

MATH 16200 SECTION 50, HOMEWORK 5

DUE DATE THURSDAY FEB 22

- (1) For the sequence $a_n = (-1)^{n+1}(3 + \frac{2}{n})$, determine $\inf\{a_n\}$, $\sup\{a_n\}$, $\overline{\lim}_{n \rightarrow \infty} a_n$ and $\underline{\lim}_{n \rightarrow \infty} a_n$. Note that the first two are suprema and infima of a *set* while the latter two are the limsup and liminf of the *sequence*.
- (2) Give an example of a sequence a_n which does not converge, but for which $\lim_{n \rightarrow \infty} (a_{n+k} - a_n) = 0$, for every positive integer k . Does this contradict theorem 9 (any Cauchy sequence of real numbers converges)?
- (3) Define a sequence a_n by $a_1 = \sqrt{2}$ followed by the recursive formula $a_{n+1} = \sqrt{2 + a_n}$. Prove that the sequence (a_n) converges, and find its limit.
- (4) Prove that if $\lim_{n \rightarrow \infty} a_n = a$, then $\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n} = a$.
- (5) Consider two strictly increasing sequences k_n and m_n of natural numbers such that for some $\ell \in \mathbf{N}$ we have

$$\{\ell, \ell + 1, \ell + 2, \dots\} \subset \{k_1, k_2, \dots\} \cup \{m_1, m_2, \dots\}.$$

Prove that a sequence of real numbers a_n converges if and only if both of the following are satisfied.

- (a) the subsequences (a_{k_n}) and (a_{m_n}) of (a_n) both converge, and
- (b) $\lim_{n \rightarrow \infty} a_{k_n} = \lim_{n \rightarrow \infty} a_{m_n}$.

Prove that in this case, the common limit of the subsequences is also the limit of the original sequence a_n .