MATH 16200 SECTION 50, HOMEWORK 3

DUE DATE THURSDAY FEB 1

William Shakespeare: In deepest Amazonian forests there are tribes of Indians, yet untouched by the civilization, who have developed far more sophisticated pedagogical and teaching skills than W.S.

Let C satisfy axioms 1, 2, and 3; let F be an ordered field; and let $\mathbb{R}$ be the real numbers.

(1) Let $A \subset B \subset C$. Show that A is open relative to B iff there exists an open set $U \subset C$ such that $A = B \cap U$. Show also the analog of this statement for the sets that are closed relative to B.

(2) Let $A, B \subset F$ and $f\colon A \to B$. Let $x \in A$; we say that f is **continuous at** x if the following holds:

> For all $\varepsilon \in F$ with $\varepsilon > 0$, there exists a $\delta \in F$ with $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon$.

Show that f is continuous (with the definition from the topology packet, using regions) iff it is continuous at x (in the above sense) for all $x \in A$. (This is Lemma C(3) in the real numbers packet.)

(3) Let $f, g\colon F \to F$ be continuous.

(a) Define a function $f + g\colon F \to F$ by $(f + g)(x) = f(x) + g(x)$. Show that $f + g$ is continuous.
(b) Define $f \circ g\colon F \to F$ by $(f \circ g)(x) = f(g(x))$. Show that $f \circ g$ is continuous.

(It is also true that products and quotients of continuous functions are continuous. If you're bored, feel free to prove this for yourself. This implies that all polynomials and rational functions are continuous on their entire domains.)

(4) Let $A \subset C$ and let $f : A \to C$ be continuous. Given $\lambda \in C$, which of the following sets are closed or open relative to A? Give a motivation for your answers.
(a) $\{x \in A;\ f(x) < \lambda\}$,
(b) $\{x \in A;\ f(x) \leq \lambda\}$,
(c) $\{x \in A;\ f(x) \geq \lambda\}$,
(d) $\{x \in A;\ f(x) > \lambda\}$,
(e) $\{x \in A;\ f(x) = \lambda\}$.

(5) The Dirichlet function (named after German mathematician Peter Gustav Dirichlet (1805-1859)), is a function defined on the whole real line $\mathbb{R}$ by

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that Dirichlet's function is discontinuous at every point on the real line.