Anchored Bicategories

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In memory of
Saunders Mac Lane

Why Anchored Bicategories?

Motivated by a desire to fix some problems arising from the use of bicategories to model certain mathematical structures (to be specified).

Three advantages over classical bicategories in these cases:

1. Correct notions of equivalence;
2. Base change functors; and
3. Allow more general variance.

What are Bicategories?

Recall that a bicategory $\mathscr{B}$ has

- Objects $A, B, C \ldots$;
- Hom-categories $\mathscr{B}(A, B)$;
- Composition $\odot : \mathscr{B}(B, C) \times \mathscr{B}(A, B) \to \mathscr{B}(A, C)$;
- Units $U_A \in \mathscr{B}(A, A)$;
- Constraint isomorphism and coherence axioms.

Examples of Bicategories - I

The bicategory $\mathscr{B}imod$ has...

- Objects = (not necessarily commutative) rings;
- $\mathscr{B}imod(A, B)$ = the category of (B, A)-bimodules.
- $\mathscr{B}imod(B, C) \times \mathscr{B}imod(A, B) \to \mathscr{B}imod(A, C)$ maps M and N to $M \otimes_B N$.
- U_A is A itself as an (A, A)-bimodule.

(More generally, take noncommutative R-algebras for some commutative ring R.)

Examples of Bicategories - II

The bicategory $\mathscr{E}x$ has...

- Objects = topological spaces;
- $\mathscr{E}x(A, B)$ = spectra parametrized over $B \times A$
- $\mathscr{E}x(B, C) \times \mathscr{E}x(A, B) \to \mathscr{E}x(A, C)$ maps E and F to $\pi_! \Delta^*(E \overline{\wedge} F)$.
- U_A is $\Delta_! S_A$.

Examples of Bicategories - III

The bicategory $\mathscr{D}ist$ has...

- Objects = small categories
- $\mathscr{D}ist(A,B)$ = the category of functors $B^{op} \times A \to \mathbf{Set}$. These are called *distributors*, also known as profunctors or bimodules.
- $\mathscr{D}ist(B,C) \times \mathscr{D}ist(A,B) \to \mathscr{D}ist(A,C)$ maps F and G to $F \otimes_B G$, where

$$(F \otimes_B G)(c,a) = \int^{b \in B} F(c,b) \times G(b,a)$$

- U_A is the hom-functor

$$A(-,-) : A^{op} \times A \to \mathbf{Set}.$$

(More generally, the categories can be enriched over anything you please.)

Examples of Bicategories - IV

The bicategory $n\mathscr{C}ob$ has

- Objects = oriented $(n-1)$-manifolds;
- $n\mathscr{C}ob(A, B)$ = the category of n-manifolds with boundary $B \sqcup A^*$, where A^* is A with the opposite orientation. The morphisms are diffeomorphisms.
- $n\mathscr{C}ob(B, C) \times n\mathscr{C}ob(A, B) \to n\mathscr{C}ob(A, C)$ is gluing of cobordisms along B.
- U_A is the cylinder $A \times [0, 1]$.

	1	2	3	4
0-cells	(noncommutative) rings	topological spaces	small categories	$(n-1)$-manifolds
horizontal 1-cells	bimodules	parametrized spectra	distributors (profunctors, bimodules)	n-cobordisms
2-cells	bimodule maps	maps of spectra	natural transformations	n-diffeomorphisms
vertical 1-cells	ring homomorphisms	continuous maps	functors	$(n-1)$-diffeomorphisms
composition $\odot$	tensor product $\otimes$	$\pi_! \Delta^* (E \overline{\wedge} F)$	tensor product of functors	gluing
units U_A	A itself	$\Delta_! S_A$	$A(-,-)$	cylinder $A \times [0,1]$
base change f^*, $f_!$	restriction/extension of scalars	f^*, $f_!$	restriction and left Kan extension	reparametrization of boundary
monoidal structure	tensor product of rings $\otimes$	cartesian product of spaces $\times$	cartesian product of categories $\times$	disjoint union $\sqcup$
pseudoduals A^*	A^{op} (A with the opposite multiplication)	A itself	the opposite category A^{op}	A^* with reversed orientation

Problems With Bicategories I: Internal Equivalence

Recall: Two 0-cells A, B in a bicategory $\mathscr{B}$ are said to be *equivalent* if there exist 1-cells

$$f : A \to B$$
$$g : B \to A$$

and 2-isomorphisms $fg \cong U_B$, $gf \cong U_A$.

Thus, two rings A and B are equivalent, in the bicategory of bimodules, if there exist

an (A, B)-bimodule M; and
a (B, A)-bimodule N

such that $M \otimes_B N \cong A$ and $N \otimes_A M \cong B$.

This is *Morita equivalence*: interesting, but not fundamental.

Problems With Bicategories II: External Equivalence

Recall: Two bicategories $\mathscr{B}$ and $\mathscr{B}'$ are *biequivalent* when we have weak 2-functors

$$F : \mathscr{B} \to \mathscr{B}'$$
$$G : \mathscr{B}' \to \mathscr{B}$$

and transformations which are equivalences

$$FG \simeq \mathrm{Id}_{\mathscr{B}'} \qquad \text{and} \qquad GF \simeq \mathrm{Id}_{\mathscr{B}}\,.$$

In particular, we have internal equivalences

$$FGA \simeq A \qquad \text{and} \qquad GFB \simeq B$$

so external equivalence inherits problems from internal equivalence.

Double Categories: Abstract Definition

A *weak double category* $\mathbb{D}$ is a weak category object in **Cat**. Thus we have two categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with functors

$$s, t : \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$$

$$U : \mathbb{D}_0 \to \mathbb{D}_1$$

$$\odot : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$$

such that $sU = tU = \mathrm{Id}_{\mathbb{D}_0}$,

$$\begin{array}{ccc} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \longrightarrow & \mathbb{D}_1 \\ & \searrow & \downarrow \\ & & \mathbb{D}_0 \times \mathbb{D}_0 \end{array}$$

and natural isomorphisms

$$(M \odot N) \odot P \cong M \odot (N \odot P)$$

$$U_A \odot M \cong M \cong M \odot U_B$$

satisfying the same coherence laws as a monoidal category or a bicategory.

Double Categories: Concrete Definition

A *weak double category* consists of

- A set of objects ($\mathrm{ob}\,\mathbb{D}_0$)
- A set of vertical arrows ($\mathrm{mor}\,\mathbb{D}_0$);
- A set of horizontal arrows ($\mathrm{ob}\,\mathbb{D}_1$);
- A set of squares ($\mathrm{mor}\,\mathbb{D}_1$). We say a square whose horizontal source and target are identities is *globular*.
- Vertical composition and identities, forming a category;
- Horizontal composition and identities, forming a category up to coherent globular isomorphism;
- Squares compose "in square ways."

Every double category $\mathbb{D}$ has an "underlying bicategory" $\mathscr{D}$ whose:

- 0-cells are the objects of $\mathbb{D}$;
- 1-cells are the horizontal arrows of $\mathbb{D}$; and whose
- 2-cells are the globular squares of $\mathbb{D}$.

All our examples naturally become double categories. For example, we have a double category $\mathbb{B}\mathbf{imod}$ whose:

- Vertical arrows are ring homomorphisms; and whose
- Squares

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & \phi & \downarrow{\scriptstyle g} \\ C & \xrightarrow[N]{} & D \end{array}$$

are (g, f)-equivariant maps $\phi : M \to N$.

The 2-category of double categories

A *(weak) double functor* $F : \mathbb{D} \to \mathbb{E}$ consists of $F_0 : \mathbb{D}_0 \to \mathbb{E}_0$, $F_1 : \mathbb{D}_1 \to \mathbb{E}_1$, commuting with s, t strictly, and natural isomorphisms

$$U_{F_0 A} \cong F_1 U_A$$

$$F_1 M \odot F_1 N \cong F_1(M \odot N)$$

satisfying obvious coherence laws.

A *(vertical) transformation* $\alpha : F \to G$ consists of, for each object A, a vertical arrow

$$\alpha_A : F_0 A \to G_0 A,$$

and for every horizontal arrow $M : A \to B$, a square

$$\begin{array}{ccc} FA & \xrightarrow{FM} & FB \\ {\scriptstyle \alpha_A}\downarrow & \alpha_M & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow[GM]{} & GB \end{array}$$

satisfying obvious naturality axioms.

Observation 1. *Double categories, functors, and transformations form a 2-category* **Dbl***.*

Equivalence of double categories is internal equivalence in this 2-category. This solves the external equivalence problem.

We can also define adjoint double functors and so on, using this 2-category.

Observe that...

- Double categories have underlying bicategories;
- Double functors have underlying weak 2-functors; but
- Vertical transformations do **not** in general have underlying 2-natural transformations.

This looks familiar from considering monoidal categories as degenerate bicategories. In fact, we have:

Observation 2. *Every monoidal category $\mathscr{C}$ gives a double category $\mathbb{C}$ with $\mathbb{C}_0 = *$ and $\mathbb{C}_1 = \mathscr{C}$. This is a 2-fully-faithful embedding*

$$\mathbf{MonCat} \hookrightarrow \mathbf{Dbl}.$$

Folding Structures

Recall that a *folding structure* on a double category consists of

- a "holonomy" $f \mapsto A_f$, which is (in our case) a weak 2-functor $\mathbb{D}_0 \to \mathscr{D}$ from the vertical category to the horizontal bicategory;
- Bijections

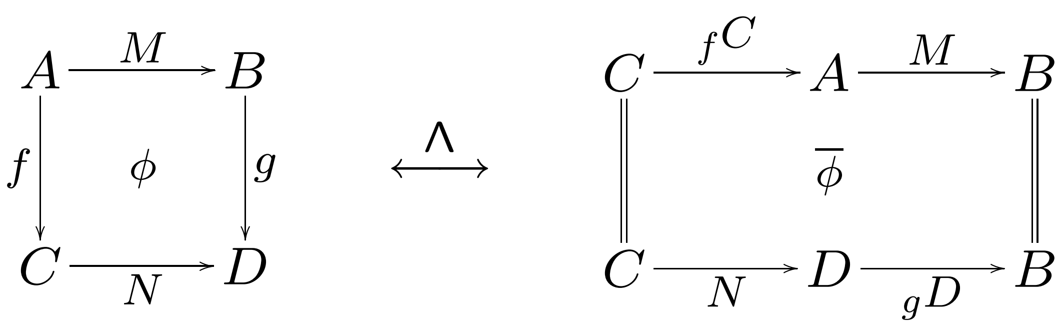
$$
\begin{array}{ccc}
A & \xrightarrow{M} & B \\
{\scriptstyle f}\downarrow & \phi & \downarrow{\scriptstyle g} \\
C & \xrightarrow[N]{} & D
\end{array}
\quad \overset{\Lambda}{\longleftrightarrow} \quad
\begin{array}{ccccc}
A & \xrightarrow{M} & B & \xrightarrow{D_g} & D \\
\| & & \overline{\phi} & & \| \\
A & \xrightarrow[C_f]{} & C & \xrightarrow[N]{} & D
\end{array}
$$

satisfying certain axioms.

Opfolding Structures

An *opfolding structure* on a double category consists of

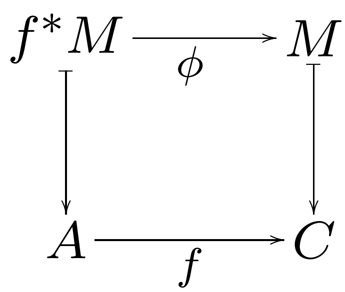
- an "opholonomy" $f \mapsto {}_f A$, which is (in our case) a *contravariant* weak 2-functor $\mathbb{D}_0{}^{op} \to \mathscr{D}$;
- Bijections



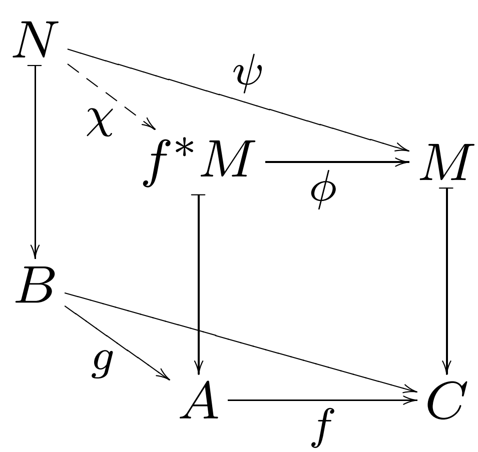
satisfying analogous axioms.

Fibrations

Let $p : \mathscr{E} \to \mathscr{B}$ be a functor between categories, let $f : A \to C$ be an arrow in $\mathscr{B}$, and let M be an object of $\mathscr{E}$ with $p(M) = C$. An arrow $\phi : f^*M \to M$ in $\mathscr{E}$ is *cartesian* if $p(\phi) = f$:



and if $\psi : N \to M$ is an arrow in $\mathscr{E}$ such that $p(\psi) = fg$ for some $g : B \to A$, then $\psi = \phi\chi$ for a unique χ with $p(\chi) = g$:



We say that p is a *fibration* if for every f and M, there exists a cartesian $\phi : f^*M \to M$.

By choosing such cartesian arrows, we obtain for each $f : A \to C$, a functor f^* from the fiber over C to the fiber over A. These are functorial:

Proposition 3. *Fibrations over $\mathscr{B}$ are essentially equivalent to weak 2-functors $\mathscr{B}^{op} \to \mathbf{Cat}$.*

Similarly, we define *opcartesian* arrows and *opfibrations*, and we have that

Proposition 4. *A fibration p is also an opfibration precisely when all the functors f^* have left adjoints $f_!$.*

Theorem 5. *A double category $\mathbb{D}$ admits both a folding structure and a opfolding structure if and only if:*

- $(s,t) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ *is a fibration and opfibration, and*
- *cartesian and opcartesian squares compose along vertical isomorphisms; i.e.*

$$\begin{array}{ccc} A & \xrightarrow{M_f} & B \\ {\scriptstyle f}\downarrow & \phi_1 & \downarrow{\scriptstyle \cong} \\ D & \xrightarrow[M]{} & E \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} B & \xrightarrow{{}_gN} & C \\ {\scriptstyle \cong}\downarrow & \phi_2 & \downarrow{\scriptstyle g} \\ E & \xrightarrow[N]{} & F \end{array}$$

are cartesian, then so is the composite

$$\begin{array}{ccc} A & \xrightarrow{{}_gN \odot M_f} & C \\ {\scriptstyle f}\downarrow & \phi_2 \odot \phi_1 & \downarrow{\scriptstyle g} \\ D & \xrightarrow[N \odot M]{} & F \end{array}.$$

and dually for opcartesian squares.

A double category with these properties we call a *framed bicategory.*

Fibrations have many useful properties, for example:

Proposition 6. *If a square of functors*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{f} & \mathscr{E} \\ {\scriptstyle p}\downarrow & \cong & \downarrow{\scriptstyle q} \\ \mathscr{C} & \xrightarrow[g]{} & \mathscr{B} \end{array}$$

commutes up to isomorphism and q is a fibration, then there is a functor $f' \cong f$ such that the square

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{f'} & \mathscr{E} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathscr{C} & \xrightarrow[g]{} & \mathscr{B} \end{array}$$

commutes strictly.

This gives another explanation of why we can require double functors to preserve s and t strictly.

Monoidal Framed Bicategories

A *monoidal double category* is a weak monoid in the 2-category **Dbl**.

Somewhat more explicitly, it is a double category $\mathbb{D}$ such that

- $\mathbb{D}_0$ and $\mathbb{D}_1$ are both monoidal categories;
- s and t are strict monoidal functors;
- U is a strong monoidal functor;
- $\odot$ is a strong monoidal functor.

In particular, we have *interchange* isomorphisms

$$(M \otimes N) \odot (P \otimes Q) \cong (M \odot P) \otimes (N \odot Q).$$

A *monoidal framed bicategory* is a framed bicategory which is a monoidal double category, and such that $\otimes$ preserves cartesian and opcartesian arrows in $\mathbb{D}_1$.

In all our examples, the monoidal structure is playing a deeper role. Namely:

- A (B, A)-bimodule is just a left $B \otimes A^{op}$-module (here A^{op} is A with the opposite multiplication);
- A parametrized spectrum from A to B is, by definition, a spectrum parametrized over $B \times A$;
- A distributor from A to B is a functor $B \times A^{op} \to \mathbf{Set}$;
- A cobordism from A to B is an n-manifold with boundary $B \sqcup A^*$.

Compact Monoidal Categories

Let $\mathscr{C}$ be a (symmetric) monoidal category. The *dual* of an object A of $\mathscr{C}$ is an object A^* together with maps

$$\varepsilon : A^* \otimes A \to I$$

$$\eta : I \to A \otimes A^*$$

satisfying the triangle identities.

This is equivalent to giving a natural isomorphism

$$\mathscr{C}(B \otimes A, C) \cong \mathscr{C}(B, C \otimes A^*).$$

A symmetric monoidal category is *compact* if all objects have duals.

Traces

Let $\mathscr{C}$ be a compact symmetric monoidal category and A an object of $\mathscr{C}$. The *trace* of an endomorphism $F : A \to A$ is the composite

$$I \xrightarrow{\eta} A \otimes A^* \xrightarrow{F \otimes 1} A \otimes A^* \xrightarrow{\cong} A^* \otimes A \xrightarrow{\varepsilon} I$$

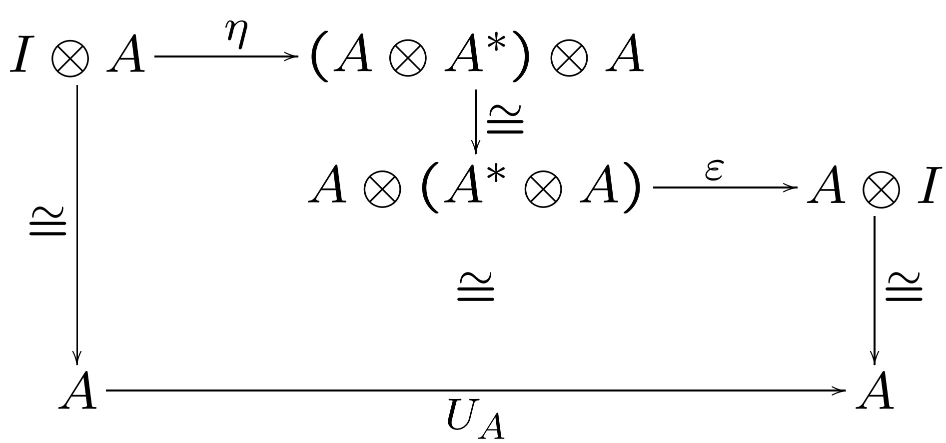
This is an endomorphism of the unit I.

We can easily check, using the triangle identities, that

$$\mathsf{Tr}(FG) = \mathsf{Tr}(GF)$$

Anchored Bicategories - I

Let $\mathbb{D}$ be a (symmetric) monoidal framed bicategory. A *pseudodual* for a 0-cell A is a 0-cell A^* equipped with horizontal 1-cells $\varepsilon : A^* \otimes A \nrightarrow I$ and $\eta : I \nrightarrow A \otimes A^*$, and iso-squares, one of which is:



satisfying some coherence axioms.

This gives rise to equivalences

$$\mathscr{D}(B \otimes A, C) \simeq \mathscr{D}(B, C \otimes A^*).$$

Anchored Bicategories - II

For example, A^{op} is the pseudodual of A in $\mathbb{B}\mathbf{imod}$.

- ε is A, considered as an $(I, A^{op} \otimes A)$-bimodule, and
- η is also A, considered as an $(A \otimes A^{op}, I)$-bimodule.

A symmetric monoidal framed bicategory is *compact* if every 0-cell has a pseudodual. We also call a compact symmetric monoidal framed bicategory an *anchored bicategory*.

2-Traces

Let $M : A \to A$ be a horizontal arrow in an anchored bicategory. Its *2-trace* is the composite

$$
\begin{array}{ccccc}
I \xrightarrow{\eta} A \otimes A^* & \xrightarrow{M \otimes 1} & A \otimes A^* & & \\
 & & \downarrow \cong & & \\
 & & A^* \otimes A & \xrightarrow{\varepsilon} & I
\end{array}
$$

This is a horizontal arrow $\mathrm{Tr}_2(M)$ from I to I.

For example, if M is an (A, A)-bimodule, then we have

$$\mathrm{Tr}_2(M) = M \otimes_{A \otimes A^{op}} A$$

which is the "underived" version of Hochschild homology.

If A is a category and H is a distributor from A to A, then

$$\mathrm{Tr}_2(H) = \int^{a \in A} H(a, a)$$

is its coend. If $H = A$, this is the codomain of the *universal trace function* for A.

Noncommutative traces

Notice that just as $\mathsf{Tr}(FG) = \mathsf{Tr}(GF)$, we always have

$$\mathsf{Tr}_2(M \odot N) \cong \mathsf{Tr}_2(N \odot M).$$

Let M be a horizontal arrow from A to B which is dualizable in the bicategory $\mathscr{B}$, i.e. it has an adjoint N from B to A.

Let $f : M \to M$ be an endomorphism of M. Its *trace* can be defined to be the composite

$$\mathsf{Tr}_2(B) \xrightarrow{\eta} \mathsf{Tr}_2(M \odot N) \xrightarrow{f \odot 1} \mathsf{Tr}_2(M \odot N)$$
$$\xrightarrow{\cong} \mathsf{Tr}_2(N \odot M) \xrightarrow{\varepsilon} \mathsf{Tr}_2(A)$$

Note that the source and target are now different.

Kate Ponto has found this sort of trace useful in fixed point theory.