# MATH 16200 SECTION 50, HOMEWORK 4

DUE DATE THURSDAY FEB 8

(1) Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$. Show that the following statements are equivalent.

(a) For all $x_0 \in A$ and for all $x_n \in A$ with $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} f(x_n) = f(x_0)$. (We say in this case that $f$ is **sequentially continuous**.)

(b) $f$ is continuous.

(2) Each of the following functions is defined from $\mathbb{R} \setminus \{0\} \to \mathbb{R}$. Which of them can be extended to a continuous function defined on the whole real line? In each case give a motivation for your answer.

(a) $f(x) = \frac{1}{x}$

(b) $f(x) = \frac{\sin x}{x}$

(c) $f(x) = \sin \frac{1}{x}$.

(3) Let $f$ be a continuous function from $\mathbb{R} \to \mathbb{R}$ such that for all $x, y$ in $\mathbb{R}$, $f(x + y) = f(x) + f(y)$. Show that

(a) $\forall n \in \mathbb{N}\ f(n) = nf(1)$,

(b) $\forall z \in \mathbb{Z}\ f(z) = zf(1)$,

(c) $\forall q \in \mathbb{Q}\ f(q) = qf(1)$,

(d) $\forall x \in \mathbb{R}\ f(x) = xf(1)$.

(4) (a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous bijective function, then its inverse $f^{-1}$ is also continuous.

(b) By definition the **integer part** of a real number $x$, denoted by $[x]$, is the unique integer $n$ such that $n \leq x < n + 1$. The **fractional part** of $x$ is by definition

$$\{x\} := x - [x] \in [0, 1).$$

Note that $\{x\}$ is a *number*, and should not be confused with the set containing only the point $x$. Now for a fixed nonzero irrational number $\alpha$, define a function $f : \mathbb{Q} \setminus \{0\} \to (0, \alpha)$ by $f(x) = \alpha\{\frac{x}{\alpha}\}$.

(i) Show that $f$ is one to one.

(ii) Show that $f$ is continuous.

(iii) If we consider $f$ as a function from $\mathbb{Q} \setminus \{0\}$ to $f(\mathbb{Q} \setminus \{0\})$ then $f$ is bijective. Show, by proving the following steps, that the inverse function of $f$ is nowhere continuous on $f(\mathbb{Q} \setminus \{0\})$:

(A) Set $J := f(\mathbb{Q} \setminus \{0\})$ and take $y \in J$. Show that for all $n \in \mathbb{N} \setminus \{0\}$ there exists a rational number $x_n$ in the region $(y + \alpha n - \frac{1}{n}, y + \alpha n + \frac{1}{n})$. What is the limit of the sequence $x_n$?

(B) Show that there is $N \in \mathbb{N} \setminus \{0\}$ such that for all $n \geq N$ one has $y_n := f(x_n) \in (y - \frac{1}{n}, y + \frac{1}{n})$. What is the limit of $y_n$?

(C) What is the conclusion of all this?