MATH 162 SECTION 50
FINAL HOMEWORK SET

DUE THURSDAY, MARCH 8

> "The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another." – Frege

Collaboration is allowed on ***some*** problems, but not on others. Pay attention! The collaboration-allowed problems should be treated like problems on an ordinary homework set. For the no-collaboration problems, you may ask any of the instructors for clarification, but you may not discuss them at all with anyone else, or consult reference materials other than our course notes and Spivak.

By the way, do not be discouraged if you cannot do all of the problems! They are intended to be difficult.

1. COLLABORATION ALLOWED

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined as follows:

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \pm\frac{p}{q},\ p, q \in \mathbb{N},\ \gcd(p, q) = 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

At what points is f continuous and at what points is it discontinuous? Prove your answer. You may assume that any region in $\mathbb{R}$ contains both rational and irrational numbers.

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and let (x_n) be a bounded sequence. Is it necessarily true that $f\left(\overline{\lim_{n\to\infty}}\, x_n\right) = \overline{\lim_{n\to\infty}}\, f(x_n)$? If so, prove it; if not, give a counterexample.

(3) Let $A \subset \mathbb{R}$ and let (x_n) and (y_n) be sequences such that for all n, $x_n \in A$ and y_n is an upper bound of A. Prove that if (x_n) and (y_n) converge to the same limit z, then $z = \sup A$.

(4) Let F be an ordered field with the following two properties:

(i) The Archimedean property (for all $x \in F$ with $x > 0$ there are $n, m \in \mathbb{N}$ such that $\frac{1}{n} < x < m$).

(ii) Any Cauchy sequence in F converges to an element of F.

Prove that F satisfies axiom 3, and therefore is the real numbers. *(Hint: use the idea of problem 3 to construct a supremum for any subset of F which is bounded above.)*

2. **NO COLLABORATION ALLOWED**

(5) Let $f\colon A \to B$ be a function, and let x be a point in A which is not a limit point of the set A. Prove that f is continuous at x.

(6) Let $A \subset \mathbb{R}$ be compact and $A \subset B_1 \cup B_2$, where B_1 and B_2 are disjoint open sets. Prove that $A \cap B_1$ is also compact. Give an example to show that this is not necessarily true if B_1 and B_2 are not disjoint.

(7) Let $f\colon [a,b] \to \mathbb{R}$ and let $|f|$ be the function defined by $|f|(x) = |f(x)|$. For each of the following statements, either prove it or give a counterexample.
   (a) If f is continuous, then $|f|$ is continuous.
   (b) If $|f|$ is continuous, then f is continuous.

(8) Let $A \subset \mathbb{R}$ be closed, let $0 < r < 1$ be a real number, and let $f\colon A \to A$ be a continuous function such that for all $x, y \in A$, we have
$$|f(x) - f(y)| \leq r|x - y|.$$
   (a) Let $x_0 \in A$. Prove that the sequence
$$\big(x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \dots\big)$$
   is Cauchy.
   (b) Prove that if A is nonempty, then f has a fixed point.

If $S = (x_i)$ is a sequence of real numbers, a real number x is said to be an **accumulation point** of S if for all regions $R \ni x$, there are infinitely many $n \in \mathbb{N}$ such that $x_n \in R$.

(9) (a) Prove that x is an accumulation point of a sequence S in $\mathbb{R}$ iff some subsequence of S converges to x.
   (b) Let $S = (x_i)$ be a sequence which is bounded above and below, and let A be the set of all accumulation points of S. Prove that $\overline{\lim}_{n\to\infty} x_n$ is the supremum of A and $\underline{\lim}_{n\to\infty} x_n$ is the infimum of A. Conclude that S converges iff it has exactly one accumulation point.
   (c) Let S and A be as above. Prove that the set A is closed (so that $\overline{\lim}\, x_n$ and $\underline{\lim}\, x_n$ are always accumulation points of S).

(10) Give examples of sequences S satisfying the following criteria. Here 'bounded' means 'bounded above and below'. You should explain briefly why each example has the desired properties, but formal proofs are not required.
   (a) S has no accumulation points.
   (b) S is bounded and has exactly three accumulation points.
   (c) S is bounded and does not converge, and every term of S is either greater than its limsup or less than its liminf.
   (d) The set of accumulation points of S is $\mathbb{N}$.
   (e) S is bounded and has infinitely many accumulation points.
   (f) Every real number is an accumulation point of S.