MATH 16200 SECTION 50, HOMEWORK 2

DUE DATE THURSDAY JAN 26

Alexander the great: Master, is there any ultimate gain or reward in studying mathematics and working so hard on the problems that you have given me?
Euclid: Jack... give this bloody brat two bucks after he has finished his homework...

(1) By definition, the identity map I_A from a set A to A is a function such that $I_A(x) = x$ for all $x \in A$. Show that if a function $f : A \to B$ is one to one and onto then there exists another function $f^{-1} : B \to A$ such that $f^{-1}(f(x)) = I_A(x)$ and $f(f^{-1}(y)) = I_B(y)$.

(2) Let $f : A \to B$ and $g : B \to C$ be one to one and onto. Define $(g \circ f)(x)$ as $g(f(x))$. Show that $(g \circ f)^{-1} : C \to A$ exists and equals $f^{-1} \circ g^{-1} : C \to A$.

(3) Let C be a set satisfying axioms 1 and 2 and the betweenness proposition, and such that every subset of C which has an upper bound has a least upper bound. Show that C satisfies axiom 3.

(4) Suppose that C' is a set satisfying axioms 1, 2, and 3 (for example, it could be the real numbers). Let a and b be two points of C' and define

$$C = \left(\overline{ab} \times \overline{ab}\right) \setminus \{(a, a), (b, b)\}$$

(Recall that for sets X, Y the notation $X \times Y$ means the set of all pairs (x, y) such that $x \in X, y \in Y$.) If (x, y) and (z, w) are two elements of C, we define $(x, y) < (z, w)$ to hold if and only if **either** of the following two conditions holds:
(i) $x < z$ in C', **or**
(ii) $x = z$ and $y < w$ in C'.
(This is called the **lexicographic order** on C.) Show that C satisfies axioms 1, 2, and 3.

When C' is taken to be the real numbers $\mathbb{R}$, the C constructed in problem 4 is not isomorphic to $\mathbb{R}$. (That is, there is no one-to-one correspondence between C and $\mathbb{R}$ which preserves order.) *Proving* this, however, is unfortunately beyond the scope of this problem set.

(5) Let $X \subset C$ (here C again satisfies axioms 1, 2, and 3). We say a collection $\mathcal{G}$ of subsets of X has the *finite intersection property* if for every finite subcollection

$$\{G_1, \ldots, G_n\}$$

of $\mathcal{G}$, the intersection $G_1 \cap \ldots \cap G_n$ is nonempty. Prove that X is compact if and only if for every collection $\mathcal{G}$ of closed subsets of X having the finite intersection property, the intersection

$$\bigcap_{G \in \mathcal{G}} G$$

is non-empty.