Unbounded quantifiers via 2-categorical logic

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Outline

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Set theory vs. topos theory

Theorem (Cole,Mitchell,Osius)

The theory of elementary topoi is equiconsistent with a weak form of set theory: (Intuitionistic) Bounded Zermelo.

Proof (Sketch).

1. BZ $\Rightarrow$ topos: The category of sets and functions in BZ is a topos.
2. topos $\Rightarrow$ BZ: The collection of "pointed well-founded trees" in the internal logic of a topos satisfies BZ. □

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This holds the promise of valuable connections between set theory and topos theory. However, most set theorists prefer ZF to BZ.

The strong axioms

What's missing from BZ?

- Full Separation: For any set a and formula $\varphi(x)$, the set $\{x \in a \mid \varphi(x)\}$ exists.
- Replacement/Collection:

$$\forall a.\ (\forall x \in a.\ \exists y.\ \varphi(x,y)) \Rightarrow$$
$$\exists b.\ (\forall x \in a.\ \exists y \in b.\ \varphi(x,y)) \wedge (\forall y \in b.\ \exists x \in a.\ \varphi(x,y))$$

(That's not expected to make sense.)

NB: In both cases φ can include so-called "unbounded" quantifiers such as "for all sets," not only "bounded" quantifiers such as "for all *elements* of the set a."
(BZ has separation for bounded formulas.)

Why unbounded quantifiers?

BZ suffices as a foundation for much of mathematics, but not all of it.

- Universal properties in large categories: "for all X and all cones with base X, ..."
- To prove full mathematical induction, we need full separation. Some instances of induction are unprovable in BZ.
- In a recursive construction like $X, TX, T^2X, T^3X, \dots,$ BZ can construct each T^nX, but not the entire sequence or its limit.
- In particular, cardinal numbers above $\aleph_\omega$ are unreachable in BZ.
- Even some "concrete" facts depend on strong axioms, e.g. Borel Determinacy.

Set-theoretic vs. category-theoretic quantifiers

- In ordinary membership-based set theory, unbounded quantifiers are very natural, while bounded ones feel like an *ad hoc* restriction.
- In categorical logic, quantifiers are adjoints to pullback along a projection, and thus range over "elements" of the projected object. Hence, they are necessarily "bounded."

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Question

Can we extend the internal logic of a (pre)topos to deal with unbounded quantifiers? Can we give a truly topos-theoretic axiom schema which is equiconsistent with ZF?

Outline

1. Unbounded quantifiers
   - The question
   - The answer: stack semantics
2. 2-categorical logic
   - Introduction to 2-logic
   - The stack semantics as 2-logic
   - Other vistas for 2-logic

Kripke-Joyal semantics

One way to define the internal logic of a (pre)topos **S** is by a forcing relation over **S.** If φ is a formula in the internal logic, whose parameters of type A are "generalized elements" $U \xrightarrow{a} A$ for some object U, we define $U \Vdash \varphi$ inductively:

$$
\begin{array}{rcl}
U \Vdash (\varphi(a) \wedge \psi(a)) & \iff & U \Vdash \varphi(a) \text{ and } U \Vdash \psi(a) \\
U \Vdash (\varphi(a) \vee \psi(a)) & \iff & U = V \cup W\text{, where } V \Vdash \varphi(a) \text{ and} \\
& & W \Vdash \psi(a). \\
U \Vdash (\exists y \in B.\ \varphi(x, y)) & \iff & \text{there exist } V \overset{p}{\twoheadrightarrow} U \text{ and } V \xrightarrow{y} B \\
& & \text{with } V \Vdash \varphi(pa, y). \\
U \Vdash (\forall y \in B.\ \varphi(x, y)) & \iff & \text{for any } V \xrightarrow{p} U \text{ and any } V \xrightarrow{y} B, \\
& & \text{we have } V \Vdash \varphi(pa, y). \\
& \vdots &
\end{array}
$$

Stack semantics

To extend this to quantifiers over objects of **S**, rather than over elements of some fixed object A, we need a notion of a "generalized object of **S** at stage U" analogous to a generalized element $U \to A$ at stage U. The obvious choice is an object of the slice category $\mathbf{S}/U$.

Now we can use the same definition for formulas φ containing variables for objects, with the analogous extension of the forcing relation for quantifiers over objects:

$$U \Vdash (\exists X.\, \varphi(X)) \iff \text{there exist } V \overset{p}{\twoheadrightarrow} U \text{ and } Y \in \mathbf{S}/V \text{ with } V \Vdash \varphi(Y).$$

$$U \Vdash (\forall X.\, \varphi(X)) \iff \text{for any } V \overset{p}{\to} U \text{ and any } Y \in \mathbf{S}/V, \text{ we have } V \Vdash \varphi(Y).$$

This is the stack semantics of **S**. We say φ is valid in the stack semantics if $1 \Vdash \varphi$.

Examples of the stack semantics

The *precise* expression of the stack semantics is seemingly new, but it is implicit all over topos theory.

- Statements asserting universal properties are valid iff they are true "locally" in **S**.
- E.g. $1 \Vdash$ "**S** is cartesian closed" iff **S** is *locally* cartesian closed.
- $1 \Vdash$ "every epimorphism splits" iff **S** satisfies the Internal Axiom of Choice (every functor Π_A preserves epimorphisms).
- $1 \Vdash$ "every subobject of 1 is initial or terminal" iff **S** is Boolean.

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We can also generalize to allow variables in arbitrary fibrations, replacing $\mathbf{S}/U$ with $\mathbb{A}^U$. Then:

- $1 \Vdash$ "every family of objects of $\mathbb{A}$ has a product" iff $\mathbb{A}$ has **S**-indexed products.
- The indexed AFT, indexed Giraud's theorem, etc. can be proven with their usual "naive" proofs in the stack semantics.

Strong axioms

- The stack semantics always satisfies the *replacement* and *collection axioms*, suitably formulated.
- It may or may not satisfy the *separation axiom*, which says that the forcing relation is *representable*: for any φ over $U \in \mathbf{S}$, there is a subobject $[\![\varphi]\!] \hookrightarrow U$ such that $V \Vdash \varphi$ iff $V \to U$ factors through $[\![\varphi]\!]$.

The ordinary Kripke-Joyal semantics is always representable; the other way to *define* the internal logic is to construct the subobjects $[\![\varphi]\!]$.

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Definition

If **S** satisfies the above separation axiom, we call it autological.

This is a first-order axiom schema for **S**, which is fully topos-theoretic.

Some autological toposes

- The topos of sets in any model of ZF (or IZF).
- Any Grothendieck topos (over an autological base).
- Any filterquotient of an autological Boolean topos.
- The gluing of two autological toposes along a “definable” lex functor.
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Theorem

The theory of autological topoi is equiconsistent with (Intuitionistic) Zermelo-Fraenkel set theory.

The proof is essentially the same.

Outline

What is 2-categorical logic?

Ordinary categorical logic has *types* and *propositions*, with much parallel structure (Curry-Howard):

Types	Propositions
$A \times B$	$\varphi \wedge \psi$
$A \to B$	$\varphi \Rightarrow \psi$
object of $\mathscr{C}$	subterminal object of $\mathscr{C}$
dependent type $B(x)$	predicate with free variable $\varphi(x)$
object of $\mathscr{C}/A$	subobject of A
$\Pi_{x:A}B(x)$	$(\forall x : A)\varphi(x)$

- In propositional logic, with models in posets (0-categories), we have *only* propositions.
- In dependent type theory without a separate logic, we have *only* types.

Sets vs. truth values

In 2-categorical logic, we have the potential for *three* sorts of types.

1-types	0-types	(-1)-types
“categories”	“sets”	“propositions”
object of $\mathcal{K}$	discrete object	subterminal object

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I prefer to say

Sets are already analogous to truth values in a 1-topos, via Curry-Howard. A 2-topos simply has *three* such analogons instead of two. In particular, 2-categorical logic still needs ordinary truth values.

Subobjects in a 2-category

Definition

A morphism $f\colon A \to B$ in a 2-category $\mathscr{K}$ is fully-faithful, or 1-monic, if for all objects X in $\mathscr{K}$, the functor

$$\mathscr{K}(X,A) \to \mathscr{K}(X,B)$$

is fully faithful.

These are the "subobjects" in a 2-category, which represent propositions. The classifier $[\![\varphi(x)]\!] \hookrightarrow A$ of a formula φ represents the *full subcategory* of A determined by the objects x such that $\varphi(x)$.

Heyting 2-categories

To interpret logic of propositions, need operations on subobjects.

- A regular 2-category has stable factorizations into fully-faithful morphisms and their orthogonal complement, called *essentially-surjective* or *strong 1-epic*.
- A coherent 2-category also has stable unions of subobjects.
- A Heyting 2-category also has dual images of subobjects.

Observation

In a Heyting 2-category, we can interpret "first-order 2-categorical logic" with 1-types, 0-types and (-1)-types.

The 2-Giraud theorem

Can also define *extensive* and *exact* 2-categories, and thereby *2-pretoposes*.

Theorem (Street 1982)

The following are equivalent for a 2-category $\mathscr{K}$.

1. *$\mathscr{K}$ is an infinitary 2-pretopos with a generating set.*
2. *$\mathscr{K}$ is the 2-category of stacks on some 2-site.*

Such a 2-category is called a Grothendieck 2-topos. It is, in particular, a Heyting 2-category.

Outline

Stacks on a topos

Let $\mathbf{S}$ be a (pre)topos, and $\mathcal{S}t(\mathbf{S})$ the 2-category of stacks for its coherent topology. Then:

- $\mathcal{S}t(\mathbf{S})$ is a Grothendieck 2-topos.
- It contains $\mathbf{S}$ as a full, generating, Heyting subcategory.
- Therefore, the internal logic of $\mathcal{S}t(\mathbf{S})$ reduces to a Kripke-Joyal-like semantics over $\mathbf{S}$.

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- $\mathcal{S}t(\mathbf{S})$ also contains the self-indexing $\mathbb{S}$, defined by $X \mapsto \mathbf{S}/X$.
- Generalized elements $U \to \mathbb{S}$, for $U \in \mathbf{S}$, are the same as objects of $\mathbb{S}^U = \mathbf{S}/U$.

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Conclusion

The stack semantics is precisely the fragment of the internal logic of the 2-category $\mathcal{S}t(\mathbf{S})$ which deals only with $\mathbb{S}$.

2-categorical logic vs. categories of classes

If we are sufficiently careful, we can also characterize the stack semantics using the ordinary 1-categorical logic of *sheaves* (or "ideals," etc.) on **S**. However:

1. The 2-category $\mathcal{S}t(\mathbf{S})$ is more intuitively satisfying to a (higher) category theorist. Outside of set theory, we usually only care about the elements of a proper class insofar as they form the objects of some large category.

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2. If **S** is autological, we can embed it in a 2-category of *definable stacks* which satisfies the separation axiom that *all* subobjects of representables are small. I don't know whether it is possible, in general, to produce a 1-category with this property.

Outline

The axiom of choice

Observation

A morphism $f\colon A \to B$ in a regular 2-category $\mathscr{K}$ is an equivalence if and only if the statement "f is fully faithful and essentially surjective" is valid in the internal logic of $\mathscr{K}$.

In particular, this has nothing to do with the axiom of choice! It is a "functor comprehension principle" or "axiom of non-choice."

This can be mimicked using internal anafunctors in a 1-category, but the 2-categorical context is more natural.

Other 2-topoi

Just as there are non-localic 1-topoi, there are (Grothendieck) 2-topoi that don't arise from 1-topoi.

Examples

- The 2-categories of categories-with-a-monad, or of adjunctions (these are "presheaf" 2-topoi).
- The 2-category of categories with an action by some monoidal category $\mathcal{V}$ (a.k.a. "$\mathcal{V}$-actegories").
- The classifying 2-topos of any "geometric 2-theory," such as the theory of monoidal categories.

The internal logic of such 2-topoi awaits exploration.

For more information...

http://ncatlab.org/michaelshulman

Including:

- A draft of the paper *Stack semantics and the comparison of material and structural set theories*.
- A partial development of general 2-categorical logic (work in progress).