ANALYSIS

In the first part of this course we studied the topology of the continuum, without reference to any algebra. We now rectify this omission.

1. The Real Numbers

We define the real numbers axiomatically. In contrast to the situation with the continuum C, our axioms for $\mathbb{R}$ *will* suffice to characterize the real numbers uniquely.

Definition I. A **field** is a set F together with binary operations $+, \cdot$ and elements $0, 1 \in F$ satisfying the following axioms.

(1) $x + (y + z) = (x + y) + z$ for all $x, y, z \in F$ (addition is associative).
(2) $x + 0 = x = 0 + x$ for all $x \in F$ (zero element).
(3) For any $x \in F$ there exists an element $-x \in F$ such that $x + (-x) = 0 = (-x) + x$ (additive inverses exist).
(4) $x + y = y + z$ for all $x, y \in F$ (addition is commutative).
(5) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in F$ (multiplication is associative).
(6) $x \cdot 1 = x = 1 \cdot x$ for all $x \in F$ (unit element).
(7) For any $x \in F$ with $x \neq 0$ there exists an element $x^{-1} \in F$ such that $x \cdot x^{-1} = 1 = x^{-1} \cdot x$ (multiplicative inverses exist).
(8) $x \cdot y = y \cdot z$ for all $x, y \in F$ (multiplication is commutative).
(9) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for all $x, y, z \in F$ (multiplication distributes over addition).
(10) $0 \neq 1$ (nontriviality).

Example I. Let F be the set $\{0, 1\}$ and define

$$0 + 0 = 0 \qquad 0 + 1 = 1 \qquad 1 + 0 = 1 \qquad 1 + 1 = 0$$
$$0 \cdot 0 = 0 \qquad 0 \cdot 1 = 0 \qquad 1 \cdot 0 = 0 \qquad 1 \cdot 1 = 1$$

Then F is a field.

Lemma A. Let F be a field.

(1) If $z \in F$ has the property that $z + x = x$ for all $x \in F$, then $z = 0$.
(2) If $z \in F$ has the property that $z \cdot x = x$ for all $x \in F$, then $z = 1$.
(3) Given x, if y and y' have the property that $x + y = 0$ and $x + y' = 0$, then $y = y'$.
(4) Given x, if y and y' have the property that $x \cdot y = 1$ and $x \cdot y' = 1$, then $y = y'$.

Definition II. An **ordered field** is a field F which is also equipped with a binary relation $<$ such that the following axioms are satisfied.

(1) $(F, <)$ satisfies Axiom 1 (linearly ordered).
(2) If $x < y$, then $x + z < y + z$ for any $z \in F$ (addition respects order).
(3) If $0 < x$ and $0 < y$, then $0 < xy$ (multiplication respects order).

Example II. The rational numbers $\mathbb{Q}$ are an ordered field.

Lemma B. Let F be an ordered field.

(1) If $0 < x$, then $-x < 0$. Similarly if $x < 0$, then $0 < -x$.
(2) For any $x \in F$, we have $0 \leq x^2$. In particular, $0 < 1$.
(3) $(F, <)$ satisfies Axiom 2.
(4) $(F, <)$ satisfies the Betweenness property (all regions are nonempty).
(5) F contains the rational numbers $\mathbb{Q}$.

In an ordered field F, we define the **absolute value** of $x \in F$ to be

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Lemma C. Let F be an ordered field and $A, B \subset F$.

(1) A is open iff for all $x \in A$, there exists an $\varepsilon > 0$ such that if $|y - x| < \varepsilon$, then $y \in A$.
(2) $x \operatorname{lp} A$ iff for all $\varepsilon > 0$ there exists a $y \in A$ with $|y - x| < \varepsilon$.
(3) A function $f \colon A \to B$ is continuous iff for all $x \in A$ and all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon$.
(4) A sequence $S = (x_n)$ in F has a sequential limit point x iff for all $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that if $k > n$, $|x_k - x| < \varepsilon$.

Definition III. The **real numbers** are an ordered field $\mathbb{R}$ such that $(\mathbb{R}, <)$ also satisfies Axiom 3.

Therefore, all of our theorems about C can be applied to $\mathbb{R}$.

Lemma D. Let $C = \mathbb{Q}$ and let C' be the set of Dedekind cuts in $\mathbb{Q}$ (defined on homeworks last quarter). Define, for Dedekind cuts A, B,

$$\begin{aligned} A + B &= \{x + y \mid x \in A, y \in B\} \\ 0 &= \{x \mid x < 0\} \\ 1 &= \{x \mid x < 1\} \end{aligned}$$

If $A > 0$, $B > 0$, define

$$A \cdot B = \{x \cdot y \mid 0 \leq x \in A, 0 \leq y \in B\} \cup \{x \mid x \leq 0\}$$

and make analogous definitions in the other cases. Then C' is the real numbers (that is, C' is an ordered field satisfying Axiom 3).

It is a fact that Definition III characterizes the real numbers up to isomorphism (that is, a one-to-one correspondence which preserves $0, 1, +, \cdot, <$).

Lemma E. Let C'' be the set of infinite decimal expansions which do not end in an infinite sequence of 9's. So

$$\begin{aligned} 2.0000000000\ldots \\ 3.1415926536\ldots \\ 29848.9027803212\ldots \\ 0.3333333333\ldots \end{aligned}$$

are all valid elements of C'', but

$$0.999999999\ldots$$

is not. Note that C'' includes the 'terminating' decimal expansions; they are those which end in an infinite sequence of 0's. We define $0, 1, +, \cdot, <$ in the usual ways; if an operation results in an infinite sequence of 9's we replace it by the corresponding terminating decimal (so $0.99999999\ldots$ would be replaced by $1.00000000\ldots$).

Then C'' is also the real numbers. An explicit isomorphism between C'' and the C' defined using Dedekind cuts is given by sending an infinite decimal expansion to the set of all rational numbers less than it is (in the order of C'').

Theorem 1 (Existence of square roots)**.** If $x \in \mathbb{R}$ and $x \geq 0$, then there exists a $y \in \mathbb{R}$ such that $y^2 = x$.

Theorem 2 (The Archimedean property)**.** If $x \in \mathbb{R}$ and $x > 0$, then there exist natural numbers $n, m \in \mathbb{N}$ such that $\frac{1}{m} < x < n$.

We will conform to standard mathematical practice as follows: if $a, b \in \mathbb{R}$ and $a < b$, then we write the region ab in $\mathbb{R}$ as (a, b). This should not be confused with an ordered pair! We also write

$$[a,b] = (a,b) \cup \{a,b\} \qquad [a,b) = (a,b) \cup \{a\} \qquad (a,b] = (a,b) \cup \{b\}$$
$$(a,\infty) = \{x : a < x\} \qquad (-\infty, b) = \{x : x < b\}$$

and similarly for $[a, \infty)$ and $(-\infty, b]$. Of course, "∞" and "$-\infty$" are not actual real numbers.

Corollary 2.1. A subset $A \in \mathbb{R}$ is open iff for all $x \in A$ there exists an $n \in \mathbb{N}$ such that $(x - \frac{1}{n}, x + \frac{1}{n}) \subset A$.