2. Sequences and series of real numbers

From now on, by a **sequence** we will mean an infinite sequence of real numbers; that is, a function $S\colon \mathbb{N} \to \mathbb{R}$. We often write $S(n) = a_n$ and call it the n**th term** of S; we sometimes write $S = (a_n)_{n\in\mathbb{N}}$. If $x \operatorname{slp} S$ we say that S **converges** to x and write $\lim_{n\to\infty} a_n = x$.

Definition IV. Let $S = (a_n)_{n\in\mathbb{N}}$ be a sequence.

- S is **bounded above** if there exists $b \in \mathbb{R}$ with $a_n \leq b$ for all n.
- S is **bounded below** if there exists $b \in \mathbb{R}$ with $b \leq a_n$ for all n.
- S is **increasing** if $n < m$ implies $a_n \leq a_m$.
- S is **decreasing** if $n < m$ implies $a_n \geq a_m$.
- S is **strictly increasing** if $n < m$ implies $a_n < a_m$.
- S is **strictly decreasing** if $n < m$ implies $a_n > a_m$.
- S is **monotone** if it is either increasing or decreasing.

Theorem 3. Let $S = (a_n)$ be increasing and bounded above. Then S has a sequential limit point (that is, S converges).

Definition V. Let $S = (a_n)$ be a sequence and let $n_1 < n_2 < \ldots$ be a strictly increasing sequence of natural numbers. Then the sequence $T = (a_{n_k})_{k\in\mathbb{N}}$ is called a **subsequence** of S.

Theorem 4. Any sequence has a subsequence which is monotonic.

Theorem 5. If S converges to x and T is a subsequence of S, then T also converges to x.

Definition VI. A sequence S is **Cauchy** if for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n, m > N$ then $|a_n - a_m| < \varepsilon$.

Cauchy sequences are named after the French mathematician Augustin Louis Cauchy (1789-1857).

Theorem 6. Any Cauchy sequence is bounded both above and below.

Theorem 7. Any sequence which converges is Cauchy.

Theorem 8. If S is Cauchy, T is a subsequence of S, and T converges to x, then S also converges to x.

Question. Is Theorem 8 true if S is not Cauchy?

Theorem 9. Any Cauchy sequence converges to some real number.

If $S = (a_n)$ is a sequence and $A \subset \mathbb{R}$, we say that S is **contained in** A if all $a_n \in A$.

Theorem 10. A set $A \subset \mathbb{R}$ is closed iff any Cauchy sequence contained in A converges to an element of A.

Theorem 11. A set $A \subset \mathbb{R}$ is compact iff any sequence contained in A has a subsequence which converges to an element of A.

Lemma F. Let $S = (a_n)$ be a sequence which is bounded above and define b_n to be the least upper bound of the *set* $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$. Then the sequence $T = (b_n)$ converges.

Definition VII. The limit of the sequence T in Lemma F is called the **limit superior** of the sequence S and written $\limsup_{n\to\infty} a_n$, or more frequently in analysis literature $\overline{\lim_{n\to\infty}}\, a_n$. If S is bounded below and we use greatest lower bounds instead, then we get the notion of **limit inferior**, written $\liminf_{n\to\infty} a_n$ or $\underline{\lim_{n\to\infty}}\, a_n$.

Theorem 12. If $S = (a_n)$ is bounded above and below, then $\underline{\lim_{n\to\infty}}\, a_n \leq \overline{\lim_{n\to\infty}}\, a_n$.

Theorem 13. S converges iff $\underline{\lim_{n\to\infty}}\, a_n = \overline{\lim_{n\to\infty}}\, a_n$.

Definition VIII. An **infinite series** is a sequence of real numbers a_n which we write

$$\sum_{n=1}^{\infty} a_n.$$

Given an infinite series, we define its **sequence of partial sums** by $s_n = a_1 + \cdots + a_n$. We say that an infinite series **converges to** x if its sequence of partial sums converges to x.

Theorem 14. If $0 < a < 1$, the series $\sum_{n=1}^{\infty} a^n$ converges to $\frac{a}{1-a}$.

Theorem 15 (The harmonic series)**.** The series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

Definition IX. We say that $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 16. If a series converges absolutely, then it converges.

Theorem 17 (Comparison test)**.** If $|a_n| \leq b_n$ for all n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 18 (Ratio test)**.** Suppose each $a_n > 0$ and that

$$\overline{\lim_{n\to\infty}}\, \frac{a_{n+1}}{a_n} < 1.$$

Then $\sum_{n=1}^{\infty} a_n$ converges.

The ratio test is also called the d'Alembert test, after French mathematician, physicist and philosopher Jean le Rond d'Alembert (1717-1783).

Theorem 19 (Root test or Cauchy's test)**.** Suppose each $a_n > 0$ and that

$$\overline{\lim_{n\to\infty}}\, (a_n)^{\frac{1}{n}} < 1.$$

Then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 20 (Alternating series test)**.** Suppose the *sequence* (a_n) is decreasing and converges to zero. Then the *series* $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

The alternating series test is also called the Leibniz criterion, after German mathematician and philosopher Gottfried Wilhelm Leibniz (1646-1716).

Theorem 21. Let $\sum_{n=1}^{\infty} a_n$ converge absolutely to x. Define $b_n = a_{2n+1}$ and $c_n = a_{2n}$. Then $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ converge to real numbers y and z and $x = y + z$.

Question. Is Theorem 21 true without the assumption of absolute convergence?