problem 6. *Let $A \subset \mathbb{R}$ be compact and $A \subset B_1 \cup B_2$, where B_1 and B_2 are disjoint open sets. Prove that $A \cap B_1$ is also compact. Give an example to show that this is not necessarily true if B_1 and B_2 are not disjoint.*

Solution to Problem 6. **First solution:** Let G be an open cover of $A \cap B_1$. Then since $A \subset B_1 \cup B_2$, $H = G \cup \{B_2\}$ is an open cover of A. Since A is compact, H has a finite subcover H'. Let $G' = H' \cap G = H' \setminus \{B_2\}$; then G' is a finite subset of G. Since no $x \in A \cap B_1$ can be in B_2, every such x must be in some element of G'; thus G' is an open cover of $A \cap B_1$. We have shown that any open cover of $A \cap B_1$ has a finite subcover, so it is compact.

Second solution: By the Heine-Borel theorem, A is closed and bounded. Since $A \cap B_1 \subset A$, it is also bounded; thus (again by Heine-Borel) it suffices to show that $A \cap B_1$ is closed. Suppose that $x \operatorname{lp}(A \cap B_1)$; we want to show that $x \in A \cap B_1$. Then $x \operatorname{lp} A$ and $x \operatorname{lp} B_1$. Since A is closed, $x \in A$, and since $A \subset B_1 \cup B_2$, we have $x \in B_1$ or $x \in B_2$. If the former, then $x \in A \cap B_1$ as desired. If the latter, then since B_2 is open, there is a region R with $x \in R \subset B_2$, and therefore $R \cap B_1 = \emptyset$. Thus x is not a limit point of B_1, which is a contradiction; hence $x \in A \cap B_1$. Thus $A \cap B_1$ is closed, so by Heine-Borel, it is compact.

Example: Let $B_1 = (0, 3)$, $B_2 = (2, 5)$, and $A = [1, 4]$. Then B_1 and B_2 are open, A is compact, $A \subset B_1 \cup B_2$, but $A \cap B_1 = [1, 3)$ is not compact, since it has 3 as a limit point which is not in it.

Remark. Both solutions to Problem 6 are equally correct, but the first is 'better' from a mathematical point of view. This is because it uses the definition of compactness rather than relying on the Heine-Borel theorem, and thus remains valid in other situations where the Heine-Borel theorem may fail.

Problem 7. *Let $f\colon [a,b] \to \mathbb{R}$ and let $|f|$ be the function defined by $|f|(x) = |f(x)|$. For each of the following statements, either prove it or give a counterexample.*

(a) *If f is continuous, then $|f|$ is continuous.*
(b) *If $|f|$ is continuous, then f is continuous.*

Solution to Problem 7. (a) This is true. Since f is continuous, we know that for any $x \in [a,b]$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $y \in [a,b]$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon$.

Claim: For any real numbers x, y we have $\Big||y| - |x|\Big| \leq |y - x|$. (This is another form of the triangle inequality.) For we can write

$$\begin{aligned} |y| &= |(y - x) + x| \\ &\leq |y - x| + |x| \end{aligned}$$

by the triangle inequality, and thus $|y| - |x| \leq |y - x|$. Similarly, we have $|x| - |y| \leq |y - x|$, so the claim follows.

The claim implies that $\Big||f(y)| - |f(x)|\Big| \leq |f(y) - f(x)|$. Thus, whenever $|y - x| < \delta$, we have

$$\begin{aligned} \Big||f(y)| - |f(x)|\Big| &\leq |f(y) - f(x)| \\ &< \varepsilon. \end{aligned}$$

Therefore, for any $x \in [a,b]$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $y \in [a,b]$ and $|y - x| < \delta$, then $\Big||f(y)| - |f(x)|\Big| < \varepsilon$. This shows that $|f|$ is continuous.

(b) This is false. A striking counterexample is given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

A familiar argument (similar to that of Problem 1) shows that f is discontinuous at every real number. However, $|f(x)| = 1$ for all x, so $|f|$ is continuous everywhere.

Remark. Some people proved things essentially equivalent to $\Big||y| - |x|\Big| \leq |y - x|$ using many different cases. A good friend of mine once said "if something seems really obvious, you're probably about to use the triangle inequality."

Problem 8. *Let $A \subset \mathbb{R}$ be closed, let $0 < r < 1$ be a real number, and let $f\colon A \to A$ be a continuous function such that for all $x, y \in A$, we have*

$$|f(x) - f(y)| \leq r|x - y|.$$

(a) *Let $x_0 \in A$. Prove that the sequence*

$$\big(x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots\big)$$

is Cauchy.

(b) *Prove that if A is nonempty, then f has a fixed point.*

Solution to Problem 8. (a) **First solution:** Write $f^n(x) = f(f(\dots f(x)\dots))$ for the n^{th} iterate of f, so that $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, and so on. By convention we define $f^0(x) = x$. We then have to show that for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n, m > N$ then $|f^n(x_0) - f^m(x_0)| < \varepsilon$. Now, by the assumption on f, for any n and any x we have

$$|f^{n+1}(x) - f^n(x)| \leq r \cdot |f^n(x) - f^{n-1}(x)|$$

and therefore, inductively,

$$|f^{n+1}(x) - f^n(x)| \leq r^n \cdot |f(x) - x|.$$

Supposing WLOG that $n > m$, we then have

$$\begin{aligned}|f^n(x) - f^m(x)| &= \left|f^n(x) - f^{n-1}(x) + f^{n-1}(x) - f^{n-2}(x) + \cdots + f^{m+1}(x) - f^m(x)\right| \\ &\leq \left|f^n(x) - f^{n-1}(x)\right| + \left|f^{n-1}(x) - f^{n-2}(x)\right| + \cdots + \left|f^{m+1}(x) - f^m(x)\right|\end{aligned}$$

(by the triangle inequality)

$$\begin{aligned}&\leq (r^{n-1} + \cdots + r^m) \cdot \left|f(x) - x\right| \\ &= r^{m-1} \cdot (r^{n-m} + \cdots + r^2 + r) \cdot \left|f(x) - x\right| \\ &< r^{m-1} \cdot \frac{r}{1-r} \cdot \left|f(x) - x\right| \\ &= r^m \cdot \frac{1}{1-r} \cdot \left|f(x) - x\right|\end{aligned}$$

Here we have used the fact that since the geometric series $\sum_{k=1}^{\infty} r^k$ converges to $\frac{r}{1-r}$, and its terms are all positive, its partial sums are all $< \frac{r}{1-r}$.

If $f(x_0) = x_0$ then the given sequence is constant and therefore Cauchy. So assume that $f(x_0) \neq x_0$. We proved in class that if $0 < r < 1$, then the sequence r^n converges to 0; thus given any $\varepsilon > 0$ we can pick $N \in \mathbb{N}$ such that if $m > N$, then

$$r^m < \frac{(1-r) \cdot \varepsilon}{|f(x_0) - x_0|}.$$

Then if $n, m > N$ with (WLOG) $n > m$, we have

$$\begin{aligned}\left|f^n(x_0) - f^m(x_0)\right| &< r^m \cdot \frac{1}{1-r} \cdot \left|f(x_0) - x_0\right| \\ &< \varepsilon\end{aligned}$$

by construction of N. Therefore, the given sequence is Cauchy.

Second solution: Let $a_n = f^n(x_0) - f^{n-1}(x_0)$. Then we have

$$\begin{aligned}|a_n| &= \left|f^n(x_0) - f^{n-1}(x_0)\right| \\ &\leq r\left|f^{n-1}(x_0) - f^{n-2}(x_0)\right| \\ &= r|a_{n-1}|.\end{aligned}$$

and therefore

$$\frac{|a_{n+1}|}{|a_n|} \leq r.$$

for all n. It follows that

$$\varlimsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \le r < 1$$

so by the ratio test, the series $\sum_{n=1}^{\infty} |a_n|$ converges, and therefore the series $\sum_{n=1}^{\infty} a_n$ converges. This implies that the sequence of partial sums $(\sum_{n=1}^{m} a_n)$ converges, and hence is Cauchy. But we have

$$\begin{aligned}\sum_{n=1}^{m} a_n &= (f(x_0) - x_0) + (f^2(x_0) - f(x_0)) + \cdots + (f^m(x_0) - f^{m-1}(x_0)) \\ &= f^m(x_0) - x_0.\end{aligned}$$

so the sequence $(f^m(x_0) - x_0)$ is Cauchy. Since adding a constant to a sequence does not change whether it is Cauchy, we conclude that the sequence $(f^m(x_0))$ is also Cauchy, as desired.

(b) Since A is nonempty, there exists some $x_0 \in A$. Let y be the limit of the Cauchy sequence constructed in part (a), so $y = \lim_{n\to\infty} f^n(x_0)$. Since A is closed and y is the limit of a Cauchy sequence contained in A, we have $y \in A$ (Theorem 10, Analysis packet). And since f is continuous, it is sequentially continuous, so we have

$$f(y) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{n+1}(x_0).$$

But the sequence $(f^{n+1}(x_0))$ is a subsequence of $(f^n(x_0))$, and therefore also converges to y. Thus $f(y) = y$ as desired.

Remark 1. Which solution you prefer here is a matter of taste. Because the proof of the ratio test uses the convergence of the geometric series, the first proof can be regarded as an 'expanded out' version of the second. I actually only thought of the first one myself, but one person came up with the second one on their final and I thought it was too clever not to mention.

Remark 2. Several people said something along the lines of

$$\text{“}\lim_{n,m\to\infty} |a_n - a_m| = 0\text{”}$$

when proving that a sequence was Cauchy. We have not defined the meaning of a limit as two different indices go to infinity, but even if we had, you would have to actually use such a definition rather than blithely making assertions about it. A better strategy is to use the definition of 'Cauchy'.

Remark 3. If f is a function such that $|f(x) - f(y)| \le r|x - y|$ for some $r \ge 0$ and all x, y, we say that f is **Lipschitz**, after the German mathematician Rudolf Lipschitz, 1832-1903. If $r < 1$, as in this problem, we say that f is a **contraction**. You may enjoy proving that any Lipschitz function is automatically continuous; thus the hypothesis in this problem that f is continuous is redundant.

Problem 9. (a) *Prove that x is an accumulation point of a sequence S in $\mathbb{R}$ iff some subsequence of S converges to x.*

(b) *Let $S = (x_i)$ be a sequence which is bounded above and below, and let A be the set of all accumulation points of S. Prove that $\varlimsup_{n\to\infty} x_n$ is the* *supremum of A and $\underline{\lim}_{n\to\infty} x_n$ is the infimum of A. Conclude that S converges iff it has exactly one accumulation point.*

(c) *Let S and A be as above. Prove that the set A is closed (so that $\overline{\lim}\, x_n$ and $\underline{\lim}\, x_n$ are always accumulation points of S).*

Solution to Problem 9. (a) Suppose that x is an acccumulation point of $S = (x_n)$. We define a subsequence (x_{n_k}) recursively as follows. Let $n_1 = 1$. Now suppose that n_k is defined for $k < \ell$. Consider the region $R = (x - \frac{1}{\ell}, x + \frac{1}{\ell})$. Since x is an accumulation point of S, this region contains x_n for infinitely many values of n. Thus, there exists an n_ℓ such that $x_{n_\ell} \in R$ and $n_\ell > n_k$ for all $k < \ell$; choose this as n_ℓ. This defines a subsequence of S, and it is easy to see that $x_{n_k} \in (x - \frac{1}{m}, x + \frac{1}{m})$ for all $k > m$, so that this subsequence converges to x.

Now suppose that S has a subsequence (x_{n_k}) which converges to x, and let $R \ni x$ be a region. Then there exists K such that if $k > K$, then $x_{n_k} \in R$. Since there are infinitely many $k > K$, there are infinitely many n such that $x_n \in R$—namely, all the $n = n_k$ for some $k > K$. Thus x is an accumulation point of S.

(b) Let $L = \overline{\lim}_{n\to\infty} x_n$. We first show that L is an upper bound of A. We do this by contradiction: suppose that there exists an accumulation point x with $x > L$. Let y be such that $L < y < x$; then there is a region $(y, z) \ni x$ which therefore contains infinitely many points of S. Therefore, any set of the form $\{x_n, x_{n+1}, \dots\}$ contains points greater than y, so its supremum is also greater than y. Thus the limit of these suprema, which is L, cannot be less than y, a contradiction. So L must be an upper bound of A. In particular, A is bounded above, hence has some supremum.

Now we show that L is itself an accumulation point of S. Let $R \ni L$ be a region, and let $b_n = \sup\{x_n, x_{n+1}, \dots\}$. Since $L = \lim_{n\to\infty} b_n$ by definition, there exists N such that if $n > N$, then $b_n \in R$. Now, if $b_n \in R$, then there must be some $m > n$ with $x_m \in R$; for if not, then the lower endpoint of R is an upper bound for $\{x_n, x_{n+1}, \dots\}$ which is less than b_n, a contradiction. Since $b_n \in R$ for all $n > N$, this implies that for any n, there exists an $m > n$ with $x_m \in R$, and therefore R contains infinitely many terms of S. This is true for any region R containing L, so L is an accumulation point of S. Hence $L \in A$, and so no upper bound of A can be less than L. Thus L must be the supremum of A.

Similarly, we show that $\underline{\lim}_{n\to\infty} x_n$ is the infimum of A. Now, we know that S converges iff its limsup is equal to its liminf. But the supremum of a nonempty bounded set is equal to its infimum precisely when the set has exactly one point, so S converges iff it has exactly one accumulation point.

(c) Let x be a limit point of A and let $R \ni x$ be a region. Since $x \operatorname{lp} A$, the region R contains some point $y \in A$. Then R is a region containing y, which is an accumulation point of S, so R contains infinitely many terms of S. This is true for any region $R \ni x$, so by definition x is an accumulation point of S, hence $x \in A$. Thus A is closed.

Remark 1. In proving part (a), many people said something like this: "if there is a subsequence T of S which converges to x, then $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that if $n > N$ then $|x_n - x| < \varepsilon$." If $S = (x_n)$ is the original sequence, then this is false; it would only be true if S itself converged to x. If you are using (x_n) to mean the subsequence T, then you should say so explicitly, and remember that if $S = (y_m)$ is the original sequence, then $x_n = y_{m_n}$ for some different index m_n.

Remark 2. In proving part (b), many people asserted that by definition, the limsup of a sequence S is the limit of a subsequence of S. This is not correct; the limsup of $S = (x_n)$ is the limit of the sequence (b_n), where $b_n = \sup\{x_n, x_{n+1}, \dots\}$, but none of the b_ns need necessarily occur at all in the sequence S. For instance, if $x_n = \frac{n-1}{n}$, then $b_n = 1$ for all n (so that the limsup is also 1), but $x_n \neq 1$ for all n. It is *true* that the limsup of S is the limit of *some* subsequence of S, since it is an accumulation point of S, but this requires a proof.

Problem 10. *Give examples of sequences S satisfying the following criteria. Here 'bounded' means 'bounded above and below'. You should explain briefly why each example has the desired properties, but formal proofs are not required.*

(a) *S has no accumulation points.*
(b) *S is bounded and has exactly three accumulation points.*
(c) *S is bounded and does not converge, and every term of S is either greater than its limsup or less than its liminf.*
(d) *The set of accumulation points of S is $\mathbb{N}$.*
(e) *S is bounded and has infinitely many accumulation points.*
(f) *Every real number is an accumulation point of S.*

Solution to Problem 10. (a) $(0, 1, 2, 3, \dots)$; it eventually leaves every bounded set, hence cannot be in any region infinitely often.

(b) $(0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, \dots)$; its only accumulation points are 0, 1, and 2. These are accumulation points because it hits them on the nose infinitely many times, and any other point is contained in a region disjoint from $\{0, 1, 2\}$.

(c) $(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{3}, \frac{4}{3}, -\frac{1}{4}, \frac{5}{4}, \dots)$. Evidently its limsup is 1 and its liminf is 0. Since $1 \neq 0$, it does not converge, and every term is either negative or greater than 1.

(d) $(1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots)$. It eventually reaches every natural number infinitely often, hence each natural number is an accumulation point. Any other real number is contained in a region disjoint from $\mathbb{N}$, hence is not an accumulation point.

(e) $(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)$. It is clearly bounded, and all numbers $\frac{1}{n}$ for $n \in \mathbb{N}$ are accumulation points, since it hits them each infinitely often. In addition, 0 is an accumulation point. Any other real number is contained in a region disjoint from $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, hence is not an accumulation point.

(f) The rational numbers $\mathbb{Q}$ are countable. Let $f : \mathbb{N} \to \mathbb{Q}$ be a surjective function. Then the sequence $(f(1), f(2), f(3), f(4), \dots)$ has every real number as an accumulation point, since every region in $\mathbb{R}$ contains infinitely many rational numbers.