Comparing composites of left and right derived functors

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January 7, 2009

Outline

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Presenting homotopy theories

We regard a model category $\mathscr{M}$ as a presentation of a homotopy theory. In particular, it has a homotopy category $\mathrm{Ho}(\mathscr{M})$.

Question

How can we use functors between model categories to present morphisms between homotopy theories?

First answer

If $F\colon \mathscr{M} \to \mathscr{N}$ preserves weak equivalences, it induces a functor

$$\mathrm{Ho}(F)\colon\ \mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{N}).$$

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But few functors preserve all weak equivalences!

Derived functors

Second answer

If F is a left Quillen functor, then it has a left derived functor

$$\mathbf{L}F\colon \mathrm{Ho}(\mathscr{M}) \to \mathrm{Ho}(\mathscr{N})$$

defined by $\mathbf{L}F(X) = F(QX)$, with QX a cofibrant replacement of X. Likewise, if F is a right Quillen functor, it has a right derived functor $\mathbf{R}F(X) = F(RX)$, where RX is a fibrant replacement of X.

Remark

Actually, less structure on F suffices, but most examples are Quillen functors.

Derived natural transformations

- If $F, G\colon \mathscr{M} \rightrightarrows \mathscr{N}$ are left Quillen, any natural transformation $\alpha\colon F \to G$ has a left derived natural transformation $\mathbf{L}\alpha\colon \mathbf{L}F \to \mathbf{L}G$ whose component at X is $\alpha_{QX}\colon F(QX) \to G(QX)$.
- Likewise, if F and G are right Quillen, any $\alpha\colon F \to G$ has a right derived natural transformation $\mathbf{R}\alpha\colon \mathbf{R}F \to \mathbf{R}G$.

Theorem (Hovey)

Let $\mathcal{LM}odel$ denote the 2-category of model categories, left Quillen functors, and natural transformations. Then there is a pseudofunctor

$$\mathbf{L}\colon \mathcal{LM}odel \longrightarrow \mathcal{C}at.$$

Likewise, we have

$$\mathbf{R}\colon \mathcal{RM}odel \longrightarrow \mathcal{C}at.$$

Why we need more

- What if $\alpha\colon F \to G$ is a natural transformation, where F is left Quillen and G is right Quillen?
- Or, what if one or both of them is the composite of two Quillen functors of different handedness?

Why we need more

- What if $\alpha\colon F \to G$ is a natural transformation, where F is left Quillen and G is right Quillen?
- Or, what if one or both of them is the composite of two Quillen functors of different handedness?

The most well-known case is a Quillen adjunction (an adjunction whose left adjoint is left Quillen and whose right adjoint is right Quillen).

Theorem (Quillen)

If $F \dashv G$ is a Quillen adjunction, then we have an adjunction $\mathbf{L}F \dashv \mathbf{R}G$.

(Here the natural transformations are $\eta\colon \mathrm{Id} \to GF$ and $\varepsilon\colon FG \to \mathrm{Id}$.)

This is extremely useful, but not always adequate.

Why we need more: an example

Suppose we have a commuting square of functors

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{f^*} & \mathscr{B} \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle k^*} \\ \mathscr{C} & \xrightarrow[g^*]{} & \mathscr{D} \end{array}$$

with adjunctions $f_! \dashv f^*$, $g_! \dashv g^*$, $h_! \dashv h^*$, and $k_! \dashv k^*$. Then we have a canonical *Beck-Chevalley* transformation

$$g_! k^* \longrightarrow g_! k^* f^* f_! \xrightarrow{\cong} g_! g^* h^* f_! \longrightarrow h^* f_!$$

which, in some important cases, is an isomorphism (the *Beck-Chevalley condition*).

Why we need more: an example (cont.)

Now suppose that the adjunctions on the last page are Quillen. Then we have derived adjunctions $\mathbf{L}f_! \dashv \mathbf{R}f^*$ and $\mathbf{L}g_! \dashv \mathbf{R}g^*$ and a Beck-Chevalley transformation

$$\mathbf{L}g_!\,\mathbf{R}k^* \longrightarrow \mathbf{L}g_!\,\mathbf{R}k^*\,\mathbf{R}f^*\,\mathbf{L}f_! \xrightarrow{\cong} \mathbf{L}g_!\,\mathbf{R}g^*\,\mathbf{R}h^*\,\mathbf{L}f_! \longrightarrow \mathbf{R}h^*\,\mathbf{L}f_!.$$

(Note $\mathbf{R}k^*\,\mathbf{R}f^* \cong \mathbf{R}g^*\,\mathbf{R}h^*$ by functoriality of $\mathbf{R}\colon \mathcal{RM}odel \to \mathcal{C}at$.)

Question

Is this the "derived natural transformation" of $g_!k^* \to h^*f_!$?

If it were, we could analyze it explicitly to find out whether it is an isomorphism (the *derived Beck-Chevalley condition*).

Outline

Double categories

A double category is like a 2-category, except that

- it has two types of morphisms, vertical and horizontal, and
- its 2-cells have the shape

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

  where f, g are horizontal morphisms and h, k are vertical morphisms. 2-cells can be composed both horizontally and vertically.

- Note: Horizontal and vertical morphisms cannot be composed. Nevertheless, we think of the domain of α as the "formal composite" kf and its codomain as the formal composite gh.

Examples of double categories

- In $\mathbb{C}$at, the objects are categories, the vertical and horizontal morphisms are both just functors, and the 2-cells are honest natural transformations.
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- In $\mathbb{C}$at, the objects are categories, the vertical and horizontal morphisms are both just functors, and the 2-cells are honest natural transformations.
- We can perform an analogous construction starting from any 2-category.
- In $\mathbb{M}$onCat, the objects are monoidal categories, the horizontal morphisms are lax monoidal functors, the vertical morphisms are colax monoidal functors, and the 2-cells are natural transformations such that a certain hexagon commutes.
- Likewise, we have a double category T-$\mathbb{A}$lg of algebras for any 2-monad.

Our favorite double category

There is a double category $\mathbb{M}$odel in which. . .

- The objects are model categories,
- The vertical morphisms are left Quillen functors,
- The horizontal morphisms are right Quillen functors, and
- The 2-cells of shape

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{F} & \mathscr{B} \\ {\scriptstyle H}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle K} \\ \mathscr{C} & \xrightarrow[G]{} & \mathscr{D} \end{array}$$

  are arbitrary natural transformations $KF \to GH$.

The main theorem

Theorem (S.)

There is a double pseudofunctor

$$\mathrm{Ho}\colon \mathbb{M}\mathrm{odel} \to \mathbb{C}\mathrm{at}$$

such that

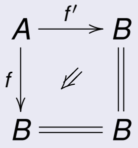
- *For a model category $\mathscr{M}$, $\mathrm{Ho}(\mathscr{M})$ is its homotopy category,*
- *For a vertical morphism F, we have $\mathrm{Ho}(F) = \mathbf{L}F$, and*
- *For a horizontal morphism F, we have $\mathrm{Ho}(F) = \mathbf{R}F$.*
- *For a 2-cell α, there is an explicit formula for $\mathrm{Ho}(\alpha)$.*

Outline

Companions

Definition

A companion pair in a double category consists of a vertical morphism $f\colon A \to B$ and a horizontal morphism $f'\colon A \to B$ together with 2-cells



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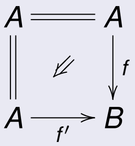
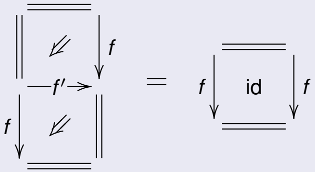


such that



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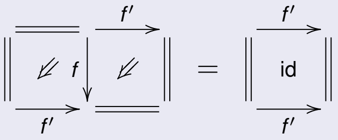

Companions, II

- A companion pair in $\mathbb{C}$at is just a natural isomorphism. In particular every morphism in $\mathbb{C}$at has a companion (itself).
- A morphism in $\mathbb{M}$onCat has a companion just when it is a strong monoidal functor.
- A morphism in $\mathbb{M}$odel has a companion just when it is both left and right Quillen.

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Theorem

Double pseudofunctors preserve companion pairs.

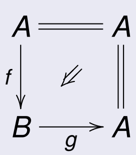
Corollary

If $F\colon \mathscr{M} \to \mathscr{N}$ is both left and right Quillen, then $\mathbf{L}F \cong \mathbf{R}F$.

Conjunctions

Definition

A conjunction in a double category consists of a vertical morphism $f\colon A \to B$ and a horizontal morphism $g\colon B \to A$ together with 2-cells



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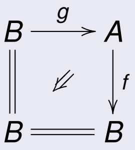
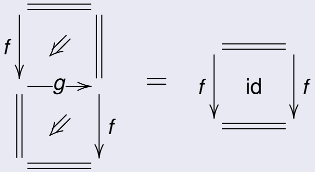


such that



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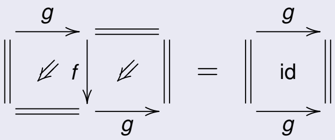

Conjoints, II

- A conjunction in $\mathbb{C}$at is an ordinary adjunction.
- A conjunction in $\mathbb{M}$onCat is a “lax/colax” adjunction.
- A conjunction in $\mathbb{M}$odel is a Quillen adjunction.

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Theorem

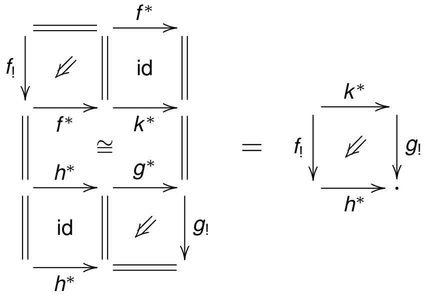
Double pseudofunctors preserve conjunctions.

Corollary

If $F \dashv G$ is a Quillen adjunction, then $\mathbf{L}F \dashv \mathbf{R}G$.

Mates

Now suppose we have conjunctions $(f_!, f^*)$ and $(g_!, g^*)$ and an isomorphism $k^* f^* \cong g^* h^*$ of horizontal arrows in a double category. The mate of this isomorphism is the composite:



In $\mathbb{C}$at, this is just the Beck-Chevalley transformation we saw before:

$$g_! k^* \longrightarrow g_! k^* f^* f_! \xrightarrow{\cong} g_! g^* h^* f_! \longrightarrow h^* f_! .$$

Mates, II

Theorem

Double pseudofunctors take mates to mates.

Corollary

Any derived Beck-Chevalley transformation is the image under Ho *of the original Beck-Chevalley transformation.*

In fact, mates are everywhere! Other applications include:

- Deriving closed monoidal functors,
- Comparing base change functors across Quillen equivalences,
- Verifying the derived projection formula,
- And so on…

Summary

- There are two different kinds of functor between model categories, which do not compose with each other. One type has left derived functors and the other has right derived functors.
- By using double categories, we can include both types of functor, together with natural transformations relating them, in a single structure.
- Since passage to homotopy categories and derived functors is functorial at the level of double categories, it preserves all the relationships between the two types of functors that can be expressed in double-categorical language. This includes isomorphisms, adjunctions, and the calculus of mates.
- Applications include many types of comparisons between composites of left and right derived functors.